2nd subdivision of a simplicial set

1 Subdivision of simplicial sets - Kan version

Δ_n is the simplicial set whose q-simplices are maps $[q] \to [n]$. For any $[p] \xrightarrow{\gamma} [q]$, for $[q] \xrightarrow{\alpha} [n]$, $\gamma^*\alpha$ is defined by precomposition. Therefore α is nondegenerate iff it is injective.

$\Delta' [n]$ is a simplicial set whose q-simplices are ordered $(q+1)$-tuples $(\alpha_0, \ldots \alpha_q)$ of injective maps with target $[n]$, such that α_i is a face of α_{i+1}, i.e. is obtained from α_{i+1} by precomposition. For any $[p] \xrightarrow{\gamma} [q]$, for $\underline{\alpha} = (\alpha_0, \ldots \alpha_q)$, $\gamma^* (\underline{\alpha}) = \left(\alpha_{\phi(0)}, \ldots \alpha_{\phi(p)}\right)$. Therefore $(\alpha_0, \ldots \alpha_q)$ is nondegenerate iff each α_i is a proper face of α_{i+1}.

Since each α_i is injective, and injective maps in $\mathbf{\Delta}$ are uniquely determined by their images, it is equivalent to define q-simplices of Δ'_n as elements of the form $(S_0, \ldots S_q)$ where each S_i is a subset of $[n]$, such that $S_i \subset S_{i+1}$ for all i. Then for any For any $[p] \xrightarrow{\gamma} [q]$, $\gamma^* (S_0, \ldots S_q) := \left(S_{\gamma(0)}, \ldots S_{\gamma(p)}\right)$. Then this simplex is nondegenerate iff $S_i \neq S_{i+1}$ for each i.

Kan subdivision is defined as $sdK := K \otimes \Delta' [-]$. The standard notation is that elements in sdK_q are in the form $x \otimes (\phi_0, \ldots \phi_q)$ where $x \in K_n$ and the ϕ_i's are each nondegenerate faces of Δ_n (i.e. injective maps $[m_i] \to [n]$) such that for each i, ϕ_i is a face of ϕ_{i+1} (i.e. can be obtained from ϕ_{i+1} via precomposition of some injective map).

These elements are under the equivalence relation generated by $\gamma^* x \otimes (\phi_0, \ldots \phi_q) \sim x \otimes (\psi_0, \ldots \psi_q)$, for any morphism γ in $\mathbf{\Delta}$, where for each i, ψ_i is the injective part of $\gamma \circ \phi_i$ (under the unique decomposition of any map in $\mathbf{\Delta}$ into a surjective map followed by an injective map).

For any $[p] \xrightarrow{\gamma} [q]$, $\gamma^* \left(x \otimes (\phi_0, \dots \phi_q)\right) := x \otimes \left(\phi_{\gamma(0)}, \dots \phi_{\gamma(p)}\right)$.

Equivalently, one can write any element of sdK as $x \otimes (S_0, \dots S_q)$ where $x \in K_n$ and each S_i is a subset of $[n]$, such that $S_i \subset S_{i+1}$ for all i. (This is equivalent since any injective map in $\mathbf{\Delta}$ is uniquely determined by its image).

In the new notation, the elements are under the equivalence relation generated by $\gamma^* x \otimes (S_0, \dots S_q) \sim x \otimes (\gamma(S_0), \dots \gamma(S_q))$, for any morphism γ in $\mathbf{\Delta}$.

And for any $[p] \xrightarrow{\gamma} [q]$, $\gamma^* \left(x \otimes (S_0, \dots S_q)\right) := x \otimes \left(S_{\gamma(0)}, \dots S_{\gamma(p)}\right)$.

Trivially these two definitions give isomorphic simplicial sets.

It is known that any element of sdK can be written uniquely in minimal form. Here an element $x \otimes (S_0, \dots S_q)$ is in minimal form if $x \in K_n$ is nondegenerate and $S_q = [n]$.

Claim 0.0.1: A simplex of sdK is degenerate if and only if when written in minimal form $x \otimes (S_0, \dots S_q)$, $S_i = S_{i+1}$ for some i.

