An Australian conspectus of higher categories*

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Much Australian work on categories is part of, or relevant to, the development of higher categories and their theory. In this note, I hope to describe some of the origins and achievements of our efforts that they might perchance serve as a guide to the development of aspects of higher-dimensional work.

I trust that the somewhat autobiographical style will add interest rather than be a distraction. For so long I have felt rather apologetic when describing how categories might be helpful to other mathematicians; I have often felt even worse when mentioning enriched and higher categories to category theorists. This is not to say that I have doubted the value of our work, rather that I have felt slowed down by the continual pressure to defend it. At last, at this meeting, I feel justified in speaking freely amongst motivated researchers who know the need for the subject is well established.

Australian Category Theory has its roots in homology theory: more precisely, in the treatment of the cohomology ring and the Künneth formulas in the book by Hilton and Wylie [71]. The first edition of the book had a mistake concerning the cohomology ring of a product. The Künneth formulas arise from splittings of the natural short exact sequences

$$0 \longrightarrow \mathrm{Ext}(HA, HB) \longrightarrow H[A, B] \xrightarrow{H} \mathrm{Hom}(HA, HB) \longrightarrow 0$$
$$0 \longrightarrow HA \otimes HB \xrightarrow{\otimes} H(A \otimes B) \longrightarrow \mathrm{Tor}(HA, HB) \longrightarrow 0$$

where A and B are chain complexes of free abelian groups; however, there are no choices of natural splittings. Wylie's former postgraduate student, Max Kelly, was intrigued by these matters and wanted to understand them conceptually.

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So stimulated, in a series of papers [87, 88, 89, 91, 93] published in *Proc. Camb. Phil. Soc.*, Kelly progressed ever more deeply into category theory. He discussed equivalence of categories and proposed criteria for when a functor should provide "complete invariants" for objects of its domain category. Moreover, Kelly invented differential graded categories and used them to show homotopy nilpotence of the kernel of certain functors [93].

Around the same time, Sammy Eilenberg invented DG-categories probably for purposes similar to those that led Verdier to derived categories. Thus began the collaboration of Eilenberg and Kelly on enriched categories. They realized that the definition of DG-category depended only on the fact that the category DGAb of chain complexes was what they called a *closed* or, alternatively, a *monoidal* category. They favoured the "closed" structure over "monoidal" since internal homs are usually more easily described than tensor products; good examples such as DGAb have both anyway.

The groundwork for the correct definition of monoidal category $\mathcal{V}$ had been prepared by Saunders Mac Lane with his *coherence theorem* for associativity and unit constraints. Kelly had reduced the number of axioms by a couple so that only the Mac Lane–Stasheff pentagon and the unit triangle remained. Enriched categories were also defined by Fred Linton; however, he had conditions on the base $\mathcal{V}$ that ruled out the examples $\mathcal{V} = \mathrm{DGAb}$ and $\mathcal{V} = \mathrm{Cat}$ that proved so vital in later applications.

The long Eilenberg–Kelly paper [46] in the 1965 LaJolla Conference Proceedings was important for higher category theory in many ways; I shall mention only two.

One of these ways was the realization that 2-categories could be used to organize category theory just as category theory organizes the theory of sets with structure. The authors provided an explicit definition of (strict) 2-category early in the paper although they used the term "hypercategory" at that point (probably just as a size distinction since, as we shall see, "2-category" is used near the end). So that the paper became more than a list of definitions with implications between axioms, the higher-categorical concepts allowed the paper to be summarized with theorems such as:

$$\mathcal{V}\text{-Cat is a 2-category} \quad \text{and} \quad (-)_*\colon \mathrm{MonCat} \longrightarrow 2\text{-Cat is a 2-functor.}$$

The other way worth mentioning here is their efficient definition of (strict) n-category and (strict) n-functor using enrichment. If $\mathcal{V}$ is symmetric monoidal then $\mathcal{V}$-Cat is too and so the enrichment process can be iterated. In particular, starting with $\mathcal{V}_0 = \mathrm{Set}$ using cartesian product, we obtain cartesian monoidal categories $\mathcal{V}_n$ defined by $\mathcal{V}_{n+1} = \mathcal{V}_n$-Cat. This $\mathcal{V}_n$ is the category n-Cat of n-categories and n-functors. In my opinion, processes like $\mathcal{V} \mapsto \mathcal{V}$-Cat are fundamental in dimension raising.

With his important emphasis on categories as mathematical structures of the ilk of groups, Charles Ehresmann [44] defined categories internal to a category $\mathcal{C}$ with pullbacks. The category $\mathrm{Cat}(\mathcal{C})$ of internal categories and internal functors also has pullbacks, so this process too can be iterated. Starting with $\mathcal{C} = \mathrm{Set}$, we obtain the category $\mathrm{Cat}^n(\mathrm{Set})$ of n-tuple categories. In particular, $\mathrm{Cat}^2(\mathrm{Set}) = \mathrm{DblCat}$ is the category of double categories; it contains 2-Cat in various way as does $\mathrm{Cat}^n(\mathrm{Set})$ contain n-Cat.

At least two other papers in the LaJolla Proceedings volume had a strong influence on Australian higher-dimensional category theory. One was the paper [115] of Bill Lawvere suggesting a categorical foundations for mathematics; concepts such as comma category appeared there. The other was the paper [57] of John Gray developing the subject of Grothendieck fibred categories as a formal theory in Cat so that it could be dualized. This meant that Gray was essentially treating Cat as an arbitrary 2-category; the duality was that of reversing morphisms (what we call $\mathrm{Cat}^{\mathrm{op}}$) not 2-cells (what we call $\mathrm{Cat}^{\mathrm{co}}$). In stark contrast with topology, Grothendieck had unfortunately used the term "cofibration" for the $\mathrm{Cat}^{\mathrm{co}}$ case.

Kelly developed the theory of enriched categories describing enriched adjunction [94] and introducing the variety of limit he called *end*. I later pointed out that Yoneda had used this concept in the special case of additive categories using an *integral notation* which Brian Day and Max Kelly adopted [33]. Following this, Mac Lane [121] discussed ends for ordinary categories.

Meanwhile, as Kelly's graduate student, I began addressing his concerns with the Künneth formulas. The main result of my thesis [134] (also see [137, 149]) was a Künneth hom formula for finitely filtered complexes of free abelian groups. I found it convenient to express the general theory in terms of DG-categories and triangulated categories; my thesis involved the development of some of their theory. In particular, I recognized that completeness of a DG-category should involve the existence of a suspension functor. The idea was consistent with the work of Day and Kelly [33] who eventually defined completeness of a $\mathcal{V}$-category $\mathcal{A}$ to include *cotensoring* $\mathcal{A}$ with objects V of $\mathcal{V}$: the characterizing property is $\mathcal{A}\big(B, A^V\big) \cong \mathcal{V}\big(V, \mathcal{A}(B, A)\big)$. The point is that, for ordinary categories where $\mathcal{V} = \mathrm{Set}$, the cotensor A^V is the product of V copies of A and so is not needed as an extra kind of limit. Cotensoring with the suspension of the tensor unit in $\mathcal{V} = \mathrm{DGAb}$ gives suspension in the DG-category $\mathcal{A}$. Experience with DG-categories would prove very helpful in developing the theory of 2-categories.

