SUBDIVISIONS OF SMALL CATEGORIES

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Let $\mathscr{A}$ be a (small) category. For example, monoids (sets with associative and unital products) can be identified with categories with a single object. Analogously, posets can be identified with those categories $\mathscr{A}$ with at most one arrow between any two objects by defining $x \leq y$ if there is an arrow $x \longrightarrow y$ between the objects x and y of $\mathscr{A}$.

In particular, we write $[n]$ for the poset $\{0 < 1 \cdots < n\}$. If we think of $[n]$ as a finite space, then a continuous map $f\colon [m] \longrightarrow [n]$ is a monotonic, or non-decreasing, function, $f(i) \leq f(j)$ if $i \leq j$. We can equally well regard such a function as a functor $[m] \longrightarrow [n]$. We are perilously close to defining the fundamental notion of a simplicial object in a possibly large category $\mathscr{C}$, so let's do so.

Let Δ denote the category of posets $[n]$ and monotonic maps between them. This category is generated by certain canonical monotonic maps. We have the "face map" $\delta_i\colon [n] \longrightarrow [n+1]$ which is the monomorphism that misses i. That is $\delta_i(j) = j$ if $j < i$ and $\delta_i(j) = j+1$ if $j \geq i$. We also have the surjection $\sigma_i\colon [n+1] \longrightarrow [n]$ that hits i twice. That is, $\sigma_i(j) = j$ if $j \leq i$ and $\sigma_i(j) = j-1$ if $j > i$. Every morphism in Δ is a composite of these morphisms, and they satisfy certain easily determined identities. A simplicial object in $\mathscr{C}$ is a *contravariant* functor $\Delta \longrightarrow \mathscr{C}$. In detail, it is a sequence of objects $C_n \in \mathscr{C}$ together with face maps $d_i\colon C_n \longrightarrow C_{n-1}$ and degeneracy maps $s_i\colon C_n \longrightarrow C_{n+1}$, $0 \leq i \leq n$, such that

$$d_i \circ d_j = d_{j-1} \circ d_i \quad \text{if} \quad i < j$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if} \quad i < j \\ \mathrm{id} & \text{if} \quad i = j \quad \text{or} \quad i = j+1 \\ s_j \circ d_{i-1} & \text{if} \quad i > j+1. \end{cases}$$

$$s_i \circ s_j = s_{j+1} \circ s_i \quad \text{if} \quad i \leq j.$$

When $\mathscr{C} = Set$ is the category of sets, we obtain $sSet$, the category of simplicial sets. For example, if S is a topological space, we obtain the simplicial set SX such that S_nX is the set of all continuous maps from the standard topological n-simplex Δ_n to X. Explicitly,

$$\Delta_n = \{(t_0, \ldots, t_n) | 0 \leq t_i \leq 1, \ \textstyle\sum t_i = 1\} \subset \mathbb{R}^{n+1}.$$

We have the "face maps"

$$\delta_i : \Delta_{n-1} \longrightarrow \Delta_n, \quad 0 \leq i \leq n,$$

specified by

$$\delta_i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

and "degeneracy maps"

$$\sigma_i : \Delta_{n+1} \longrightarrow \Delta_n, \quad 0 \leq i \leq n,$$

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specified by

$$\sigma_i(t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1}).$$

Precomposing with these maps, we obtain the maps d_i and s_i that make SX into a simplicial set. It has long been known that we can use simplicial sets pretty much interchangeably with topological spaces when studying homotopy theory.

For a simplicial set K, we define a space $|K|$, called the "geometric realization" of K, as follows. As a set

$$|X| = \coprod_{n\geq 0}(K_n \times \Delta_n)/(\sim),$$

where the equivalence relation $\sim$ is generated by

$$(k, \delta_i u) \sim (d_i(k), u) \quad \text{for} \quad k \in K_n \quad \text{and} \quad u \in \Delta_{n-1}$$

and

$$(k, \sigma_i v) \sim (s_i(k), v) \quad \text{for} \quad K \in K_n \quad \text{and} \quad v \in \Delta_{n+1}.$$

Topologize $|K|$ by giving

$$|K|^n \equiv \coprod_{0\leq q\leq n}(K_q \times \Delta_q)/(\sim)$$

the quotient topology and then giving $|K|$ the topology of the union, so that a subset is closed if it intersects each $|K|^n$ in a closed subset. Write $|k, u|$ for points of $|K|$. Say that (k, u) is nondegenerate if $k \in K_n$ is not of the form $s_i j$ for any i and any $j \in K_{n-1}$ and if $u \in \Delta_n$ is an interior point. Every (k, u) is equivalent to one and only one nondegerate point.