Proof: 'If' is trivial; 'only if' can be proved as follows:

Take any degenerate simplex; this means that the simplex can be represented by $Y := y \otimes (T_0, \dots T_q)$ where for some j, $T_j = T_{j+1}$. Write $y \in K_n$. Let $m = |T_q| - 1$, and find injective $[m] \xrightarrow{\gamma} [n]$ such that $\gamma([m]) = T_q \in [n]$. Then define $T'_i := \gamma^{-1}(T_i)$ for all i; note that $T'_q = [m]$. So, $Y \sim y' \otimes \left(T'_0, \dots T'_q\right)$ where $y' = \gamma^* y \in K_m$. Now write $y' = \sigma^* z$ for unique surjective σ and nondegenerate $z \in K_r$; $Y \sim z \otimes \left(\sigma(T'_0), \dots \sigma\left(T'_q\right)\right)$. Note that $\sigma\left(T'_q\right) = [r]$ so this is now in minimal form. And, $T_j = T_{j=1} \Rightarrow \sigma\left(T'_j\right) = \sigma\left(T'_{j+1}\right)$. This proves the claim.

$\square$.

Note that sd is a functor $\mathcal{S} \to \mathcal{S}$; for any map of simplicial sets $K \xrightarrow{f} L$, define the map $sd(f)$ to send $x \otimes (S_0, \dots S_q) \mapsto f(x) \otimes (S_0, \dots S_q)$. Trivially this is a map of simplicial sets, and the functor satisfies laws of identity and composition.

Lemma 0.0.2: If $x \in K_n$ is nondegenerate then there exists a nondegenerate q-simplex X in sdK with qth vertex $x \otimes ([n])$ iff $q \leq n$.

Proof: If $q \leq n$, set $X = x \otimes ([n-q], [n-q+1], \ldots [n])$; by Claim 0.0.1, X is nondegenerate. Conversely if X is a nondegenerate q-simplex with qth vertex x then in minimal form, we must have $X = x \otimes (S_0, \ldots S_q = [n])$ with S_i a proper subset of S_{i+1} for all i; therefore $q \leq n$.
□.

Claim 0.0.3: An isomorphism $sdK \cong sdL$ induces a natural isomorphism $K_n \cong L_n$ for each n, and for any $x \in K_n$ corresponding to $y \in L_n$, the faces of x correspond to the faces of y.

(However, I will show in the 'Non-examples' section that K and L are not necessarily isomorphic).

Proof: Given a map of simplicial sets $sdK \xrightarrow{f} sdL$, define a function $K_n \xrightarrow{g_n} L_n$ as follows. For nondegenerate $x \in K_n$, we have simplex $x \otimes ([n]) \in sdK_0$, written in minimal form. f sends this to some element Y of sdL_0; write Y in minimal form, $y \otimes ([m])$. But by Lemma 0.0.2, $m = n$, and so $y \in L_n$. Define $g_n(x) = y$. Note that in particular, g_n sends nondegenerate simplices to nondegenerate simplices.

For degenerate $x \in K_n$, there is a unique surjective map σ in $\mathbf{\Delta}$ and nondegenerate simplex x' in K_m (for some m), such that $x = \sigma^* x'$. Define $g_n(x) = \sigma^* g_m(x')$.

Constructing an analagous map $L_n \to K_n$ will show that g is bijective (this follows from the fact that for any x, σ and x' as defined above are unique).

For nondegenerate $x \in K_n$, the $(n-1)$-faces of x are the simplices $x' \in K_{n-1}$ such that there is some $X \in sdK_1$ with $d_0^* X$ and $d_1^* X$ yielding x' and x, respectively. By uniqueness of minimal form, there are $n+1$ such simplices x' (although some may repeat - count them with multiplicities). Since this property is preserved by the isomorphism $sdK \cong sdL$, it follows that for any nondegenerate $x \in K_n$ corresponding via g_n to $y \in L_n$, the $(n-1)$-faces of x correspond (with multiplicities) via g_{n-1} to the $(n-1)$-faces of y.
□.

2 Property A:

Say that K has property A if for any non-degenerate simplex x, all faces of x are also non-degenerate.

Claim 0.1: K has property A iff sdK has property A.

Proof: First suppose K has property A. Take an element $X = x \otimes (S_0, \ldots S_q)$ in sdK_q. Assume X is in minimal form and $x \in K_n$.