In 1968–9 I was a postdoctoral fellow at the University of Illinois (Champaign–Urbana) where John Gray worked on 2-categories. To construct higher-dimensional comprehension schema [58], Gray needed lax limits and even lax Kan extensions [61]. He also worked on a closed structure for the category 2-Cat for which the internal hom $[A, B]$ of two 2-categories A and B consisted of 2-functors from A to B, lax natural transformations, and modifications. (By "lax" we mean the insertion of compatible morphisms in places where there used to be equalities. We use "pseudo" when the inserted morphisms are all invertible.) The next year at Tulane University, Jack Duskin and I had one-year (1969–70) appointments where we heard for a second time Mac Lane's lectures that led to his book [121]; we had all been at Bowdoin College (Maine) over the Summer. Many category theorists visited Tulane that year. Duskin and Mac Lane convinced Gray that this closed category structure on 2-Cat should be monoidal. Thus appeared the (lax) Gray tensor product of 2-categories that Gray was able to prove satisfied the coherence pentagon using Artin's braid groups (see [62, 63]).

Meanwhile Jean Bénabou [12] had invented weak 2-categories, calling them *bicategories*. He also defined a weak notion of morphism that I like to call *lax functor*. His convincing example was the bicategory $\mathrm{Span}(\mathcal{C})$ of spans in a category $\mathcal{C}$ with pullbacks; the objects are those of the category $\mathcal{C}$ while it is the morphisms of $\mathrm{Span}(\mathcal{C})$ that are spans; composition of spans requires pullback and so is only associative up to isomorphism. He pointed out that a lax functor from the terminal category $\mathbf{1}$ to Cat was a category $\mathcal{A}$ equipped with a "standard construction" or "triple" (that is, a monoid in the monoidal category $[A, A]$ of endofunctors of $\mathcal{A}$ where the tensor product is composition); he introduced the term *monad* for this concept. Thus we could contemplate monads in any bicategory. In particular, Bénabou observed that a monad in $\mathrm{Span}(\mathcal{C})$ is a category internal to $\mathcal{C}$.

The theory of monads (or "triples" [47]) became popular as an approach to universal algebra. A monad T on the category Set of sets can be regarded as an algebraic theory and the category Set^T of "T-modules" regarded as the category of models of the theory. Michael Barr and Jon Beck had used monads on categories to define an abstract cohomology that included many known examples.

The category $\mathcal{C}^T$ of T-modules (also called "T-algebras") is called, after its inventors, the Eilenberg–Moore category for T. The underlying functor $U^T\colon \mathcal{C}^T \longrightarrow \mathcal{C}$ has a left adjoint which composes with U^T to give back T. There is another category $\mathcal{C}_T$, due to Kleisli, equivalent to the full subcategory of $\mathcal{C}^T$ consisting of the free T-modules; this gives back T in the same way. In fact, whenever we have a functor $U\colon \mathcal{A} \longrightarrow \mathcal{C}$ with left adjoint F, there is a "generated" monad $T = UF$ on $\mathcal{C}$. There are comparison functors $\mathcal{C}_T \longrightarrow \mathcal{A}$ and $\mathcal{A} \longrightarrow \mathcal{C}^T$; if the latter functor is an equivalence, the functor U is said to be *monadic*. See [121] for details. Beck [11] established necessary and sufficient conditions for a functor to be monadic. Erny Manes showed that compact Hausdorff spaces were the modules for the ultrafilter monad β on Set (see [121]). However, Bourbakifying the definition of topological space via Moore–Smith convergence, Mike Barr [7] showed that general topological spaces were the relational modules for the ultrafilter lax monad on the 2-category Rel whose objects are sets and morphisms are relations. (One of my early Honours students at Macquarie University baffled his proposed Queensland graduate studies supervisor who asked whether the student knew the definition of a topological space. The aspiring researcher on dynamical systems answered positively: "Yes, it is a relational β-module!" I received quite a bit of flak from colleagues concerning that one; but the student Peter Kloeden went on to become a full professor of mathematics in Australia then Germany.)

I took Bénabou's point that a lax functor $W\colon \mathcal{A} \longrightarrow \mathrm{Cat}$ became a monad when $\mathcal{A} = \mathbf{1}$ and in [136] I defined generalizations of the Kleisli and Eilenberg–Moore constructions for a lax functor W with any category $\mathcal{A}$ as domain. These constructions gave two universal methods of assigning strict functors $\mathcal{A} \longrightarrow \mathrm{Cat}$ to a lax one; I pointed out the colimit- and limit-like nature of the constructions. I obtained a generalized Beck monadicity theorem that we have used recently in connection with natural Tannaka duality. The Kleisli-like construction was applied by Peter May to spectra under the recommendation of Robert Thomason.

When I was asked to give a series of lectures on universal algebra from the viewpoint of monads at a Summer Research Institute at the University of Sydney, I wanted to talk about the lax functor work. Since the audience consisted of mathematicians of diverse backgrounds, this seemed too ambitious so I set out to develop the theory of monads in an arbitrary 2-category $\mathcal{K}$, reducing to the usual theory when $\mathcal{K} = \mathrm{Cat}$. This "formal theory of monads" [135] (see [113] for new developments) provides a good example of how 2-dimensional category theory provides insight into category theory. Great use could be made of duality: comonad theory became rigorously dual to monad theory under 2-cell reversal while the Kleisli and Eilenberg–Moore constructions became dual under morphism reversal. Also, a distributive law between monads could be seen as a monad in the 2-category of monads.

In 1971 Bob Walters and I began work on *Yoneda structures* on 2-categories [108, 165]. The idea was to axiomatize the deeper aspects of categories beyond their merely being algebraic structures. This worked centred on the Yoneda embedding $A \longrightarrow \mathcal{P}A$ of a category A into its presheaf category $\mathcal{P}A = [A^{\mathrm{op}}, \mathrm{Set}]$. We covered the more general example of categories enriched in a base $\mathcal{V}$ where $\mathcal{P}A = [A^{\mathrm{op}}, \mathcal{V}]$. Clearly size considerations needed to be taken seriously although a motivating size-free example was preordered sets with $\mathcal{P}A$ the inclusion-ordered set of right order ideals in A. Size was just an extra part of the structure. With the advent of elementary topos theory and the stimulation of the work of Anders Kock and Christian Mikkelsen, we showed that the preordered objects in a topos provided a good example. We were happy to realize [108] that an elementary topos was precisely a finitely complete category with a *power object* (that is, a relations classifier). This meant that my work with Walters could be viewed as a higher-dimensional version of topos theory. As usual when raising dimension, what we might mean by a 2-dimensional topos could be many things, several of which could be useful. I looked [139, 141] at those special Yoneda structures where $\mathcal{P}A$ classified *two-sided discrete fibrations*.

