JOURNAL OF ALGEBRA 149, 438–453 (1992)

Derived Functors of I-adic Completion and Local Homology

J. P. C. GREENLEES AND J. P. MAY

Department of Mathematics, University of Chicago, Chicago, Illinois 60637

Communicated by Richard G. Swan

Received June 1, 1990

In recent topological work [2], we were forced to consider the left derived functors of the I-adic completion functor, where I is a finitely generated ideal in a commutative ring A. While our concern in [2] was with a particular class of rings, namely the Burnside rings $A(G)$ of compact Lie groups G, much of the foundational work we needed was not restricted to this special case.

The essential point is that the modules we consider in [2] need not be finitely generated and, unless G is finite, say, the ring $A(G)$ is not Noetherian. There seems to be remarkably little information in the literature about the behavior of I-adic completion in this generality. We presume that interesting non-Noetherian commutative rings and interesting non-finitely generated modules arise in subjects other than topology. We have therefore chosen to present our algebraic work separately, in the hope that it may be of value to mathematicians working in other fields.

One consequence of our study, explained in Section 1, is that I-adic completion is exact on a much larger class of modules than might be expected from the key role played by the Artin–Rees lemma and that the deviations from exactness can be computed in terms of torsion products.

However, the most interesting consequence, discussed in Section 2, is that the left derived functors of I-adic completion usually can be computed in terms of certain local homology groups, which are defined in a fashion dual to the definition of the classical local cohomology groups of Grothendieck. These new local homology groups may well be relevant to algebraists and algebraic geometers.

In particular, we obtain a universal coefficients theorem for calculating these groups from local cohomology in Section 3; the classical local duality spectral sequence is a very special case.

The brief Section 4 gives an analysis of the behavior of composites of left derived functors of I-adic completion. The still briefer Section 5 describes

0021-8693/92 $5.00

the right derived functors of I-adic completion, which are much less interesting (and irrelevant to our topological applications).

We restrict ourselves to the main points here, and the arguments are quite elementary. Commutative ring theorists will see that we have left many very natural questions unanswered. In particular, we have left sheaf theoretic generalizations to the reader.

0. Preliminaries

To establish notations and context, we recall briefly the definitions of left derived functors and of some basic constructions that we shall use. Let $A\mathcal{M}$ be the category of modules over a commutative ring A.

A ∂-functor $\mathcal{D}$ is a sequence $\{D_i \mid i \geqslant 0\}$ of covariant functors $D_i: A\mathcal{M} \to A\mathcal{M}$ together with natural connecting homomorphisms $\partial_i: D_i(M'') \to D_{i-1}(M')$ for short exact sequences

$$0 \to M' \to M \to M'' \to 0$$

such that the following are zero sequences (all composites are zero):

$$\begin{aligned} \cdots &\to D_i(M') \to D_i(M) \to D_i(M'') \to D_{i-1}(M') \\ &\to \cdots \to D_0(M) \to D_0(M'') \to 0; \end{aligned}$$

$\mathcal{D}$ is exact if these sequences are exact. $\mathcal{D}$ is effaceable if, for each M, there is an epimorphism $N \to M$ such that $D_i N \to D_i M$ is zero for $i > 0$. This obviously holds if $D_i F = 0$ for $i > 0$ when F is free.

Let Γ: $A\mathcal{M} \to A\mathcal{M}$ be an additive functor. Its left derived functors are given by an exact and effaceable ∂-functor $\mathcal{L}\Gamma = \{L_i\Gamma\}$ together with a natural transformation ε: $L_0\Gamma \to \Gamma$, which is an isomorphism on free modules. The functor $L_0\Gamma$ is right exact and its left derived functors for $i > 0$ are the same as those of Γ. For any ∂-functor $\mathcal{D}$, a natural tranformation f_0: $D_0 \to L_0\Gamma$ extends uniquely to a map $\{f_i\}$: $\mathcal{D} \to \mathcal{L}\Gamma$ of ∂-functors. Moreover, $\{f_i\}$ is an isomorphism if and only if $\mathcal{D}$ is exact and effaceable and f_0 is an isomorphism on free modules. For an A-module M, $\mathcal{L}\Gamma M$ can be constructed by taking the homology of the complex obtained by applying Γ to a free resolution of M. Details may be found in [1, V, Sects. 2–3].

Define the cone, or cofiber, Ck of a chain map k: $X \to Y$ by $(Ck)_i = Y_i \oplus X_{i-1}$, with differential $d_i(y, x) = (d_i(y) + k_{i-1}(x), -d_{i-1}(x))$. Define the suspension ΣX and desuspension $\Sigma^{-1}X$ by $(\Sigma X)_i = X_{i-1}$ and $(\Sigma^{-1}X)_i = X_{i+1}$, with the differential $-d$. We have a short exact sequence $0 \to Y \to Ck \to \Sigma X \to 0$, and the connecting homomorphism of the derived long exact sequence in homology is k_*. It is convenient to define the fiber of k to be $Fk = \Sigma^{-1}C(-k)$.

Given a sequence of chain maps $f^r\colon X^r \to X^{r+1}$, $r \geqslant 0$, define a map ι: $\bigoplus X^r \to \bigoplus X^r$ by $\iota(x) = x - f^r(x)$ for $x \in X^r$. Define the homotopy colimit, or telescope, of the sequence $\{f^r\}$ to be $C\iota$ and denote it $\mathrm{Tel}(X^r)$. Then $H_i(\mathrm{Tel}(X^r)) = \mathrm{Colim}\, H_i(X^r)$. The composite of the projection from $C\iota$ to its first variable and the canonical map $\bigoplus X^r \to \mathrm{Colim}\, X^r$ is a homology isomorphism $\zeta\colon \mathrm{Tel}(X^r) \to \mathrm{Colim}\, X^r$.

We shall need an observation about the behavior of telescopes with respect to tensor products. Given two sequences $f^r\colon X^r \to X^{r+1}$ and $g^s\colon Y^s \to Y^{s+1}$, we obtain a sequence $f^r \otimes g^r\colon X^r \otimes Y^r \to X^{r+1} \otimes Y^{r+1}$.