Define $\gamma : |SX| \longrightarrow X$ by

$$\gamma|f, u| = f(u) \quad \text{for} \quad f : \Delta_n \longrightarrow X \quad \text{and} \quad u \in \Delta_n.$$

It is a fact that γ is a weak homotopy equivalence for every space X, although we shall not prove that. There is also a map $\iota\colon K \longrightarrow S|K|$ of simplicial sets specified by $\iota(k)(u) = |k, u|$ for $k \in K_n$ and $u \in \Delta_n$. Again, as we also shall not prove, $|\iota|\colon |K| \longrightarrow |S|K||$ is a homotopy equivalence.

There is a neat relationship between $|-|$ and S. They are left and right adjoint functors, meaning that there is a bijection, natural in both variables, between morphism sets:

$$\text{Top}(|K|, X) \cong \text{sSet}(K, SX).$$

It is specified by letting f correspond to g if $f(|k, u|) = g(k)(u)$.

There is also a construction that assigns a simplicial *set* K^s to a simplicial *complex* K. The idea is to allow repeated elements in the sets of simplices. It is easiest to define $\mathscr{K}(X)^s$ for a poset X, and then the construction gives a functor, but one can use any arbitrarily chosen ordering of the set of vertices to apply the definition more generally. One lets the set $\mathscr{K}(X)^s_n$ of n-simplices be the set of sequences $x_0 \leq x_1 \leq \cdots \leq x_n$. Deleting x_i gives d_i, and repeating x_i gives s_i. The geometric realization gives the geometric realization $|\mathscr{K}(X)^s|$. When X is finite this is homeomorphic to any choice of geometric realization as we defined it earlier, but the definition $|\mathscr{K}(X)| = |\mathscr{K}(X)^s|$ works in general and gives a functor of X.

With this as background, we turn to the homotopy theory of small categories $\mathscr{A}$. We construct a simplicial set $N\mathscr{A}$ called the nerve of $\mathscr{A}$. Regarding $[n]$ as a category, we define the set $N_n\mathscr{A}$ of n-simplices to be the set of functors $[n] \longrightarrow \mathscr{A}$. Regarding a monotonic function $f\colon [m] \longrightarrow [n]$ as a functor, precomposition with f gives us the required contravariant functoriality on Δ. The definition should look very similar to the definition of the total singular functor S from spaces to simplicial sets. It gives us a functor N from $\mathscr{C}at$, the category of small categories and functors between them, to simplicial sets. We define $B\mathscr{A} = |N\mathscr{A}|$. This is called the *classifying space* of the category $\mathscr{A}$. When G is a group regarded as a category with a single object, BG is called the classsifying space of the group G. The space BG is often written as $K(G,1)$ and called an Eilenberg-Mac Lane space. It is characterized (up to homotopy type) as a connected space with $\pi_1(K(G,1)) = G$ and with all higher homotopy groups $\pi_q(K(G,1)) = 0$. These are fundamentally important constructions in topology and its applications.

The nerve functor N is accompanied by a functor $\tau_1 \colon sSets \longrightarrow \mathscr{C}at$, called the "fundamental category" functor. It is left adjoint to N, meaning that

$$\mathscr{C}at(\tau_1 K, \mathscr{A}) \cong \mathrm{sSet}(K, N\mathscr{A}).$$

This means that it is conceptually sensible, but it does not have good homotopical properties. For a simplicial set K, the objects of the category $\tau_1 K$ are the vertices (= 0-simplices) of K. To construct the morphisms, one starts by thinking of the 1-simplices y as maps $d_1 y \longrightarrow d_0 y$. One forms all words (formal composites) that make sense, that is, whose targets and sources match up. Then one imposes the relations on morphisms determined by

$$s_0 x = \mathrm{id}_x \quad \text{for} \quad x \in K_0 \quad \text{and} \quad d_1 z = d_0 z \circ d_2 z \quad \text{for} \quad z \in K_2.$$

This makes good sense since if $K = N\mathscr{A}$, then a 0-simplex is an object x of $\mathscr{A}$, a 1-simplex y is a map $d_1 y \longrightarrow d_0 y$ and $s_0 x = \mathrm{id}_x$, and a 2-simplex z is given by a pair of composable morphisms $d_2 z$ and $d_0 z$ together with their composite $d_1 z$. Therefore there is a natural map $\tau_1 N\mathscr{A} \longrightarrow \mathscr{A}$ that is the identity on zero simplices and is induced by the identity on 1-simplices. In fact, it is an isomorphism of categories: it is the identity on objects, and it presents the category in terms of generators given by the morphism sets modulo relations determined by the category axioms.