At the same time, having made significant progress with Mac Lane on the coherence problem for symmetric closed monoidal categories [106, 107], Kelly was developing a general approach to coherence questions for categories with structure. In fact, Max Kelly and Peter May were in the same place at the same time developing the theories of "clubs" and "operads"; there was some interaction. As I have mentioned, clubs [95, 96, 97, 98, 99] were designed to address coherence questions in categories with structure; however, operads were initially for the study of topological spaces bearing homotopy invariant structure. Kelly recognized that at the heart of both notions were monoidal categories such as the category $\mathbf{P}$ of finite sets and permutations. May was essentially dealing with the category $[\mathbf{P}, \mathrm{Top}]$ (also written $\mathrm{Top}^{\mathbf{P}}$) of functors from $\mathbf{P}$ to the category Top of topological spaces; there is a tensor product on $[\mathbf{P}, \mathrm{Top}]$, called "substitution", and a monoid for this tensor product is a *symmetric topological operad*. Kelly was dealing with the slice 2-category $\mathrm{Cat}/\mathbf{P}$ with its "substitution" tensor product; a monoid here Kelly called a *club* (a special kind of 2-dimensional theory). There is a canonical functor $[\mathbf{P}, \mathrm{Top}] \longrightarrow [\mathrm{Top}, \mathrm{Top}]$ and a canonical 2-functor $\mathrm{Cat}/\mathbf{P} \longrightarrow [\mathrm{Cat}, \mathrm{Cat}]$; each takes substitution to composition. Hence each operad gives a monad on Top and each club gives a 2-monad on Cat. The modules for the 2-monad on Cat are the categories with the structure specified by the club. Kelly recognized that complete knowledge of the club solved the coherence problem for the club's kind of structure on a category.

That was the beginning of a lot of work by Kelly and colleagues on "2-dimensional universal algebra" [17]. There is a lot that could be said about this with some nice results and I recommend looking at that work; homotopy theorists will recognize many analogues. One theme is the identification of structures that are essentially unique when they exist (such as "categories with finite products", "regular categories" and "elementary toposes") as against those where the structure is really extra (such as "monoidal categories"). A particular class of the essentially unique case is those structures that are modules for a Kock–Zöberlein monad [111, 172]. In this case, the action of the monad on a category is provided by an adjoint to the unit of the monad. It turns out that these monads have an interesting relationship with the simplicial category [142]. It is well known (going right back to the days when monads were called *standard constructions*) that the coherence problem for monads is solved by the (algebraic) simplicial category $\overline{\Delta}_{\mathrm{alg}}$: the monoidal category of finite ordinals (including the empty ordinal) and order-preserving functions. A monad on a category $\mathcal{A}$ is the same as a strict monoidal functor $\overline{\Delta}_{\mathrm{alg}} \longrightarrow [\mathcal{A}, \mathcal{A}]$. In point of fact, $\overline{\Delta}_{\mathrm{alg}}$ is the underlying category of a 2-category $\mathrm{Ord}_{\mathrm{fin}}$ where the 2-cells give the pointwise order to the order-preserving functions. There are nice strings of adjunctions between the face and degeneracy maps. A Kock–Zöberlein monad on a 2-category $\mathcal{K}$ is the same as a strict monoidal 2-functor $\mathrm{Ord}_{\mathrm{fin}} \longrightarrow [\mathcal{K}, \mathcal{K}]$; see [142, 112]. My main example of algebras for a Kock–Zöberlein monad in [138, 142] was fibrations in a 2-category. The monad for fibrations needed an idea of John Gray that I will describe.

In the early 1970s, Gray [60] was working on 2-categories that admitted the construction which in Cat forms the arrow category $A^{\rightarrow}$ from a category A. This rang a bell, harking me back to my work on DG-categories: Gray's construction was like suspension. I saw that its existence should be part of the condition of completeness of a 2-category. A 2-category is complete if and only if it admits products, equalizers and cotensoring with the arrow category $\rightarrow$.

Walters and I had a general concept of limit for an object of a 2-category bearing a Yoneda structure. As a special case I looked at what this meant for limits in 2-categories. Several people and collaborators had come to the same conclusion about what limit should mean for enriched categories. Borceux and Kelly called the notion "mean cotensor product". I used the term "indexed limit" for the 2-category case and Kelly adopted that name in his book on enriched categories. When preparing a talk to physicists and engineers in Milan, I decided a better term was *weighted limit*: roughly, the "weighting" J should provide the number of copies JA of each object SA in the diagram S whose limit we seek. Precisely, for $\mathcal{V}$-categories, the limit $\lim(J, S)$ of a $\mathcal{V}$-functor S:

$\mathcal{A} \longrightarrow \mathcal{X}$ weighted by a $\mathcal{V}$-functor $J\colon \mathcal{A} \longrightarrow \mathcal{V}$is an object of $\mathcal{X}$ equipped with a $\mathcal{V}$-natural isomorphism

$$\mathcal{X}\big(X, \lim(J,S)\big) \cong [\mathcal{A},\mathcal{V}]\big(J, \mathcal{X}(X,S)\big).$$

Products, equalizers and cotensors are all examples. Conversely, if $\mathcal{X}$ admits these three particular examples, it admits all weighted limits; despite this, individual weighted limits can occur without being thus constructible.

The $\mathcal{V} = \mathrm{Cat}$ case is very interesting. Recall that a $\mathcal{V}$-category in this case is a 2-category. As implied above, it turns out that all weighted limits can be constructed from products, equalizers and cotensoring with the arrow category. Yet there are many interesting constructions that are covered by the notion of weighted limit: good examples are the Eilenberg–Moore construction on a monad and Lawvere's "comma category" of two morphisms with the same codomain.

Gray had defined what we call lax and pseudo limits of 2-functors. Mac Lane says that a limit is a universal cone; a cone is a natural transformation from a constant functor. A lax limit is a universal lax cone. A pseudo limit is a universal pseudo cone. Although these concepts seemed idiosyncratic to 2-category theory, I showed that all lax and pseudo limits were weighted limits and so were covered by "standard" enriched category theory. For example, the lax limit of a 2-functor $F\colon \mathcal{A} \longrightarrow \mathcal{X}$ is precisely $\lim(L_{\mathcal{A}}, F)$ where $L_{\mathcal{A}}\colon \mathcal{A} \longrightarrow \mathrm{Cat}$ is the 2-functor defined by $L_{\mathcal{A}}A = \pi_{0*}(\mathcal{A}/A)$; here $\mathcal{A}/A$ is the obvious slice 2-category of objects over A and π_{0*} applies the set-of-path-components functor $\pi_0\colon \mathrm{Cat} \longrightarrow \mathrm{Set}$ on the hom categories of 2-categories. Gray then pointed out that, for $\mathcal{V} = [\Delta^{\mathrm{op}}, \mathrm{Set}]$ (the category of simplicial sets), *homotopy limits* of $\mathcal{V}$-functors could be obtained as limits weighted by the composite $\mathcal{A} \xrightarrow{L_{\mathcal{A}}} \mathrm{Cat} \xrightarrow{\mathrm{Nerve}} [\Delta^{\mathrm{op}}, \mathrm{Set}]$.

In examining the limits that exist in a 2-category admitting finite limits (that is, admitting finite products, equalizers, and cotensors with $\rightarrow$) I was led to the notion of *computad*. This is a 2-dimensional kind of graph: it has vertices, edges and faces. Each edge has a source and target vertex; however, each face has a source and target directed path of edges. More 2-categories can be presented by finite computads than by finite 2-graphs. Just as for 2-graphs, the forgetful functor from the category of 2-categories to the category of computads is monadic: the monad formalizes the notion of pasting diagram in a 2-category while the action of the monad on a 2-category encapsulates the operation of *pasting in a 2-category*. Later, Steve Schanuel and Bob Walters pointed out that these computads form a presheaf category.