LEMMA 0.1. *There is a natural homology isomorphism*

$$\xi\colon \mathrm{Tel}(X^r \otimes Y^r) \to \mathrm{Tel}(X^r) \otimes \mathrm{Tel}(Y^s).$$

Proof. Using an ordered pair notation for elements of the relevant cofibers, we specify ξ by the explicit formula

$$\xi(x' \otimes y', x \otimes y) = (0, x) \otimes (y, 0) + (-1)^{\deg(x)}(f(x), 0) \otimes (0, y)$$
$$+ (x', 0) \otimes (y', 0).$$

A tedious computation shows that ξ commutes with differentials. It is a homology isomorphism because the diagram

$$\begin{array}{ccc}
\mathrm{Tel}(X^r \otimes Y^r) & \xrightarrow{\qquad\qquad} & \mathrm{Tel}(X^r) \otimes \mathrm{Tel}(Y^s) \\
\downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \zeta \otimes \zeta} \\
\mathrm{Colim}_r(X^r \otimes Y^r) \to \mathrm{Colim}_{r,s}(X^r \otimes Y^s) & \cong & \mathrm{Colim}_r(X^r) \otimes \mathrm{Colim}_s(Y^s)
\end{array}$$

commutes. Here the bottom left arrow is the diagonal cofinality isomorphism.

Dually to the telescope, given chain maps $f^r\colon X^r \to X^{r-1}$ for $r \geqslant 1$, define a map $\pi\colon \times X^r \to \times X^r$ by $\pi(x^r) = (x^r - f^{r+1}(x^{r+1}))$. Define the homotopy limit, or microscope, of the sequence $\{f^r\}$ to be $F\pi$ and denote it $\mathrm{Mic}(X^r)$. Then there are short exact sequences

$$(0.2) \qquad 0 \to \mathrm{Lim}^1\, H_{i+1}(X^r) \to H_i(\mathrm{Mic}(X^r)) \to \mathrm{Lim}\, H_i(X^r) \to 0.$$

Observe that a degreewise short exact sequence

$$0 \to \{X^r\} \to \{Y^r\} \to \{Z^r\} \to 0$$

of systems of chain complexes gives rise to a short exact sequence

$$0 \to \mathrm{Mic}(X^r) \to \mathrm{Mic}(Y^r) \to \mathrm{Mic}(Z^r) \to 0$$

and thus to a long exact sequence of homology groups.

Here Lim^1 denotes the first right derived functor of the inverse limit functor. We shall be concerned only with inverse sequences, for which the higher right derived functors of Lim vanish. Thus a short exact sequence of inverse sequences gives a six term exact sequence of Lim's and Lim^1's. We say that an inverse sequence $\{M^r\}$ is pro-zero if, for each r, there exists $s > r$ such that $M^s \to M^r$ is zero; of course, if $\{M^r\}$ is pro-zero, then $\mathrm{Lim}\, M^r = 0$ and $\mathrm{Lim}^1\, M^r = 0$.

1. The Left Derived Functors of I-adic Completion

Let I be an ideal in our commutative ring A. For an A-module M, define $M_I^\wedge = \mathrm{Lim}\, M/I^rM$. Let L_i^I denote the i^{th} left derived functor of I-adic completion. We begin by obtaining a construction of these functors that leads to a description of the $L_i^I(M)$ in terms of torsion products. Let X^r be a free resolution of A/I^r and construct chain maps f^r: $X^r \to X^{r-1}$ over the quotient maps $A/I^r \to A/I^{r-1}$.

Proposition 1.1. *The functors $L_i^I(M)$ are computable as the homology groups of the complexes* $\mathrm{Mic}(X^r \otimes M)$. *Therefore, by* (0.2), *there are short exact sequences* (*the rightmost term in the zeroth being $M_I^\wedge$*)

$$0 \to \mathrm{Lim}^1\, \mathrm{Tor}_{i+1}^A(A/I^r, M) \to L_i^I \to \mathrm{Lim}\, \mathrm{Tor}_i^A(A/I^r, M) \to 0.$$

Proof. The $H_*(\mathrm{Mic}(X^r \otimes M))$ clearly give an exact ∂-functor. If M is free, the evident natural map ε: $H_0(\mathrm{Mic}(X^r \otimes M)) \to \mathrm{Lim}(A/I^r \otimes M) = M_I^\wedge$ is an isomorphism and $H_i(\mathrm{Mic}(X^r \otimes M)) = 0$ for $i > 0$.

We need some restrictive hypotheses to proceed further. In the rest of the paper, all ideals are assumed to be finitely generated.

Definition 1.2. Let $\alpha \in A$. For an A-module M, let $\Gamma(\alpha; M)$ denote the kernel of α: $M \to M$ and observe that $\Gamma(\alpha^r; M) \subset \Gamma(\alpha^{r+1}; M)$ for $r \geqslant 1$. Say that M has bounded α-torsion if this increasing sequence stabilizes, for example if A is Noetherian and M is finitely generated.

Remarks 1.3. (i) Observe that α: $M \to M$ restricts to a map $\Gamma(\alpha^{r+1}; M) \to \Gamma(\alpha^r; M)$ for each r. It is easily checked that M has bounded α-torsion if and only if the inverse sequence $\{\Gamma(\alpha^r; M)\}$ is pro-zero. Thus $\mathrm{Lim}\, \Gamma(\alpha^r; M) = 0$ and $\mathrm{Lim}^1\, \Gamma(\alpha^r; M) = 0$ if M has bounded α-torsion.

(ii) If $N \subset M$, then $\Gamma(\alpha^r; N) = \Gamma(\alpha^r; M) \cap N$, so that N has bounded α-torsion if M does. If each of a set M_k of A-modules has bounded α-torsion with a common bound r, then the sum and product of the M_k have bounded α-torsion. In particular, if A itself has bounded α-torsion, then so does every submodule of any free A-module.