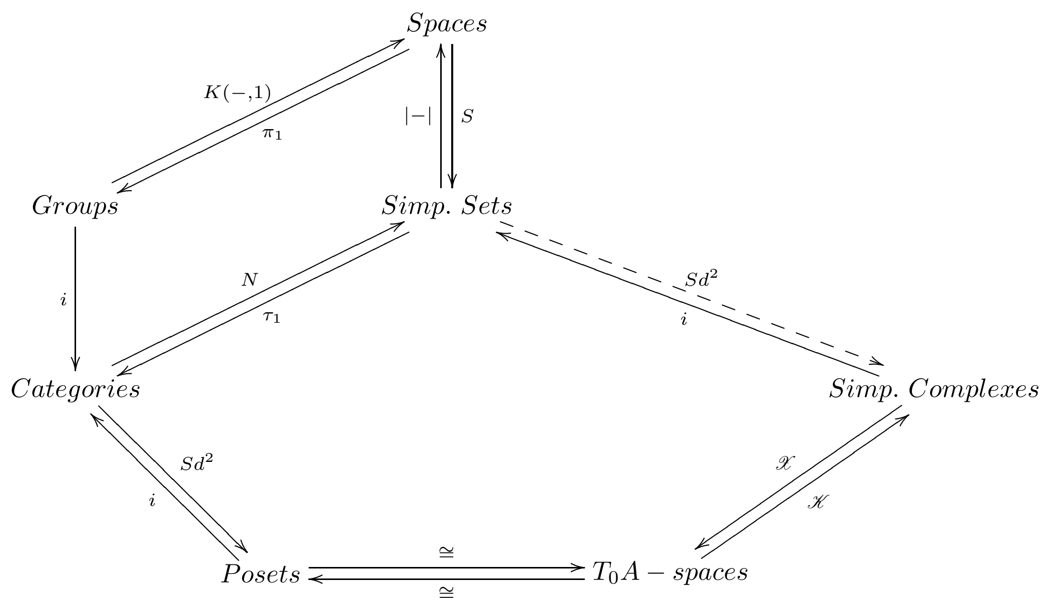
For the adjunction, a functor $F \colon \tau_1 K \longrightarrow \mathscr{A}$ is constructed from a map of simplicial sets $g \colon K \longrightarrow N\mathscr{A}$ by letting F be the unique functor that agrees with g on objects (= 0-simplices) and equivalence classes of morphisms (= 1-simplices). Applying the adjunction to the identity map of $\tau_1 K$, we obtain a natural map $\eta \colon K \longrightarrow N\tau_1 K$. Backing up, we define the standard simplicial n-simplex $\Delta[n]$ to be the simplicial set whose q-simplices are the monotonic functions $\sigma \colon [q] \longrightarrow [n]$; precomposition with monotonic functions $\xi \colon [p] \longrightarrow [q]$ gives the required contravariant functoriality on Δ. The nondegenerate q-simplices in $\Delta[n]$ are the monomorphisms (= strictly monotonic functions) $[q] \longrightarrow [n]$, and there is one for each subset of $[n]$ of cardinality $q+1$. We may identify the set of all non-degenerate simplices with the poset of non-empty subsets of the set $[n]$ of $n+1$ elements, ordered by inclusion. In other words, $\Delta[n] = (\mathscr{K}([n]))^s$ is the simplicial set determined by the simplicial complex $\mathscr{K}([n])$. A monotonic function $\alpha \colon [m] \longrightarrow [n]$ gives a map $\alpha \colon \Delta[m] \longrightarrow \Delta[n]$ of simplicial sets that sends $\sigma \colon [q] \longrightarrow [m]$ to $\alpha \circ \sigma$. Thus $\Delta[-]$ is a covariant functor from Δ to simplicial sets.

The n-skeleton K^n of a simplicial set K is the subsimplicial set generated by the q-simplices for all $q \leq n$. Visibly, $\tau_1 K$ depends only on the 2-skeleton K^2. Therefore the inclusion $K^2 \longrightarrow K$ of simplicial sets induces an isomorphism of categories $\tau_1 K^2 \longrightarrow \tau_1 K$ for any K. In particular, τ_1 takes the inclusion $\iota \colon \partial\Delta[n] \longrightarrow \Delta[n]$ of the boundary of the n-simplex to the identity functor when $n > 2$. Thus τ_1 loses homotopical information: upon realization, $|\iota|$ is equivalent to the inclusion

$S^{n-1} \longrightarrow D^n$. What is amazing is that this extreme loss of information disappears after subdividing twice. This is something I am trying to better understand myself.