The step across from limits in 2-categories to limits in bicategories is fairly obvious. For bicategories $\mathcal{A}$ and $\mathcal{X}$, the limit $\lim(J,S)$ of a pseudofunctor $S\colon \mathcal{A} \longrightarrow \mathcal{X}$ weighted by a pseudofunctor $J\colon \mathcal{A} \longrightarrow \mathrm{Cat}$ is an object of $\mathcal{X}$ equipped with a pseudonatural equivalence

$$\mathcal{X}\big(X, \lim(J,S)\big) \simeq \mathrm{Psd}(\mathcal{A},\mathrm{Cat})\big(J, \mathcal{X}(X,S)\big).$$

It is true that every bicategorical weighted limit can be constructed in a bicategory that has products, iso-inserters (or "pseudoequalizers"), and cotensoring with the arrow category (where the universal properties here are expressed by equivalences rather than isomorphisms of categories); however, the proof is a little more subtle than in the 2-category case. It is also a little tricky to determine which 2-categorical limits give rise to bicategorical ones: for example pullbacks and equalizers are not bicategorical limits *per se*; the weight needs to be *flexible* in a technical sense that would be natural to homotopy theorists.

Now I would like to say more about 2-dimensional topos theory. We have mentioned that Yoneda structures can be seen as a 2-dimensional version of elementary topos theory. However, given that a topos is a category of sheaves, there is a fairly natural notion of "2-sheaf", called *stack*, and a 2-topos should presumably be a 2-category of stacks. After characterizing Grothedieck toposes as categories possessing certain limits and colimits with exactness properties, Giraud developed a theory of stacks in connection with his non-abelian 2-dimensional cohomology. He expressed this in terms of fibrations over categories. Grothendieck had pointed out that a fibration $P\colon \mathcal{E} \longrightarrow \mathcal{C}$ over the category $\mathcal{C}$ was the same as a pseudofunctor $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$ where, for each object U of $\mathcal{C}$, the category FU is the fibre of P over U. If $\mathcal{C}$ is a site (that is, it is a category equipped with a Grothendieck topology) then the condition that F should be a stack is that, for each covering sieve $R \longrightarrow \mathcal{C}(-,U)$, the induced functor

$$FU \longrightarrow \mathrm{Psd}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat})(R, F)$$

should be an equivalence of categories. We write $\mathrm{Stack}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat})$ for the full sub-2-category of $\mathrm{Psd}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat})$ consisting of the stacks. I developed this direction a little by defining 2-dimensional sites and proved a Giraud-like characterization of bicategories of stacks on these sites. Perhaps one point is worth mentioning here. In sheaf theory there are various ways of approaching the associated sheaf. Grothendieck used a so-called "L" construction. Applying L to a presheaf gave a separated presheaf (some "unit" map became a monomorphism) then applying it again gave the associated sheaf (the map became an isomorphism). I found that essentially the same L works for stacks. This time one application of L makes the unit map *faithful*, two applications make it *fully faithful*, and the associated stack is obtained after three applications when the map becomes an *equivalence*.

Just as Kelly was completing his book [101] on enriched categories, a remarkable development was provided by Walters who linked enriched category theory with sheaf theory. First, he extended the theory of enriched categories to allow a bicategory $\mathcal{W}$ (my choice of letter!) as base: *a category $\mathcal{A}$ enriched in $\mathcal{W}$* (or *$\mathcal{W}$-category*) has a set $\mathrm{Ob}\,\mathcal{A}$ of objects where each object A is assigned an object $\mathrm{e}(A)$ of $\mathcal{W}$; each pair of objects A and B is assigned a morphism $\mathcal{A}(A,B)\colon \mathrm{e}(A) \longrightarrow \mathrm{e}(B)$ in $\mathcal{W}$ thought of as a "hom" of $\mathcal{A}$; and "composition" in $\mathcal{A}$ is a 2-cell $\mu^{B}_{A,C}\colon \mathcal{A}(B,C) \circ \mathcal{A}(A,B) \Rightarrow \mathcal{A}(A,C)$ which is required to be associative and unital. Walters regards each object A as a copy of "model pieces" $\mathrm{e}(A)$ and $\mathcal{A}$ as a presentation of a structure that is made up of model pieces that are glued together according to "overlaps" provided by the homs. For example, each topological space T yields a bicategory $\mathcal{W} = \mathrm{Rel}(T)$ whose objects are the open subsets of T, whose morphisms $U \longrightarrow V$ are open subsets $R \subseteq U \cap V$, and whose 2-cells are inclusions. Each presheaf P on the space T yields a $\mathcal{W}$-category $el(P)$ whose objects are pairs (U, s) where U is an open subset of T and s is an element of PU; of course, $\mathrm{e}(U, S) = U$. The hom $el(P)\big((U, s), (V, t)\big)$ is the largest subset $R \subseteq U \cap V$ such that the "restrictions" of s and t to R are equal. As another example, for any monoidal category $\mathcal{V}$, let $\Sigma\mathcal{V}$ denote the bicategory with one object and with the endohom category of that single object being $\mathcal{V}$; then a $\mathcal{V}$-category in the Eilenberg–Kelly sense is exactly a $\Sigma\mathcal{V}$-category in Walters' sense.

For each Grothendieck site $(\mathcal{C}, J)$, Walters constructed a bicategory $\mathrm{Rel}_J\mathcal{C}$ such that the category of symmetric Cauchy-complete $\mathrm{Rel}_J\mathcal{C}$-categories became equivalent to the category of set-valued sheaves on $\mathcal{C}, J$. This stimulated the development of the generalization of enriched category theory to allow a bicategory as base. We established a higher-dimensional version of Walters' result to obtain stacks as enriched categories. Walters had been able to ignore many coherence questions because the base bicategories he needed were locally ordered (no more than one 2-cell between two parallel morphisms). However the base for stacks is not locally ordered.

I have mentioned the 2-category $\mathcal{V}$-Cat of $\mathcal{V}$-categories; the morphisms are $\mathcal{V}$-functors. However, there is another kind of "morphism" between $\mathcal{V}$-categories. Keep in mind that a category is a "monoid with several objects"; monoids can act on objects making the object into a module. There is a "several objects" version of module. Given $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, we can speak of left $\mathcal{A}$-, right $\mathcal{B}$-bimodules [117]; I call this a *module from $\mathcal{A}$ to $\mathcal{B}$* (although earlier names were "profunctor" and "distributor" [13]). Provided $\mathcal{V}$ is suitably cocomplete, there is a bicategory $\mathcal{V}$-Mod whose objects are $\mathcal{V}$-categories and whose morphisms are modules. This is not a 2-category (although it is biequivalent to a fairly natural one) since the composition of modules involves a colimit that is only unique up to isomorphism. The generalization $\mathcal{W}$-Mod for a base bicategory $\mathcal{W}$ was explained in [147] and, using some monad ideas, in [15].

Also in [15] we showed how to obtain prestacks as Cauchy complete $\mathcal{W}$-categories for an appropriate base bicategory $\mathcal{W}$. This has some relevance to algebraic topology since Alex Heller and Grothendieck argue that homotopy theories can be seen as suitably complete prestacks on the category cat of small categories. I showed in [146] (also see [147] and [162]) that stacks are precisely the prestacks possessing colimits weighted by torsors. In [145] (accessible as [164]), I show that stacks on a (bicategorical) site are Cauchy complete $\mathcal{W}$-categories for an appropriate base bicategory $\mathcal{W}$.