EXAMPLES 1.4. If A is the quotient of the polynomial ring generated by $\{\alpha, y_r \mid r \geqslant 1\}$ by the ideal generated by $\{\alpha^r y_r \mid r \geqslant 1\}$, then A has unbounded α-torsion. As pointed out by Swan, if k is a field and if α, β, and x_s, $s \geqslant 1$, are indeterminates, then the sub k-algebra A of $k(\alpha, \beta, x_s)$ which is generated by α, β, the x_s, and the elements $y_{s,r} = \alpha^s x_s/\beta^r$ for $s \geqslant r \geqslant 1$ is an example of an integral domain in which $A/(\beta^r)$ has unbounded α-torsion for every r.

PROPOSITION 1.5. *Let $I = (\alpha)$ and assume that A has bounded α-torsion. If* Lim $\Gamma(\alpha^r; M) = 0$ *and* $\mathrm{Lim}^1\, \Gamma(\alpha^r; M) = 0$, *for example if M has bounded α-torsion, then $L_0^I(M) \cong M_I^\wedge$* and *$L_i^I(M) = 0$ for $i > 0$. Moreover, the following conclusions hold for any A-module M.*

(i) *There is a short exact sequence*

$$0 \to \mathrm{Lim}^1\, \mathrm{Tor}_1^A(A/I^r, M) \to L_0^I(M) \to M_I^\wedge \to 0.$$

(ii) $L_1^I(M) \cong \mathrm{Lim}\, \mathrm{Tor}_1^A(A/I^r, M)$.

(iii) $L_i^I(M)$ *for* $i \geqslant 2$.

Proof. Tensoring M with the diagram

$$\begin{array}{ccccccccc} 0 \to & \Gamma(\alpha; A) & \to & A & \to & (\alpha) & \to 0 \\ & & & \downarrow{\scriptstyle \alpha} & & & \\ & 0 \to (\alpha) & \to & \mathrm{A} & \to & A/(\alpha) & \to 0 \end{array}$$

and inspecting, we see that $\mathrm{Tor}_1(A/I, M) \cong \Gamma(\alpha; M)/\Gamma(\alpha; A)\, M$. There results an exact sequence

$$0 \to \Gamma(\alpha^r; A)\, M \to \Gamma(\alpha^r; M) \to \mathrm{Tor}_1(A/I^r, M) \to 0$$

of inverse systems. If $\mathrm{Lim}\, \Gamma(\alpha^r; M) = 0$ and $\mathrm{Lim}^1\, \Gamma(\alpha^r; M) = 0$, the six term Lim–$\mathrm{Lim}^1$ exact sequence and our hypothesis on A imply that $\mathrm{Lim}\, \mathrm{Tor}_1(A/I^r, M) = 0$ and $\mathrm{Lim}^1\, \mathrm{Tor}_1(A/I^r, M) = 0$. In view of Proposition 1.1, it remains to show that $\mathrm{Lim}\, \mathrm{Tor}_i(A/I^r, M) = 0$ and $\mathrm{Lim}^1\, \mathrm{Tor}_i(A/I^r, M) = 0$ for all M when $i \geqslant 2$. If $0 \to N \to F \to M \to 0$ is exact, where F is free, then N has bounded α-torsion. The conclusion follows inductively from the connecting isomorphisms

$$\mathrm{Tor}_{i+1}(A/I^r, M) \cong \mathrm{Tor}_i(A/I^r, N), \qquad i \geqslant 1.$$

To generalize to arbitrary finitely generated ideals, we need to understand the behavior of composites of completions. We begin with the following observation (in which J need not be finitely generated).

LEMMA 1.6. *Let $I = (J, \alpha)$ and suppose that*

$$\operatorname*{Lim}_s \operatorname*{Lim}_r{}^1 \Gamma(\alpha^s; M/J^rM) = 0.$$

Then $M_I^\wedge$ is isomorphic to $(M_J^\wedge)_\alpha^\wedge$.

Proof. For each r and s, we have the two short exact sequences

$$\begin{array}{ccccccccc} 0 \to \Gamma(\alpha^s; M/J^rM) & \to & M/J^rM & \to & \alpha^s(M/J^rM) & \to 0 \\ & & & & \downarrow{\scriptstyle \alpha^s} & \\ 0 \to \quad \alpha^s(M/J^rM) & \to & M/J^rM & \to & M/(\alpha^s, J^r)M & \to 0 \end{array}$$

For each fixed s, the Lim–Lim¹ exact sequence gives exact sequences

$$\begin{array}{l} 0 \to \operatorname{Lim} \Gamma(\alpha^s; M/J^rM) \to M_J^\wedge \to \operatorname{Lim} \alpha^s(M/J^rM) \to \operatorname{Lim}^1 \Gamma(\alpha^s; M/J^rM) \to 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \alpha^s} \\ \quad 0 \to \operatorname{Lim} \alpha^s(M/J^rM) \to M_J^\wedge \to \operatorname{Lim} M/(\alpha^s, J^r)M \to 0 \end{array}$$

This diagram implies the short exact sequence

$$0 \to \alpha^s M_J^\wedge \to \operatorname{Lim} \alpha^s M/J^rM \to \operatorname{Lim}^1 \Gamma(\alpha^s; M/J^rM) \to 0 \qquad (*)$$

and the map of short exact sequences

$$\begin{array}{ccccccc} 0 \to & \alpha^s M_J^\wedge & \to & M_J^\wedge & \to & M_J^\wedge/\alpha^s M_J^\wedge & \to 0 \\ & \downarrow & & \| & & \downarrow & \\ 0 \to & \operatorname{Lim} \alpha^s M/J^rM & \to & M_J^\wedge & \to & \operatorname{Lim} M/(\alpha^s, J^r)M & \to 0. \end{array} \qquad (**)$$