Suppose for some $[p] \xrightarrow{\gamma} [q]$, $\gamma^* X = x \otimes \left(S_{\gamma(0)}, \ldots S_{\gamma(p)}\right)$ is degenerate. Let α be the unique injective map $[m] \to [n]$ with image $S_{\gamma(p)}$. Then $\gamma^* X \sim Y := \alpha^* x \otimes \left(S'_{\gamma(0)}, \ldots S'_{\gamma(p)} = [m]\right)$. (Here $\alpha\left(S'_{\gamma(i)}\right) = S_{\gamma(i)}$).

Since x is nondegenerate and K has property A, $\alpha^* x$ is nondegenerate, and so Y is in minimal form. By Claim 0.0.1, for some i, $S'_{\gamma(i)} = S'_{\gamma(i+1)}$. Therefore $S_{\gamma(i)} = S_{\gamma(i+1)}$, proving that X is degenerate.

To prove the converse, suppose K does NOT have property A. Let x be a non-degenerate n-simplex with degenerate m-face y. Then find nondegenerate k-simplex z, injective α, and surjective $\sigma \neq id$ such that $\alpha^* x = y = \sigma^* z$. Find injective right inverse for σ, β.

In sdK, we have 2-simplex $x \otimes (\alpha \circ \beta([k]), \alpha([m]), [n])$. It is written in unique minimal form, therefore by above, it is nondegenerate. However, a face of this 2-simplex is $x \otimes (\alpha \circ \beta([k]), \alpha([m])) \sim \alpha^* x \otimes (\beta([k]), [m]) = \sigma^* z \otimes (\beta([k]), [m]) \sim z \otimes (\sigma \circ \beta([k]), \sigma([m])) = z \otimes ([k], [k])$, a degenerate 1-simplex.

Therefore sdK does not have property A.
$\square$.

3 Property B

Say K has property B if for any nondegenerate $x \in K_n$, x has $n+1$ distinct vertices (meaning 0-faces).

Claim 1.1: K has property B if and only if for any n and nondegenerate $x \in K_n$, for any injective α, β in $\mathbf{\Delta}$ with target $[n]$, $\alpha^* x = \beta^* x \Rightarrow \alpha = \beta$.

Proof: Clearly if the second statement is true then K has property B.

To prove the converse I will first introduce an alternate notation. For $x \in K_n$ and $S \subset [n]$ let $S^* x$ denote $\alpha^* x$ where $[m] \xrightarrow{\alpha} [n]$ is the unique injective map in $\mathbf{\Delta}$ with target $[n]$ and image S.

Now the second statement is equivalent to saying that for any n and nondegenerate $x \in K_n$, if S and T are distinct subsets of $[n]$ then $S^* x \neq T^* x$.

Suppose K has property B, and for some nondegenerate $x \in K_n$ and $S, T \subset [n]$ we have $S^* x = T^* x$. This implies $|S| = |T|$. Write $S = \{s_0, \dots s_m\}$, $T = \{t_0, \dots t_m\}$, each in strictly increasing order. For $0 \leq i \leq m$, we have $\{s_i\}^* x = \{i\}^* (S^* x) = \{i\}^* (T^* x) = \{t_i\}^* x$. But since K has property B, this implies $s_i = t_i$ for all i. Therefore $S = T$.
$\square$.

Claim 1.2: Property B implies property A.

Proof: Suppose property A is false for K. Then find a nondegenerate n-simplex x with a degenerate face. Any degenerate simplex has a degenerate 1-face, therefore x has a degenerate 1-face. Since two of x's vertices lie in this 1-face and are therefore equal, x cannot have $n+1$ distinct vertices. Then property B is false for K.
$\square$.

Claim 1.3: If K has property A then sdK has property B.

Proof: Take any $X \in sdK_q$, $X = x \otimes (S_0, \dots S_q)$, in minimal form, with

$x \in K_n$. Suppose $x \otimes (S_a) \approx x \otimes (S_b)$ for some $0 \leq a < b \leq q$. Let α and β be injective maps with target $[n]$ and images S_a and S_b, respectively. We have $\alpha^* x \otimes ([m_a]) \sim \beta^* x\,([m_b])$. But since K has property A, $\alpha^* x$ and $\beta^* x$ are nondegenerate, and therefore by uniqueness of minimal form, $m_a = m_b$. Since also $S_a \subset S_b$, this implies $S_a = S_b$, and therefore X is degenerate. Therefore sdK has property B.
$\square$.