Earlier (see [139] and [142]) I had concocted a construction on a bicategory $\mathcal{K}$ to obtain a bicategory $\mathcal{M}$ such that, if $\mathcal{K}$ is $\mathcal{V}$-Cat, then $\mathcal{M}$ is $\mathcal{W}$-Mod; the morphisms of $\mathcal{M}$ were *codiscrete two-sided cofibrations* in $\mathcal{K}$. I had used this as an excuse in [142] to develop quite a bit of bicategory theory: the bicategorical Yoneda Lemma, weighted bicategorical limits, and so on. The need for tricategories was also implicit.

The mathematical physicist John Roberts had asked Peter Freyd whether he knew how to recapture a compact group from its monoidal category of finite-dimensional unitary representations. While visiting the University of New South Wales in 1971, Freyd lectured on his solution of the finite group case. A decade and a half later Roberts with Doplicher did the general case using an idea of Cuntz: this is an analytic version of Tannaka duality. In 1977–8, Roberts visited Sydney. He spoke in the Australian Category Seminar (ACS) about non-abelian cohomology. It came out that he had worked on (strict) n-categories because he thought they were what he needed as coefficient structures in non-abelian cohomology. In the tea room at the University of Sydney, Roberts explained to me what the nerve of a 2-category should be: the dimension 2 elements should be triangles of 1-cells with 2-cells in them and the dimension 3 elements should be commutative tetrahedra. Furthermore, he had defined structures he called *complicial sets*: these were simplicial sets with distinguished elements (he originally called them "neutral" then later suggested "hollow", but I am quite happy to use Dakin's term [31] "thin" for these elements) satisfying some conditions, most notably, unique "thin horn filler" conditions. The important point was which horns need to have such fillers. Roberts believed that the category of complicial sets was equivalent to the category of n-categories.

I soon managed to prove that complicial sets, in which all elements of dimension greater than 2 were thin, were equivalent to 2-categories. I also obtained some nice constructions on complicial sets leading to new complicial sets. However the general equivalence seemed quite a difficult problem.

I decided to concentrate on one aspect of the problem. How do we rigorously define the nerve of an n-category? After unsuccessfully looking for an easy way out using multiple categories and multiply simplicial sets (I sent several letters to Roberts about this), I realized that the problem came down to defining the free n-category $\mathcal{O}_n$ on the n-simplex. Meaning had to be given to the term "free" in this context: free on what kind of structure? How was an n-simplex an example of the structure? The structure required was *n-computad*. The definition of n-computad and free n-category on an n-computad is done simultaneously by induction on n (see [150], [127, 154, 155, 162]). An element of dimension n of the nerve $\mathrm{N}(A)$ of an ω-category A is an n-functor from $\mathcal{O}_n$ to A. Things began to click once I drew the following picture of the 4-simplex.

big diagram

I was surprised to find out that Roberts had not drawn this picture in his work on complicial sets! It was only in studying this and the pictures for the 5- and 6-simplex that I understood the horn filler conditions for the nerve of an n-category. The resemblence to Stasheff's associahedra was only pointed out much later (I think by Jim Stasheff himself).

I think of the n-category $\mathcal{O}_n$ as a simplex with oriented faces; I call it the nth *oriental*. The problem in constructing it inductively starting with small n is *where to put that highest dimensional cell*. What are that cell's source and target? Even in the case of $\mathcal{O}_4$ above, the description of the 3-source and 3-target of the cell (01234), in terms of composites of lower dimensional cells, takes some work to write explicitly. To say a 4-functor out of $\mathcal{O}_4$ takes (01234) to the identity is the *non-abelian 3-cocycle condition.*

In mid-1982 I circulated a conjectural description of the free ω-category $\mathcal{O}_\omega$ on the infinite-dimensional simplex; the objects were to be the natural numbers and $\mathcal{O}_n$ would be obtained by restricting to the objects no greater than n. The description is very simple: however, it turns out to be hard even to prove $\mathcal{O}_\omega$ is an ω-category, let alone prove it free.

The starting point for my description is the fact that a path in a circuit-free (directed) graph is determined by the finite **set** of edges in the path: the edges order themselves using source and target. The set must be "well formed": there should be no two edges with the same source and no two with the same target. Moreover, the source of the path is the unique vertex which is a source of some edge but not the target of any edge in the set. What a **miracle** that this should work in higher dimensions.

Meanwhile, on the enriched category front, Walters had pointed out that in order for $\mathcal{W}$-Mod to be monoidal, the base bicategory $\mathcal{W}$ should be monoidal. You will recall that, in order to define tensor products and duals for $\mathcal{V}$-categories, Eilenberg–Kelly [46] had assumed $\mathcal{V}$ to be symmetric. In a talk in the ACS, Bob Walters reported on a discussion Carboni, Lawvere and he had had about the possibility of using an Eckmann–Hilton argument to show that a monoidal bicategory with one object was a symmetric monoidal category in the same way that a monoidal category with one object is a commutative monoid. It is perhaps not surprising that they did not pursue the calculation to completion at that time since monoidal bicategories had not appeared in print except for the locally ordered case. I was so taken by how much I could do *without* a monoidal structure on $\mathcal{W}$-Mod that I did not follow up the idea then either.

Duskin returned to Australia at the end of 1983 and challenged me to draw $\mathcal{O}_6$; this took me a weekend. The odd-faces-source and even-faces-target convention forces the whole deal!

By the end of 1984 I had prepared the oriented simplexes paper [150]. My conjectured description is correct. (Actually, Verity pointed out an error in the proof written in [150] which I corrected in [156].) The heart of proving things about $\mathcal{O}_\omega$ is the algorithm I call *excision of extremals* for deriving the non-abelian n-cocycle condition "from the top down".

The paper [150] has several other important features. Perhaps the most obvious are the diagrams of the orientals; they resemble Stasheff associahedra with some oriented faces and some commuting faces. I give the 1-sorted definition of ω-category and show the relationship between the 1-sorted definition of n-category and the inductive one in terms of enrichment. I make precise some facts about the category ω-Cat of ω-categories such as its cartesian closedness. I say what it means for a morphism in an n-category to be a weak equivalence.

The paper [150] defines what it means for an n-category to be free. I define the nerve of an n-category and make a conjecture about characterizing those nerves as "stratified" (or filtered) simplicial sets satisfying horn-filler conditions. The horns I suggested should be filled were a wider class than those of Roberts' complicial sets; I called my horns "admissible" and Roberts' "complicial". However, I really believed the admissible horns would still lead to complicial sets.

That there is a weaker notion of n-category than the strict ones was an obvious consequence of the introduction of weak 2-categories (bicategories) by Bénabou [12]. I later was reminded that Mac Lane, in 1969, had suggested tricategories as a possible area of study [120]. As a kind of afterthought in [150], I suggest a characterization of weak n-categories as stratified simplicial sets with horn filler conditions. My intuition was that, even in a strict n-category, the same horns should be fillable by only insisting that our thin elements be simplexes whose highest dimensional cell is a weak equivalence rather than a strict identity. So the same horns should have fillers even in a weak n-category. Of course, the fillers now would not be unique.