As s varies, the sequences above all give exact sequences of inverse systems. By hypothesis, the Lim–Lim¹ exact sequence, and the fact that Lim¹ Lim¹ is always zero for bi-countably indexed systems (e.g., by a spectral sequence in Roos [7]), the exact sequences $(*)$ give rise to isomorphisms

$$\operatorname{Lim} \alpha^s M_J^\wedge \to \operatorname{Lim} \operatorname{Lim} \alpha^s M/J^rM$$

and

$$\mathrm{Lim}^1\, \alpha^s M_J^\wedge \to \mathrm{Lim}^1\, \mathrm{Lim}\, \alpha^s M/J^r M.$$

Now application of the Lim–Lim^1 exact sequence to the diagram (**) gives the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Lim}\, \alpha^s M_J^\wedge & \to & M_J^\wedge & \to & (M_J^\wedge)_\alpha^\wedge & \to & \mathrm{Lim}^1 \alpha^s M_J^\wedge & \to 0 \\ & \downarrow \cong & & \| & & \downarrow & & \downarrow \cong & \\ 0 \to & \mathrm{Lim}\,\mathrm{Lim}\, \alpha^s M/J^r M & \to & M_J^\wedge & \to & M_I^\wedge & \to & \mathrm{Lim}^1\, \mathrm{Lim}\, \alpha^s M/J^r M & \to 0 \end{array}$$

By the five lemma, $(M_J^\wedge)_\alpha^\wedge \to M_I^\wedge$ is an isomorphism.

We need a conveniently verifiable criterion for checking that the hypothesis of the previous lemma holds. The following observation gives us one (and, here again, J need not be finitely generated). It also gives a means of verifying the hypothesis of Proposition 1.5 for modules of the form $M_J^\wedge$ that does not require boundedness of their α-torsion.

LEMMA 1.7. *Let $I = (J, \alpha)$. Multiplication by α and the quotient map $M/J^{r+1} M \to M/J^r M$ induce a map*

$$\Gamma(\alpha^{r+1}; M/J^{r+1} M) \to \Gamma(\alpha^r; M/J^r M).$$

If the resulting inverse system $\{\Gamma(\alpha^r; M/J^r M)\}$ is pro-zero, for example if each $M/J^r M$ has bounded α-torsion, then

$$\mathrm{Lim}_s\, \mathrm{Lim}^1_r\, \Gamma(\alpha^s, M/J^r M) = 0,$$

$$\mathrm{Lim}_r\, \Gamma(\alpha^r; M_J^\wedge) = 0, \qquad \text{and} \qquad \mathrm{Lim}^1_r\, \Gamma(\alpha^r; M_J^\wedge) = 0.$$

Proof. The left exactness of Lim implies that, for any α, J, and M,

$$\Gamma(\alpha; M_J^\wedge) = \Gamma(\alpha; \mathrm{Lim}\, M/J^s M) \cong \mathrm{Lim}\, \Gamma(\alpha; M/J^s M).$$

Write $Y_{r,s} = \Gamma(\alpha^r; M/J^s M)$. This is a bi-indexed system, and the diagonal system $Y_{r,r}$ is cofinal in it. We have an isomorphism

$$\mathrm{Lim}_r\, \mathrm{Lim}_s\, Y_{r,s} \cong \mathrm{Lim}_{r,s}\, Y_{r,s}.$$

By a spectral sequence of Roos [7], we also have a short exact sequence

$$0 \to \mathrm{Lim}^1_r\, \mathrm{Lim}_s\, Y_{r,s} \to \mathrm{Lim}^1_{r,s}\, Y_{r,s} \to \mathrm{Lim}_r\, \mathrm{Lim}^1_s\, Y_{r,s} \to 0,$$

and similarly with the roles of r and s reversed. The result follows.

DEFINITION 1.8. Let $\boldsymbol{a} = \{\alpha_1, ..., \alpha_n\}$ be a sequence of elements of A. Write $I(0) = 0$ and $I(i) = (\alpha_1, ..., \alpha_i)$. Say that $\boldsymbol{a}$ is a pro-regular sequence for M if the inverse sequence $\Gamma(\alpha_i^r; M/I(i-1)^r M)$ is pro-zero for $1 \leqslant i \leqslant n$. Say that the ring A is good if every $\boldsymbol{a}$ is a pro-regular sequence for A. Clearly A is good if A/J has bounded α-torsion for every finitely generated ideal J (including 0) and every element α.

THEOREM 1.9. *Let $I = (\alpha_1, ..., \alpha_n)$ and write $J = (\alpha_1, ..., \alpha_{n-1})$ and $\alpha = \alpha_n$. Assume that A has bounded α_i-torsion for each i and that $\boldsymbol{a}$ is a pro-regular sequence for A. If $\boldsymbol{a}$ is a pro-regular sequence for an A-module M, then $L_0^I(M) \cong M_I^\wedge$ and $L_i^I(M) = 0$ for $i > 0$. Moreover, the following conclusions hold for any A-module M.*

(i) $L_0^I(M) \cong L_0^\alpha(L_0^J(M))$.

(ii) *For $1 \leqslant i \leqslant n-1$, there is a short exact sequence*

$$0 \to L_0^\alpha((L_i^J(M)) \to L_i^I(M) \to L_1^\alpha(L_{i-1}^J(M)) \to 0.$$

(iii) $L_n^I(M) \cong L_1^\alpha(L_{n-1}^J(M))$.

(iv) $L_i^I(M) = 0$ *for* $i \geqslant n+1$.