The reader will find it easy to believe that there is a subdivision functor on simplicial sets that generalizes the subdivision functor Sd on simplicial complexes in the sense that $(SdK)^s \cong Sd(K^s)$ for a simplicial complex K. This allows one to define a subdivision functor on categories by setting $Sd\mathscr{A} = \tau_1 SdN\mathscr{A}$. One can iterate subdivision, forming functors Sd^2 on both simplicial sets and categories. What is mind blowing at first is that the iterated subdivision $Sd^2\mathscr{A}$ is actually a poset whose classifying space $BSd^2\mathscr{A}$ is homotopy equivalent to $B\mathscr{A}$. I will explain at least the construction in a slow way to try to make the idea transparent.

However, before heading for that, let us summarize a schematic and technically oversimplified global picture of all of the big categories we are constructing and comparing by functors. There is an interesting picture of lots of kinds of mathematics that come together with a focus on simplicial sets.



Our earlier talks focused on finite spaces, but the basic theory generalizes with the finiteness removed, provided we understand simplicial complexes to mean abstract simplicial complexes. As noted above, we didn't define geometric realization in general earlier, but we have done so now. The equivalence of posets with T_0 Alexandroff spaces and the constructions $\mathscr{K}$ and $\mathscr{X}$ that we worked out in detail for finite spaces work in exactly the same way when we no longer restrict ourselves to the finite case. The functors i in the diagram are thought of as inclusions of categories. We have defined all of the categories and functors exhibited in the diagram except for Sd^2.

As very often, there are two ways to define subdivision, a conceptual one and a concrete combinatorial one. Which one prefers is a matter of taste. We will start conceptually and then come back down to earth.

For a set C and a simplicial set L, one can form a new simplicial set $C \times L$ by letting $(C \times L)_q = C \times L_q$, and similarly letting the faces and degeneracies be induced by those of L. A simplicial set K can be reconstructed from the disjoint union over n of the simplicial sets $K_n \times \Delta[n]$ for $n \geq 0$ by taking equivalence classes under the equivalence relation generated by

$$(\alpha^*(k), \sigma) \simeq (k, \alpha_*(\sigma)) \quad (1)$$

for $k \in K_n$, $\sigma \in \Delta[m]_q$, and $\alpha\colon [m] \longrightarrow [n]$. Here $\alpha^*(k) \in K_m$ is given by the fact that K is a contravariant functor from Δ to sets and $\alpha_*(\sigma) \in \Delta[n]_q$ is given by the fact that $\Delta[-]$ is a covariant functor from Δ to simplicial sets. The simplicial structure is induced from the simplicial structure on the $\Delta[n]$. The point is that an arbitrary pair (k, τ) in $K_n \times \Delta[n]_q$ is equivalent to the pair $(\tau(k), \iota_q)$ in $K_q \times \Delta[q]_q$, where $\iota_q\colon [q] \longrightarrow [q]$ is the identity map viewed as a canonical q-simplex in $\Delta[q]$, and $\tau\colon [q] \longrightarrow [n]$ is viewed as a morphism of Δ, so that $\tau = \tau_*(\iota_q)$. Identifying equivalence classes of q-simplices with elements of K_q in this faction, we find that the faces and degeneracies agree. Indeed, for $\xi\colon [p] \longrightarrow [q]$, $\xi \circ \iota_p = \iota_q \circ \xi$ and

$$(k, \xi^*(\iota_q)) = (k, \xi_*(\iota_p)) \simeq (\xi^*(k), \iota_p).$$

We define $Sd\Delta[n] = (\mathscr{K}'[n])^s$. That is, we take the simplicial set associated to the barycentric subdivision of the simplicial complex $\mathscr{K}[n]$, where we again regard $[n]$ as a poset. Just like $\Delta[-]$, this gives a covariant functor $Sd\Delta[-]$ from Δ to simplicial sets. We use it in exactly the same way as above to construct SdK as a quotient of the disjoint union of the $\mathscr{K}[n] \times Sd\Delta[n]$ by the equivalence relation of the same form as (**??**). This is obviously sensible! We then define $Sd\mathscr{A} = \tau_1 SdN\mathscr{A}$.

MORE TO COME.