While travelling in North America, I submitted the preprint of [150] to expatriate Australian Graeme Segal as editor of *Topology*. I thought Graeme might have some interest in higher nerves as a continuation of his work in [132]. He rejected the paper without refereeing on grounds that it would not be of sufficient interest to topologists. I think this IMA Summer Program proves he was wrong. To make things worse, his rejection letter went to the institution I was visiting when I submitted and it was not forwarded to me at Macquarie University. I waited a year or so before asking Segal what happened! He sent me a copy of his short letter.

In April 1985, all excited about higher nerves, I began a trip to North America that would trigger two wonderful collaborations: one with Sammy Eilenberg and one with André Joyal. The first stop was a conference organized by Freyd and Scedrov at the University of Pennsylvania. After my talk, Sammy told me of his work on rewriting systems and that, in my orientals, he could see higher rewriting ideas begging to be explained. I left Philadelphia near the end of April as spring was beginning to bloom only to arrive in Montréal during a blizzard. Michael Barr had invited me to McGill where Robin Cockett was also visiting.

During my talk in the McGill Category Seminar, André Joyal started quizzing me on various aspects of the higher nerves. We probably remember that discussion differently. My memory is that André was saying that the higher associativities were not the important things as they could be coherently ignored; the more important things were the higher commutativities. In arguing that commutativities were already present in the "middle-of-four interchange", I was harking back to Walters' talk about applying an Eckmann–Hilton-like argument to a one-object monoidal bicategory. That night I checked what I could find out about a monoidal object (or pseudomonoid) in the 2-category of monoidal categories and strong monoidal functors. It was pretty clear that some kind of commutativity was obtained that was not as strong as a symmetry.

When I reported my findings to André the next day, he already knew what was going on. He told me about his work with Myles Tierney on homotopy 3-types as groupoids enriched in 2-groupoids with the Gray tensor product. I told André that I was happy enough with weak 3-groupoids as homotopy types and that ordinary cartesian product works just as well as the Gray tensor product when dealing with bicategories rather than stricter things. In concentrating on this philosophy, I completely put out of my memory the claim that André recently reminded me he made at that time about Gray-categories being a good 3-dimensional notion of weak 3-category. I believed we should come to grips with the fully weak n-categories and this dominated my thinking.

There had been other weakenings of the notion of symmetry for monoidal categories but this kind had not been considered by category theorists. I announced a talk on joint work with Joyal for the Isle of Thornes (Sussex, England) conference in mid-1985: the title was "Slightly incoherent symmetries for monoidal categories". Before the actual talk, we had settled on the name *braiding* for this kind of commutativity. I talked about the a coherence theorem for braided monoidal categories based on the braid groups just as Mac Lane had for symmetries based on the symmetric groups.

After this talk, Sammy Eilenberg told me about his use of the *braid monoid with zero* to understand the equivalence of derivations in rewriting systems. This was the basis of our unpublished work some of which is documented in [48]. We were going to finish the work after he finished his books on cellular spaces with Eldon Dyer.

I returned to Australia where Peter Freyd was again visiting. He became very excited when I lectured on braided monoidal categories in the ACS. He knew about his ex-student David Yetter's monoidal category of tangles. Freyd and Yetter had already entered low-dimensional topology with their participation in the "homfly" polynomial invariant for links. By the next year (mid-1986) at the Cambridge category meeting, I heard that Freyd was announcing his result with Yetter about the freeness of Yetter's category of tangles as a compact braided monoidal category. Their idea was that duals turned braids into links.

In the mid-1980s the low-dimensional topologist Iain Aitchison (Masters student of Hyam Rubenstein and PhD student of Robion Kirby) was my first postdoctoral fellow. He reminded me in more detail of the string diagrams for tensor calculations used by Roger Penrose. Max Kelly had mentioned these at some point, having seen Penrose using them in Cambridge. Moreover, Aitchison [1] developed an algorithm for the non-abelian n-cocycle condition "from the bottom up", something Roberts and I had failed to obtain. He did the same for oriented cubes in place of oriented simplexes. The algorithm is a kind of "Pascal's triangle" where a given entry is derived from two earlier ones; the simplex case is less symmetric because of the different lengths of sources and targets in that case. The algorithm appeared in combinatorial form in a Macquarie Math. Preprint but was nicely represented in terms of string diagrams drawn by hand with coloured pens.

Aitchison and I satisfied ourselves that the arguments of [150] carried over to cubes in place of simplexes but this was not published. That work was subsumed by my parity complexes [153, 156] and Michael Johnson's pasting schemes [73] which I intend to discuss below.

Following my talks on orientals in the ACS, Bob Walters and his student Mike Johnson obtained [74] a variant of my construction of the nerve of a (strict) category. The cells in their version of $\mathcal{O}_n$ were actual subsimplicial sets of a simplicial set and the compositions were all unions; they thought of these cells as simplicial "pasting diagrams". The cells in my $\mathcal{O}_n$ were only generators for the Walters–Johnson simplicial sets and so, while smaller objects to deal with, required some deletions from the unions defining composition.

Around this time I set my student Michael Zaks the problem of proving the equivalence between complicial sets and categories. To get him started I proved [152] that the nerve of an category satisfies the unique thin filler condition for admissible (and hence complicial) horns. So nerves of categories are complicial sets. Zaks fell in love with the simplicial identities and came up with a construction he believed to be the zero-composition needed to make an n-category from a complicial set. We showed that this composition was the main ingredient required by using an induction based on showing an equivalence

$$\mathrm{Cmpl}_n \simeq \mathrm{Cmpl}_{n-1}\text{-Cat}$$

where the left-hand side is the category of n-trivial complicial sets; a stratified simplicial set is *n-trivial* when all elements of dimension greater than n are thin. Zaks did not complete the proof that his formula worked and we still do not know whether it does. In 1990, Dominic Verity was motivated by my paper [150] to work on this problem. Unaware of [152], Dominic independently came up with the machinery Zaks and I had developed. By mid-1991 Dominic had proved, amongst other things, that the nerve was fully faithful; he completed the details of the proof of the equivalence

$$\omega\text{-Cat} \simeq \mathrm{Cmpl}$$

in 1993; the details are still being written [169].

Knowing the nerve of an n-category, we now knew the non-abelian cocycle conditions. So I turned attention to understanding the full cohomology. The idea was that, given a simplicial object X and an category object A, there should be an ω-category to be called the *cohomology of X with coefficients in A*. Jack Duskin pointed out that this should be part of a general *descent* construction which obtains an ω-category $\mathrm{Desc}\,\mathcal{C}$ from any cosimplicial ω-category $\mathcal{C}$. For the cohomology case, the cosimplicial ω-category is $\mathcal{C} = \mathrm{Hom}(X, A)$ taken in the ambient category. Furthermore, Jack drew a few low-dimensional diagrams.

It took me some time to realize that the diagrams Jack had drawn were really just products of globes with simplexes. I then embarked on a program of abstracting the structure possessed by simplexes, cubes and globes, and to show the structure was closed under products. For his PhD, Mike Johnson was also working on abstracting the notion of pasting diagram. In an ACS, I explained my idea about descent and gave an overly-simplistic description of the product of parity complexes. By the next week's ACS Mike Johnson had corrected my definition of product based on the usual tensor product of chain complexes. The next step was to find the right axioms on a parity complex in order for it to be closed under product. For this I invented a new order that I denoted by a solid triangle: let me denote it now by $\prec$. I need to give more detail.