Proof. Proposition 1.5 handles the case $n = 1$. Assume inductively that the conclusion holds for J. By [1, XVII, Sect. 7], there is a pair of composite functor spectral sequences, $\{E_{p,q}^r\}$ and $\{'E_{p,q}^r\}$, which both converge to the same hyperhomology groups $\mathscr{L}_*$. They have E^2-terms

$$E_{p,q}^2 = L_p(L_q^\alpha \circ (\ _J^\wedge\ ))\,(M) = H_p(L_q^\alpha(X_J^\wedge\ )),$$

where X is a free resolution of M, and

$$'E_{p,q}^2 = L_p^\alpha(L_q^J(M)).$$

It is clear that $\boldsymbol{a}$ is a pro-regular sequence for any free A-module. Thus Proposition 1.5 and the previous two lemmas give that

$$E_{p,q}^2 = 0 \text{ for } q \geqslant 1 \qquad \text{and} \qquad E_{p,0}^2 = H_p((X_J^\wedge\ )_\alpha^\wedge\ ) = H_p(X_I^\wedge\ ).$$

Therefore $\mathscr{L}_p = L_p^I M$. For the first statement of the theorem, the induction hypothesis, Proposition 1.5, and the previous two lemmas give that

$$E_{p,q}^2 = 0 \text{ for } q > 0, \quad 'E_{p,0}^2 = L_p^\alpha(M_J^\wedge\ ) = 0 \text{ for } p > 0, \quad \text{and} \quad 'E_{0,0}^2 = M_I^\wedge\ .$$

It follows that $\mathscr{L}_p = 0$ for $p > 0$ and that $\mathscr{L}_0 = M_I^\wedge$. For the second statement, the induction hypothesis implies that $'E_{p,q}^2 = 0$ for $p > 1$ and for $q > n-1$. Thus $'E^2 = {}'E^\infty$, and (i) through (iv) follow.

Note that (i) holds even though $M_I^\wedge$ need not be isomorphic to $(M_J^\wedge)_\alpha^\wedge$ in general. The point is that these two functors agree on free modules and so have the same derived functors. Theorems 3.3 and 3.4 below imply better vanishing results than (iv) for Noetherian rings and Burnside rings. For a good ring A, we conclude from the first statement that I-adic completion is an exact functor when restricted to those A-modules M for which $\boldsymbol{\alpha}$ is a pro-regular sequence. It is obvious that Noetherian rings are good, and so are all Burnside rings $A(G)$ [2]. Some bad rings are exhibited in Examples 1.4.

2. Local Homology and Derived Functors

We begin by recalling Grothendieck's definition and calculation of local cohomology groups [4; 5, Sect. 2].

Definitions 2.1. For $\alpha \in A$, let $K_\bullet(\alpha)$ be the chain complex $\alpha\colon A \to A$, where the two copies of A are in degrees 1 and 0, respectively. For a sequence $\boldsymbol{\alpha} = \{\alpha_1, ..., \alpha_n\}$, let $K_\bullet(\boldsymbol{\alpha}) = K_\bullet(\alpha_1) \otimes \cdots \otimes K_\bullet(\alpha_n)$. The identity map in degree 0 and multiplication by α in degree 1 give a chain map $K_\bullet(\alpha^{r+1}) \to K_\bullet(\alpha^r)$, and thus, by tensoring, a chain map $K_\bullet(\boldsymbol{\alpha}^{r+1}) \to K_\bullet(\boldsymbol{\alpha}^r)$. Let M be an A-module and define the local cohomology groups of M at the ideal $I = (\alpha_1, ..., \alpha_n)$ to be

$$H_I^*(M) = H^*(\operatorname{Colim} \operatorname{Hom}(K_\bullet(\boldsymbol{\alpha}^r), M)).$$

For a space X, a closed subspace Y, and a sheaf $\mathscr{F}$ of Abelian groups over X, let $\Gamma_Y(X; \mathscr{F})$ be the group of sections of $\mathscr{F}$ with support in Y. The functor $\Gamma_Y(X; ?)$ on sheaves is left exact, and its right derived functors are denoted $H_Y^*(X; \mathscr{F})$.

Theorem 2.2. *Let* $X = \operatorname{Spec}(A)$ *and* $Y = V(I)$. *Then*

$$H_I^*(M) \cong H_Y^*(X; \tilde{M}),$$

where $\tilde{M}$ *is the associated sheaf of* M. *If* A *is Noetherian, then*

$$H_I^*(M) \cong \operatorname{Colim} \operatorname{Ext}^*(A/I^r, M).$$

This identifies local cohomology groups as right derived functors. We shall define certain local homology groups and verify that they agree with the left derived functors $L_*^I(M)$ under mild hypotheses. We begin with a reformulation of the definition of local cohomology.

Remarks 2.3. For $\alpha \in A$, let $K^{\bullet}(\alpha)$ be the cochain complex $\alpha\colon A \to A$, where the two copies of A are in degrees 0 and 1, respectively. For a sequence $\boldsymbol{a} = \{\alpha_1, ..., \alpha_n\}$, let $K^{\bullet}(\boldsymbol{a}) = K^{\bullet}(\alpha_1) \otimes \cdots \otimes K^{\bullet}(\alpha_n)$. The identity map in degree 0 and multiplication by α in degree 1 give a cochain map $K^{\bullet}(\alpha^r) \to K^{\bullet}(\alpha^{r+1})$, and thus, by tensoring, a cochain map $K^{\bullet}(\boldsymbol{a}^r) \to K^{\bullet}(\boldsymbol{a}^{r+1})$. These cochain complexes and cochain maps are obtained by applying $\mathrm{Hom}(?, A)$ to the chain complexes and chain maps in Definitions 2.1, and we have an isomorphism of direct systems

$$\mathrm{Hom}(K_{\bullet}(\boldsymbol{a}^r), M) \cong K^{\bullet}(\boldsymbol{a}^r) \otimes M.$$

Define $K^{\bullet}(\boldsymbol{a}^{\infty}) = \mathrm{Colim}\, K^{\bullet}(\boldsymbol{a}^r)$, and observe that $K^{\bullet}(\alpha^{\infty})$ is just the cochain complex $A \to A[1/\alpha]$. We have an evident isomorphism

$$H_I^*(M) \cong H^*(K^{\bullet}(\boldsymbol{a}^{\infty}) \otimes M).$$

The homology isomorphism $\mathrm{Tel}\, K^{\bullet}(\boldsymbol{a}^r) \to K^{\bullet}(\boldsymbol{a}^{\infty})$ gives a projective approximation of the flat cochain complex $K^{\bullet}(\boldsymbol{a}^{\infty})$. By the Künneth spectral sequence, this approximation induces an isomorphism

$$H_I^*(M) \cong H^*(\mathrm{Tel}\, K^{\bullet}(\boldsymbol{a}^r) \otimes M).$$

This suggests the following definition, which seems to be new.