Note that Claims 0.1, 1.2, and 1.3 combined prove that sdK has property B iff K has property A.

4 Property C

Say K has property C if for any distinct 0-simplices $v_0, \ldots v_n$, there is at most one n-simplex x with vertex set $\{v_0, \ldots v_n\}$.

Claim 2.2: K has property B if and only if sdK has property C.
Proof: First suppose K has property B. Suppose that in sdK_q, nondegenerate simplices $X = x \otimes (S_0, \ldots S_q)$ and $Y = y \otimes (T_0, \ldots T_q)$ each have $q+1$ distinct vertices, and their vertex sets are equal. Assume without loss of generality that X and Y are in minimal form. Then by property B, the vertices of X and of Y can be written in minimal form as $S_i^* x \otimes ([n_i])$ and $T_i^* y \otimes ([m_i])$.

Note that since X and Y are nondegenerate, we have that $n_0 < \ldots < n_q$ and $m_0 < \ldots < m_q$.

Now suppose for some i, j, we have $S_i^* x \otimes ([n_i]) \sim T_j^* x \otimes ([m_j])$. By minimal form this would imply $n_i = m_j$. If $i < j$ then it follows by the pigeonhole principle that for some $j' < j$ and $i' > i$, we would have $n_{i'} = m_{j'}$. Then we would have $n_i < n_{i'} = m_{j'} < m_j = n_i$, a contradiction. Therefore the ORDERED vertex sets of X and Y are equal; that is, for each i, $n_i = m_i$ and $S_i^* x = T_i^* y$.

But by minimal form we also have $x = y$, and so by property B, $S_i = T_i$ for all i, therefore $X = Y$. Therefore property C holds.

Next suppose K does not have property B. Then find nondegenerate $x \in K_n$ and distinct $[0] \xrightarrow{\alpha,\beta} [n]$ such that $\alpha^* x = \beta^* x$ (i.e. repeated vertices). Then by uniqueness of minimal form, $x \otimes (\alpha([0]), [n]) \not\sim x \otimes (\beta([0]), [n])$ but $x \otimes (\alpha([0])) \sim \alpha^* x \otimes ([0]) = \beta^* x \otimes ([0]) \sim s \otimes (\alpha([0]))$, so we have a vertex set $\{v_0, v_1\}$ and two distinct nondegenerate 1-simplices with this vertex set. (Note that as x is nondegenerate and by definition $n \neq 0$, $v_0 \neq v_1$). Therefore sdK does not have property C.
□.

Note that the combination of the claims above proves that if K has property A, $sd(sdK)$ has properties B and C. (And conversely, if $sd^n K$ has property A, B, or C for any $n > 0$, then K has property A).

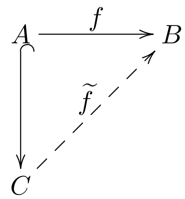
5 Relationship to simplicial complexes

There is a standard full embedding of the category of simplicial complexes, $\mathcal{SC}plx$, into $\mathcal{S}$, by creating all the necessary degenerate simplices. Call this functor D. It is known that for simplicial set K, $K \in D(\mathcal{SC}plx)$ if and only if the vertex sets of nondegenerate simplices of K satisfy the criteria for a simplicial complex; that is, if each n-simplex's vertex set contains $n+1$ vertices, and it is uniquely determined by its vertex set.

Claim 3.1: For $K \in \mathcal{S}$, $K \in D(\mathcal{SC}plx)$ iff K has properties B and C.
Proof: This follows directly from definitions of properties B and C.
□.

6 Relationship to quasicategories and Kan complexes

A 'quasicategory' is a simplicial set K such that for any n, $0 < k < n$, any map $\Lambda_n^k \to F$ extends to Δ_n as in the diagram below:



Here Λ_n^k is the $(n-1)$-faces of Δ_n minus the kth $(n-1)$-face, where Δ_n is the simplicial set whose m-simplices are maps $[m] \to [n]$ with faces and degeneracies via precomposition. Note that there are $\binom{n+1}{m+1}$ nondegenerate m-simplices for each m.

A Kan complex is a simplicial set K such that for any n, $0 \leq k \leq n$, any map $\Lambda_n^k \to F$ extends to Δ_n. In other words, a quasicategory is a simplicial set with extensions of inner horns, while a Kan complex is a simplicial set with extensions of both inner and outer horns.