A *parity complex* is a graded set $\dim\colon P \longrightarrow \mathrm{N}$ together with functions

$$(-)^-, (-)^+\colon P \longrightarrow \mathcal{P}_{\mathrm{fin}} P,$$

where $\mathcal{P}_{\mathrm{fin}} S$ is the set of finite subsets of the set S, such that

$$x \in y^- \cup y^+ \text{ implies } \dim x + 1 = \dim y.$$

For x in the fibre P_n we think of x^- as the set of elements in the source of x and x^+ as the set of elements in the target of x. For a subset S of P_n, put $S^\varepsilon = \bigcup_{x\in S} x^\varepsilon$ for $\varepsilon \in \{+,-\}$. There are some further conditions such as

$$x^- \cap x^+ = \emptyset, \quad x^{-+} \cap x^{+-} = \emptyset, \quad x^{--} \cap x^{++} = \emptyset, \quad x^{-+} \cup x^{+-} = x^{--} \cup x^{++}.$$

These conditions imply that we obtain a positive chain complex $\mathbf{Z}P$ consisting of the free abelian groups $\mathbf{Z}P_n$ with differential defined on generators by

$$\mathrm{d}(x) = x^+ - x^-$$

where we write S for the formal sum of the elements of a finite subset S of P_n. The order on P is the smallest reflexive transitive relation $\prec$ such that

$$x \prec y \text{ if either } x \in y^- \text{ or } y \in x^+.$$

The amazing axiom we require is that this order should be ***linear***.

If the functions $(-)^-, (-)^+ \colon P \longrightarrow \mathcal{P}_{\mathrm{fin}}P$ land in singleton subsets of P, the parity complex is a globular set which represents a *globular pasting diagram*. As later shown by Michael Batanin, these globular sets hold the key to free n-categories on all globular sets. A very special globular pasting diagram is the "free-living globular k-cell"; it is a parity complex $\mathbf{G}_k$ with $2k+1$ elements.

The original example of a parity complex is the infinite simplex $\Delta[\omega]$ whose elements of dimension n are injective order-preserving functions $x \colon [n] \longrightarrow \omega$; we write such an x as an ordered $(n+1)$-tuple $(x_0, x_1, \ldots x_n)$. Also $\partial_i \colon [n-1] \longrightarrow [n]$ is the usual order-preserving function whose image does not contain i in $[n] = \{0, 1, \ldots, n\}$. Then

$$x^- = \{x\partial_i \mid i \,\mathrm{odd}\} \text{ and } x^+ = \{x\partial_i \mid i \,\mathrm{even}\}.$$

We obtain a parity complex $\Delta[k]$, called the *parity k-simplex*, by restricting attention to those x that land in $[k]$. In particular, $\Delta[1]$ is the parity interval and also denoted by $\mathbf{I}$.

The product of two parity complexes P and Q is the cartesian product $P \times Q$ with

$$\dim(x, y) = \dim x + \dim y \text{ and } (x, y)^\varepsilon = x^\varepsilon \times \{y\} \cup \{x\} \times y^{\varepsilon(m)}$$

where $\dim x = m$ and $\varepsilon(m)$ is the sign ε when m is even and the opposite of ε when m is odd. It can be shown that $P \times Q$ is again a parity complex. In particular, there is a parity k-cube

$$\mathbf{I}^k = \overbrace{\mathbf{I} \times \cdots \times \mathbf{I}}^{k}.$$

There is a canonical isomorphism of chain complexes

$$\mathbf{Z}(P \times Q) \cong \mathbf{Z}P \otimes \mathbf{Z}Q.$$

A parity complex P generates a free ω-category $\mathcal{O}(P)$. The description is rather simple because the conditions on a parity complex ensure sufficient "circuit-freeness" for the order of composition to sort itself out. The detailed description can be found in [153] or [162].

We shall now describe a monoidal structure on ω-Cat that was considered by Richard Steiner and Sjoerd Crans. It turns out that the full subcategory of ω-Cat, consisting of the free ω-categories $\mathcal{O}\big(\mathbf{I}^k\big)$ on the parity cubes, is dense in ω-Cat. The tensor product of the free ω-categories $\mathcal{O}\big(\mathbf{I}^h\big)$ and $\mathcal{O}\big(\mathbf{I}^k\big)$ is defined by

$$\mathcal{O}\big(\mathbf{I}^h\big) \otimes \mathcal{O}\big(\mathbf{I}^k\big) = \mathcal{O}\big(\mathbf{I}^{h+k}\big).$$

This is extended to a tensor product on ω-Cat by Kan extension along the inclusion. A result of Brian Day applies to show this is a monoidal structure. We call it the *Gray monoidal structure* on ω-Cat although John Gray only defined it on 2-Cat by forcing all cells of dimension higher than 2 to be identities. Dominic Verity has shown that, for a wide class of parity complexes P and Q, we have an isomorphism of categories

$$\mathcal{O}(P) \otimes \mathcal{O}(Q) \cong \mathcal{O}(P \times Q).$$

These ingredients allow us to define the descent ω-category $\operatorname{Desc}\mathcal{E}$ of a cosimplicial category $\mathcal{E}$ as follows. The functor $\mathrm{Cell}_n\colon \omega\text{-Cat} \longrightarrow \mathrm{Set}$, which assigns the set of n-cells to each ω-category, is represented by the free n-category $\mathcal{O}(\mathbf{G}^n)$ on the n-globe; that is,

$$\mathrm{Cell}_n(A) = \omega\text{-Cat}(\mathcal{O}(\mathbf{G}^n), A).$$

From this we see that $\mathcal{O}(\mathbf{G}^n)$ is a *co-n-category in* ω-Cat. Since the functor $-\otimes A$ preserves colimits, it follows that $\mathcal{O}(\mathbf{G}^n) \otimes A$ is a co-n-category in ω-Cat for all categories A. In particular,

$$\mathcal{O}(\mathbf{G}^n) \otimes \mathcal{O}_m = \mathcal{O}(\mathbf{G}^n) \otimes \mathcal{O}(\Delta[m]) \cong \mathcal{O}(\mathbf{G}^n \times \Delta[m])$$

is a co-n-category in ω-Cat. Allowing m to vary, we obtain a co-n-category $\mathcal{O}(\mathbf{G}^n \times \Delta)$ in the category $[\Delta, \omega\text{-Cat}]$ of cosimplicial ω-categories; so we define

$$\operatorname{Desc}\mathcal{E} = [\Delta, \omega\text{-Cat}](\mathcal{O}(\mathbf{G}^n \times \Delta), \mathcal{E}).$$

As a special case, the *cohomology ω-category* of a simplicial object X with coefficients in an category object A (in some fixed category) is defined by

$$\mathcal{H}(X, A) = \operatorname{Desc}\operatorname{Hom}(X, A).$$

During 1986–7, André Joyal and I started to hear about Yang–Baxter operators from the Russian School. Drinfeld lectured on quantum groups at the World Congress in 1986. We attended Yuri Manin's lectures on quantum groups at the University of Montréal. My opinion at first was that, as far as monoidal categories were concerned, braidings were the good notion and Yang–Baxter operators were only their mere shadow. André insisted that we also needed to take these operators seriously. The braid category is not only the free braided monoidal category on a single object, it is the free monoidal category on an object bearing a Yang–Baxter operator. While we were at the Louvain-la-nerve category conference in mid-1987, Iain Aitchison brought us a paper by Turaev that had been presented at an Isle of Thorns low-dimensional topology meeting the week before. Turaev knew about Yetter's monoidal category of tangles and gave a presentation of it using Yang–Baxter operators. I had the impression that Turaev did not know about braided monoidal categories at the time. All André and I had put out in print were a handwritten Macquarie Mathematics Report at the end of 1985 and a typed revision about a year later.