DEFINITION 2.4. Define the local homology groups of M at I by

$$H_*^I(M) = H_*(\mathrm{Hom}(\mathrm{Tel}\, K^{\bullet}(\boldsymbol{a}^r), M)).$$

A formal duality argument shows that

$$\mathrm{Hom}(\mathrm{Tel}\, K^{\bullet}(\boldsymbol{a}^r), M) \cong \mathrm{Mic}\, \mathrm{Hom}(K^{\bullet}(\boldsymbol{a}^r), M),$$

and clearly $\mathrm{Hom}(K^{\bullet}(\boldsymbol{a}^r), M) \cong K_{\bullet}(\boldsymbol{a}^r) \otimes M$. Putting these isomorphisms together, we obtain the alternative description

$$H_*^I(M) \cong H_*(\mathrm{Mic}(K_{\bullet}(\boldsymbol{a}^r) \otimes M)).$$

The resemblance to the description

$$L_*^I(M) \cong H_*(\mathrm{Mic}(X^r \otimes M))$$

in Proposition 1.1 is obvious, and the following result should now come as no surprise.

THEOREM 2.5. *Let $I = (\alpha_1, ..., \alpha_n)$. Assume that A has bounded α_i-torsion for each i and that $\boldsymbol{a}$ is a pro-regular sequence for A. Then*

$$H_*^I(M) \cong L_*^I(M).$$

Since the $K_\bullet(\boldsymbol{a}^r)$ are free chain complexes, the $H^I_*(M)$ certainly give an exact ∂-functor. We need only construct a natural map f_0: $H^I_0(M) \to L^I_0(M)$ and show that f_0 is an isomorphism and $H^I_i(M)=0$ for $i>0$ when M is free. We proceed in three steps, first handling the case $n=1$, next constructing a spectral sequence that will allow induction, and then completing the proof.

LEMMA 2.6. *Let $I=(\alpha)$, where A has bounded α-torsion. Then*

$$H^I_*(M) \cong L^I_*(M).$$

Proof. The free complex $K_\bullet(\alpha^r)$ over $A/(\alpha^r)$ is not a resolution, but it gives the first two terms of a free resolution X^r. We thus obtain a map of inverse systems $K_\bullet(\alpha^r) \to X^r$ and thus a map of microscopes. The homology of $K_\bullet(\alpha^r) \otimes M$ is $M/\alpha^r M$ in degree zero and $\Gamma(\alpha^r; M)$ in degree one. If M is free, the system $\Gamma(\alpha^r; M)$ is pro-zero and thus $H^I_0(M) = M^\wedge_I$ and $H^I_i(M)=0$ for $i>0$ by the short exact sequence for the computation of the homology of microscopes.

LEMMA 2.7. *Let $I=J+K$. Then there is a spectral sequence $\{E^r\}$ which converges to $H^I_*(M)$ and has $E^2_{p,q} = H^J_p(H^K_q(M))$.*

Proof. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be sequences of generators for J and K. By Lemma 0.1 and the evident adjunction, we have a homology isomorphism

$$\xi^*\colon \operatorname{Hom}(\operatorname{Tel} K^\bullet(\boldsymbol{\alpha}^r), \operatorname{Hom}(\operatorname{Tel} K^\bullet(\boldsymbol{\beta}^s), M)) \to \operatorname{Hom}(\operatorname{Tel} K^\bullet(\boldsymbol{\alpha}^r, \boldsymbol{\beta}^r), M).$$

A standard argument with double complexes yields the conclusion.

Proof of Theorem 2.5. Let $J=I(n-1)$ and $\alpha=\alpha_n$. Lemma 2.6 gives the result for (α) and we may assume it for J. By the previous result, the induction hypothesis, and Theorem 1.9(i), we have

$$H^I_0(M) \cong H^\alpha_0(H^J_0(M)) \cong L^\alpha_0(L^J_0(M)) \cong L^I_0(M).$$

If M is free, Theorem 1.9 gives that $H^J_q(M) \cong L^J_q(M)$ is zero for $q>0$ and is $M^\wedge_J$ for $q=0$, and Proposition 1.5 gives that $H^\alpha_p(M^\wedge_J) \cong L^\alpha_p(M^\wedge_J)$ is zero for $p>0$. Thus, when M is free, $E^2_{p,q}=0$ unless $p=q=0$ and therefore $H^I_n(M)=0$ for $n>0$. This completes the proof.

3. A UNIVERSAL COEFFICIENTS SPECTRAL SEQUENCE

We can use the relationship between local homology and local cohomology to obtain a duality, or universal coefficients, spectral sequence. It is the most useful tool for explicit calculation of local homology groups.

PROPOSITION 3.1. *There is a fourth quadrant spectral sequence*

$$\{E_r; d_r: E_r^{p,q} \to E_r^{p+r,q-r+1}\}$$

which converges to $H_*^I(M)$ in total (homological) degree $-(p+q)$ and has

$$E_2^{p,q} = \mathrm{Ext}^p(H_I^{-q}(A), M).$$

Proof. Replace M in Hom(Tel $K^\bullet(\boldsymbol{a}^r)$, M) by an injective resolution Y of M. To keep track of the grading, think of Tel $K^\bullet(\boldsymbol{a}^r)$ as a complex graded in non-positive degrees, so that Hom(Tel $K^\bullet(\boldsymbol{a}^r)$, Y) is a (cohomological) bicomplex. Filtering so as to take the homology of Y first we obtain Hom(Tel $K^\bullet(\boldsymbol{a}^r)$, M) on the E_1-level and $H_*^I(M)$ on the E_2-level, with no further differentials and with trivial extensions. Filtering so as to take the homology of Tel $K^\bullet(\boldsymbol{a}^r)$ first, we obtain the spectral sequence we want.