Claim 4.1: For any simplicial set K, if K does not have property A, then sdK is not a quasicategory.

Proof: Assume K does not have property A. I will construct a map $\Lambda_3^2 \xrightarrow{f} sdK$ that cannot be extended to a map $\Delta_3 \xrightarrow{\tilde{f}} sdK$.

As in examples above, take a nondegenerate simplex x with degenerate face $y = \sigma^* z = \alpha^* x$ where σ is surjective and α is injective, and z is non-degenerate. Find injective right inverse β for σ. $x \in K_n$, $y \in K_m$, $z \in K_k$.

Write ι_3 for the generating 3-simplex of Δ_3. The three 2-faces of ι_3 in the inner horn Λ_3^2 are $d_0^*\iota_3$, $d_1^*\iota_3$, and $d_3^*\iota_3$.

Let f map $d_0^*\iota_3$ to $x \otimes (\alpha \circ \beta([k]), \alpha([m]), [n])$, map $d_1^*\iota_3$ to $x \otimes (\alpha([m]), \alpha([m]), [n])$, and map $d_3^*\iota_3$ to $z \otimes ([k], [k], [k])$. This gives a consistent definition of f on the 0- and 1-faces of the horn (a straightforward calculation).

However, f cannot be extended to the last 2-face $d_2^*\iota_3$. This is because any possible image, in minimal form, would have to be in the form $x \otimes (S, T, [n])$ (unique minimal form of the last vertex). Then we would need $x \otimes (S, [n]) \sim x \otimes (\alpha([m]), [n])$ and $x \otimes (T, [n]) \sim x \otimes (\alpha \circ \beta([k]), [n])$. By uniqueness of minimal form, we see that $S = \alpha([m])$ and $T = \alpha \circ \beta([k])$. Therefore, since $k < m$,

T is a propert subset of S. But by definition $S \subset T$. This is a contradiction, therefore no such extension exists.

Therefore sdK does not have extensions for all inner horns, and is therefore not a quasicategory.
$\square$.

Claim 4.2: sdK **is not a Kan complex, unless** K **is a trivial simplicial set with no non-degenerate simplices of positive degree.**

Proof: Assume K has a non-degenerate n-simplex x, where $n > 0$. Let y be a vertex of x with injective α such that $\alpha^* x = y$. Then map $\Lambda_2^2 \to sdK$ by sending the vertices to x, y, and x, respectively, and sending the edge between 1 and 2 to $x \otimes (\alpha([0]), [n])$ and the edge between 0 and 2 to $x \otimes ([n], [n])$. There is no 1-simplex with first vertex x and second vertex y (since $y \in K_0$), so there is no way to extend this map to the third edge. Therefore sdK does not have extension from outer horns, and is therefore not a Kan complex.
$\square$.

7 Relationship to categories and the nerve functor

Now let N be the nerve functor, $\mathcal{C}at \xrightarrow{N} \mathcal{S}$. For $\mathcal{C} \in \mathcal{C}at$, $N(\mathcal{C})_n = \{A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \ldots A_{n-1} \xrightarrow{f_n} A_n\}$, where the A_i's and f_i's are objects and morphisms in $\mathcal{C}$. It is known that N is a full embedding, i.e. it is injective on objects and full and faithful on morphism sets. Therefore we can regard $\mathcal{C}at$ as a full subcategory of $\mathcal{S}$. It is also known that a quasicategory K is the nerve of a category if and only if any map $\Lambda_n^k \to F$ extends uniquely to a map from Δ_n.

It is also known that simlicial set K is the nerve of a category iff for any $x_1, \ldots x_n \in K_1$ such that $d_1^* x_{i-1} = d_0^* x_i$ for each i, there exists a unique $y \in K_n$ such that for $[1] \xrightarrow{\epsilon_i} [n]$ the injective map with image $\{i-1, i\}$, $\epsilon_i^* y = x_i$ (for $1 \leq i \leq n$).

Claim 5.1: If K has property A, then sdK is the nerve of a category.

Proof: Take $X_1 = x_1 \otimes (A_1, B_1), \ldots X_q = x_q \otimes (A_q, B_q) \in sdK_1$ such that for each i, $d_1^* X_{i-1} \sim d_0^* X_i$. Assume that each X_i is in minimal form, with $x_i \in [n_i]$ and injective $[m_i] \xrightarrow{\alpha_i} [n_i]$ with image A_i for each i.