I set my student Mei Chee Shum on the project of "adapting" Kelly–Laplaza's coherence for compact symmetric monoidal categories [105] to the braided case. She soon detected a problem with our understanding of the Freyd–Yetter result. Meanwhile, Joyal and I continued working on braided monoidal categories; there was a variant we called *balanced* monoidal categories based on braids of ribbons (not just strings). We started developing the appropriate string diagrams for calculating in the various monoidal categories with extra structure [76]; this could be seen as a formalization of the Penrose notation for calculating with tensors but now deepened the connection with low-dimensional topology. The notion of *tortile* monoidal category was established; Shum's thesis became a proof (based on Reidermeister calculus) that the free tortile monoidal category was the category of tangles on ribbons. Joyal and I proved in [77], just using universal properties, that this category was also freely generated as a monoidal category by a tortile Yang–Baxter operator. In doing this we introduced the notion of *centre* of a monoidal category $\mathcal{C}$: it is a braided monoidal category $\mathcal{ZC}$. This construction can be understood from the point of view of higher categories. For any bicategory $\mathcal{D}$, the braided monoidal category $\mathrm{Hom}(\mathcal{D},\mathcal{D})(1_{\mathcal{D}},1_{\mathcal{D}})$, whose objects are pseudo-natural transformations of the identity of $\mathcal{D}$, whose morphisms are modifications, and whose tensor product is either of the two compositions, might be called the *centre* of the bicategory $\mathcal{D}$. If $\mathcal{D}$ is the one-object bicategory $\Sigma\mathcal{C}$ with hom monoidal category $\mathcal{C}$ then $\mathrm{Hom}(\mathcal{D},\mathcal{D})(1_{\mathcal{D}},1_{\mathcal{D}})$ is the centre $\mathcal{ZC}$ of $\mathcal{C}$ in the sense of [77].

In statistical mechanics there are higher dimensional versions of the Yang–Baxter equations. The next one in the list is the Zamolodchikov equation. I began to hear about this from various sources; I think first from Aitchison who showed me the string diagrams. I talked a little about this at the category meeting in Montréal in 1991. This is where I was given a copy of Dominic Verity's handwritten notes on complicial sets. Moreover, Bob Gordon and John Power asked me whether I realized that my bicategorical Yoneda lemma in [142] could be used to give a one-line proof that every bicategory is biequivalent to a 2-category. I remembered that I had thought about using that lemma for some kind of coherence but it was probably along the lines of the Giraud result that every fibration was equivalent to a strict one (in the form that every pseudo-functor into Cat is equivalent to a strict 2-functor). Gordon and Power had been looking at categories on which a monoidal category acts and (I imagine) examined the "Cayley theorem" in that context, and then realized the connection with the bicategorical Yoneda lemma. Since this coherence theorem for bicategories was so easy, we decided we would use it as a model for a coherence theorem for trictegories. Tricategories had not been defined in full generality at that point. Our theorem was that every tricategory was triequivalent to a Gray-category; the latter is a little more general than a 3-category (there is an isomorphism instead of equality for the middle-four law). In fact, Gray-categories are categories enriched in 2-Cat with a Gray-type tensor product. John Power has briefly described at this conference the rest of the story behind [54] so I shall say no more about that here.

Of course a tricategory is a "several object version" of a monoidal bicategory. The need for this had already come up in the Australian School: a monoidal structure was needed on the base bicategory $\mathcal{W}$ to obtain a tensor product of enriched $\mathcal{W}$-categories. Kapranov had also sent us rough notes on his work with Voevodsky (see [83, 84, 85]). Their monoidal bicategories were not quite as general as our one-object tricategories but they had ideas about braided monoidal bicategories and the relationship with the Zamolodchikov equation. Larry Breen and Martin Neuchl independently realized that Kapranov–Voevodsky needed an extra condition on their higher braiding. Kapranov–Voevodsky called Gray-categories *semi-strict* 3*-categories* and were advising us that they were writing a proof of coherence; I do not think that ever appeared.

By 1993, with Dominic Verity and Todd Trimble at Macquarie University, many interesting ideas were developed about monoidal and higher-order categories. Amongst other things Verity contributed vitally to the completion of work I had begun with other collaborators: modulated bicategories [23] and traced monoidal categories [81]. Todd was interested in operads and was establishing a use of Stasheff's associahedra to define weak n-categories. He seemed to know what was going on but could not write the general definition formally. I challenged him to write down a definition of weak 4-category which he did [167] against his better judgement: it is horrendous. Moreover, at Macquarie, Todd and Margaret McIntyre worked out the surface diagrams for monoidal bicategories; the paper was submitted to *Advances* and is still in revision limbo. I should point out that Todd was married just before taking the postdoctoral fellowship at Macquarie University. His wife stayed in the U.S. with her good job. So it was natural that, after two years (and only a couple of visits each way), he had to return to the U.S. This left one year of the fellowship to fill. Tim Porter mentioned a chap from Novosibirsk (Siberia). So Michael Batanin was appointed to Macquarie and began working on higher categories.

This brings me to the point of the letter John Baez and James Dolan sent me concerning their wonderful definition of weak ω-category. I think Michael Makkai caught on to their idea much quicker than me and I shall skip over the history in that direction.

I have learnt that when Michael Batanin comes to me starting a new topic with: "Oh Ross, have you . . . ?", that something serious is about to come. If it is mathematics, it is something he has thought deeply about already. A few months after he arrived at Macquarie University, after returning to the

Macquarie carpark from an ACS at Sydney University, Michael popped me one of these questions:

> "Oh Ross, have you ever thought of the free strict n-category on the terminal globular set?"

My response was that the terminal globular set is full of loops, so my approach to free n-categories using parity complexes did not apply. The loops frankly scared me! Soon after, Michael described the monad for ω-categories on globular sets. The clue was his answer to the carpark question: it involved plane trees which he used to codify globular pasting diagrams. Then the solution is like using what I tell undergraduates is my favourite mathematical object, the geometric series, to obtain free monoids.

Batanin's full fledged theory of higher (globular) operads quickly followed, including the operad for weak ω-categories and the natural monoidal environment for the operads; see [8, 158]. He also developed a theory of computads for the algebras of any globular operad [9]: the computads for weak n-categories differ from the ones for the strict case since you need to choose a pasting order for the source and target before placing your generating cell. (Verity's PhD thesis had a coherence theorem for bicategories that pointed out the need for this kind of thing.)

Let me finish with one further development I see as a highlight and a reference which contains many precise details of topics of interest to this conference. The highlight, arising from the development of the theory of monoidal bicategories jointly with Brian Day, is the realization of the connection among the concepts of quantum groupoids, $*$-autonomy in the sense of Michael Barr, and Frobenius algebras (see [38, 163]). The reference for further reading is [162] which I prepared for the Proceedings of the Workshop on "Categorical Structures for Descent and Galois Theory, Hopf Algebras and Semiabelian Categories" at the Fields Institute, Toronto 2002; it represents an improved and updated version of notes of three lectures presented at Oberwolfach in September 1995.

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