This spectral sequence looks a little strange. If $H_I^k(A) = 0$ for $k > n$, then the non-zero terms of $E_2^{p,q}$ lie on the 0^{th} through $(-n)^{\text{th}}$ rows of the fourth quadrant, while the non-zero terms of $E_\infty^{p,q}$ lie on the 0^{th} through n^{th} diagonals in the seventh octant; that is, $E_2^{p,q} = 0$ if $q < -n$ or $q > 0$ and $E_\infty^{p,q} = 0$ if either $-(p+q) < 0$ or $-(p+q) > n$. The differentials wipe out all but finitely many of the non-zero terms present in E_2. The following immediate observation is quite useful.

COROLLARY 3.2. *If $H_I^i(A) = 0$ for $i > k$, then $H_i^I(M) = 0$ for $i > k$.*

This gains force from the following theorem of Grothendieck (see [3, 3.6.5] or [6, 2.7]).

THEOREM 3.3. *If A is Noetherian, then $H_I^i(A) = 0$ for $i > \dim A$.*

Even though Burnside rings need not be Noetherian, the same conclusion holds for them [2]; recall that they have dimension one.

THEOREM 3.4. *If A is the Burnside ring of a compact Lie group, then $H_I^i(A) = 0$ for $i > 1$.*

COROLLARY 3.5. *Suppose that A is a Noetherian ring of dimension one or a Burnside ring. Then there is an exact sequence*

$$0 \to \mathrm{Ext}_A^1(H_I^1(A), M) \to H_0^I(M) \to \mathrm{Hom}(H_I^0(A), M) \to \mathrm{Ext}_A^2(H_I^1(A), M) \to 0$$

and an isomorphism

$$H_1^I(M) \cong \mathrm{Hom}(H_I^1(A), M).$$

The spectral sequence of Proposition 3.1 generalizes Grothendieck's local duality spectral sequence. To see this, we consider modules of the form $M = \mathrm{Hom}(N, Q)$, where Q is injective. Here our E_2-term and abutment take the following alternative forms.

LEMMA 3.6. *For modules L and N and injective modules Q, there is a natural isomorphism*

$$\mathrm{Ext}^p(L, \mathrm{Hom}(N, Q)) \cong \mathrm{Ext}^p(N, \mathrm{Hom}(L, Q)).$$

Proof. There is an evident natural isomorphism

$$\mathrm{Hom}(L, \mathrm{Hom}(N, Q)) \cong \mathrm{Hom}(N, \mathrm{Hom}(L, Q)).$$

If X is a projective resolution of L, then $\mathrm{Hom}(X, Q)$ is an injective resolution of $\mathrm{Hom}(L, Q)$.

LEMMA 3.7. *For modules N and injective modules Q, there is a natural isomorphism*

$$\mathrm{Hom}(H_I^i(N), Q) \cong H_i^I(\mathrm{Hom}(N, Q)).$$

Proof. Apply homology to the evident isomorphisms

$$\mathrm{Hom}(\mathrm{Tel}\, K^{\bullet}(\mathfrak{a}^r) \otimes N, Q) \cong \mathrm{Hom}(\mathrm{Tel}\, K^{\bullet}(\mathfrak{a}^r), \mathrm{Hom}(N, Q)).$$

After the second degree is raised by n so as to put the non-zero terms in the first quadrant, the spectral sequence of Proposition 3.1 takes the same form as the local duality spectral sequence.

PROPOSITION 3.8. *Write $DN = \mathrm{Hom}(N, Q)$, where Q is injective, and assume that $H_I^q(A) = 0$ for $q > n$. There is a spectral sequence*

$$\{E_r; d_r: E_r^{p,q} \to E_r^{p+r, q-r+1}\}$$

which converges to $DH_I^*(N)$ in total degree $n - q - p$ and has

$$E_2^{p,q} = \mathrm{Ext}^p(N, DH_I^{n-q}(A)).$$

Here A is any commutative ring, I is any finitely generated ideal, N is any A-module, and Q is any injective A-module. In the special case when A is a complete local ring of dimension n, I is its maximal ideal, N is finitely generated, and Q is a dualizing module, this is precisely [5, Theorem 6.8].

4. Composites of Derived Functors

Let ε: $L_0^I M \to M_I^\wedge$ be the natural epimorphism. We also have a natural map γ: $M \to M_I^\wedge$, and γ is an isomorphism if $M = N_I^\wedge$. Since the zero$^{\text{th}}$ left derived functor of the identity functor is the identity functor, there results a natural map η: $M \to L_0^I M$ such that $\varepsilon \circ \eta = \gamma$. In our topological work in [2], the map η appears naturally and plays a far more central role than the more intuitive map γ. In fact, we were led there to say that M is "I-complete" if η: $M \to L_0^I M$ is an isomorphism. With this sense of the term "I-complete," the following result shows that $M_I^\wedge$ and all of the $L_q^I M$ are I-complete; it also shows that $L_p^I N = 0$ for $p \geqslant 1$ when N is I-complete.

THEOREM 4.1. *Assume the hypotheses of Theorem* 2.5, *so that*

$$H_*^I(M) \cong L_*^I(M).$$

Let N be either $M_I^\wedge$ or $L_q^I M$ for some $q \geqslant 0$. Then η: $N \to L_0^I N$ is an isomorphism and $L_p^I N = 0$ for $p \geqslant 1$.

Proof. We agree to write L_p for L_p^I throughout the proof. It suffices to prove that η: $N \to L_0 N$ is an isomorphism for the specified N and that $L_p L_0 M = 0$ for $p \geqslant 1$ and any M.

We can let $I = J = K$ in Lemma 2.7, using the same list of generators twice, and so obtain a spectral sequence $\{E^r\}$ converging from $L_* L_* M$ to $L_* M$. In total degree zero, the spectral sequence collapses to an isomorphism $L_0 M \cong L_0 L_0 M$. Writing down an explicit construction of η and using the proof of Theorem 2.5, we easily check that the isomorphism is in fact given by η. Since $L_0 \varepsilon$: $L_0 L_0 M \to L_0 M_I^\wedge$ is an epimorphism, it follows by a little diagram chase that η: $M_I^\wedge \to L_0 M_I^\wedge$ is an epimorphism, and η is certainly a monomorphism since $\varepsilon \circ \eta$ is the isomorphism γ. We will prove at the end that η: $L_q M \to L_0 L_q M$ is an isomorphism for $q > 0$.

Suppose next that F is a free module. Then the E_2-term of the spectral sequence above is zero unless $q = 0$, when it is $L_* L_0 F = L_* F_I^\wedge$, while the limit term is zero except in degree 0. Thus $L_p L_0 F = 0$ for $p \geqslant 1$. Given a general module M, construct a short exact sequence

$$0 \to R \to F \to M \to 0,$$

where F is free. We first show that $L_1 L_0 M = 0$.

Since $L_1 F = 0$, we have an exact sequence

$$0 \to L_1 M \to L_0 R \to L_0 F \to L_0 M \to 0.$$

Let K be the kernel of $L_0F \to L_0M$ and break this sequence into the two short exact sequences

$$0 \to L_1M \to L_0R \to K \to 0 \quad \text{and} \quad 0 \to K \to L_0F \to L_0M \to 0.$$

The first gives an epimorphism $L_0L_0R \to L_0K$, and the fact that η: $L_0 \to L_0L_0$ is an isomorphism implies that η: $K \to L_0K$ is an epimorphism. Using the second and the fact that $L_1L_0F = 0$, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & 0 \to & K & \to & L_0F & \to & L_0M & \to 0 \\ & & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} & \\ 0 \to & L_1L_0M & \to & L_0K & \to & L_0L_0F & \to & L_0L_0M & \to 0 \end{array}$$

Chasing the diagram, we see that η: $K \to L_0K$ is an isomorphism, hence that $L_0K \to L_0L_0F$ is a monomorphism, hence that $L_1L_0M = 0$.

Since $L_{p+1}L_0M \cong L_pK \cong L_pL_0K$, it follows inductively that $L_pL_0M = 0$ for all $p \geqslant 1$. Finally, η: $L_qM \to L_0L_qM$ is an isomorphism for $q = 1$ since η: $L_0R \to L_0L_0R$ and η: $K \to L_0K$ are isomorphisms; it is an isomorphism for $q \geqslant 2$ since, inductively, η: $L_{q-1}R \to L_0L_{q-1}R$ is an isomorphism and $L_{q-1}R$ is isomorphic to L_qM.

5. The Right Derived Functors of I-adic Completion

Let $I = (\alpha_1, ..., \alpha_n)$ and let R_I^i be the ith right derived functor of I-adic completion. These functors are much less interesting than the functors L_i^I. The main reason is the following observation, which surely must be known. For an A-module M, define $\Gamma(I, M)$, the annihilator of I in M, to be $\{m \mid I \cdot m = 0\} \subset M$. Write $\Gamma(I) = \Gamma(I, A)$.

Lemma 5.1. *For an injective A-module N, $IN = \Gamma(\Gamma(I), N)$; in particular, if A is an integral domain, then $IN = N$.*

Proof. Clearly $\Gamma(\Gamma(I), N) = \{n \mid a \cdot I = 0 \text{ implies } a \cdot n = 0\}$ contains IN. The injectivity of N implies the reverse inclusion. To see this, note that $\Gamma(I) = \Gamma(\alpha_1) \cap \cdots \cap \Gamma(\alpha_n)$ is the kernel of the map $A \to (\alpha_1) \oplus \cdots \oplus (\alpha_n)$ with coordinates α_i. Thus we have inclusions

$$A/\Gamma(I) \to (\alpha_1) \oplus \cdots \oplus (\alpha_n) \to A \oplus \cdots \oplus A.$$

We may identify $\Gamma(\Gamma(I), N)$ with $\operatorname{Hom}(A/\Gamma(I), N)$. By extending maps over $(\alpha_1) \oplus \cdots \oplus (\alpha_n)$ and then over $A \oplus \cdots \oplus A$, we see that

$$\Gamma(\Gamma(I), N) = \sum \Gamma(\Gamma(\alpha_i), N) = \sum \alpha_i N = IN.$$

Now assume that A has bounded α_i-torsion for all i. Using that $\Gamma(I) = \Gamma(\alpha_1) \cap \cdots \cap \Gamma(\alpha_n)$ and that I^r is generated by the monomials of degree r in the α_i, we see that A has bounded I-torsion. That is, there exists r such that $\Gamma(I^s) = \Gamma(I^r)$ for all $s \geqslant r$. We conclude from the lemma that $N_I^{\wedge} = N/\Gamma(\Gamma(I^r), N)$ for injective A-modules N. For an arbitrary A-module M, the right derived A-modules $R_I^i M$ are computed by applying I-adic completion to an injective resolution of M and then taking homology. In particular, if A is an integral domain, then $N_I^{\wedge} = 0$ for any injective module N and we conclude that $R_I^i M = 0$ for any A-module M and all $i \geqslant 0$.

Note that the functor $RM = M/\Gamma(\Gamma(I^r), M)$ of M preserves monomorphisms and epimorphisms but fails to be half exact in general. For a short exact sequence,

$$0 \to M' \to M \to M'' \to 0,$$

the middle homology group measuring the deviation from exactness is

$$\{m \mid aI^r = 0 \text{ implies } am \in M'\}/M' + \Gamma(\Gamma(I^r), M).$$

Of course, when the functor R is exact, $R_I^0 = R$ and $R_I^i = 0$ for $i > 0$.

Acknowledgments

It is a pleasure to thank Dick Swan for his very careful reading of an earlier draft, which uncovered several errors. We are also grateful to Gennady Lyubeznik and Bill Dwyer for helpful conversations.

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