Then the condition on the vertices is equivalent to saying that $x_{i-1} \otimes ([n_{i-1}]) \sim \alpha_i^* x_i \otimes ([m_i])$. Since K has property A, $\alpha_i^* x_i$ is nondegenerate for all i, and therefore by uniqueness of minimal form, $x_{i-1} = \alpha_i^* x_i$ and $n_{i-1} = m_i$ for all i.

Define $X = x_q \otimes (\alpha_q \circ \ldots \alpha_1 ([m_1]), \alpha_q \circ \ldots \alpha_2 ([m_2]), \ldots \alpha_q ([m_q]), [n_q])$. Then $\epsilon_i^* X = x_q \otimes \big(\alpha_q \circ \ldots \alpha_i ([m_i]), \alpha_q \circ \ldots \alpha_{i+1} ([m_{i+1}]) \sim \alpha_{i+1}^* \circ \ldots \alpha_q^* x \otimes (\alpha_q ([m_i]), [m_{i+1}])\big) = x_i \otimes (A_i, [n_i])$. Therefore $\epsilon_i^* X \sim X_i$ for all i as desired.

Now suppose there exists another extension $Y = y \otimes (S_0, \ldots S_q) \in sdK_q$. Assume Y is in minimal form; then $y \in K_{n_q}$ and $S_q = [n_q]$. Let $\beta_1, \ldots \beta_q$ be injective maps, with $[n_q]$ the target of β_q, such that the image of $\beta_q \circ \ldots \beta_i$ is S_{i-1} for all i.

First consider the vertices of Y. The ith vertex is $y \otimes (S_i) \sim \beta_{i+1}^* \circ \ldots \beta_q^* y \otimes ([|S_i| - 1]) \sim x_i \otimes ([n_i])$. K has property A, so $\beta_{i+1}^* \circ \ldots \beta_q^* y$ is nondegenerate; by uniqueness of minimal form, $\beta_{i+1}^* \circ \ldots \beta_q^* y = x_i$ and $|S_i| = n_i + 1$ for all i; in particular we have $y = x_q$.

Next consider the edges of Y. For each i, $\epsilon_i^* Y = y \otimes (S_{i-1}, S_i) \sim \beta_{i+1}^* \circ \ldots \beta_q^* y \otimes (\beta_i ([n_{i-1}]), [n_i]) \sim x_i \otimes (A_i, [n_i])$. By uniqueness of minimal form, $\beta_i ([n_i]) = A_i$ for all i. Therefore $\beta_i = \alpha_i$ for all i; therefore $S_{i-1} = \alpha_q \circ \ldots \alpha_i ([m_i])$ for all i; therefore $Y = X$.

$\square$.

Let $\mathcal{S} \xrightarrow{\tau_1} \mathcal{C}at$ be the adjoint to N. It is defined as follows:

For $K \in \mathcal{S}$, the object set of $\tau_1 K$ is K_0. For $x, y \in K_0$, $\tau_1 K (x, y) = \{z_n \star \ldots \star z_1 \mid n \geq 1, z_i \in K_1 \ \forall i,\ d_0^* z_1 = x,\ d_1^* z_n = y,\ d_1^* z_i = d_0^* z_{i+1} \ \forall 1 \leq i < n\} \backslash \approx$, where $\star$ represents formal composition, with $s_0^* x$ labeled as id_x, with composition defined as concatenation of strings (written in the order one writes compositions), and with $\approx$ the equivalence relation generated by $d_1^* w \approx d_0^* w \circ d_2^* w$

for all $w \in K_2$ and $f \circ id_x \approx id_x \circ f \approx f$ for any f in the morphism set. (I will write $\approx$ to distinguish it from the relation $\sim$ used to define sd).

It is known that for any $\mathcal{C} \in \mathcal{C}at$, $\tau_1 \circ N(\mathcal{C}) \cong \mathcal{C}$.

Lemma 5.1.1: For any K, any morphism in $\tau_1 (sdK)$ can be represented by a 1-simplex in sdK.

Proof: By definition, a morphism in $\tau_1 (sdK)$ can be represented by a formal string of 1-simplices, $X_q \star \ldots \star X_q$, with $d_1^* X_i = d_0^* X_{i+1}$ for all i. Write each X_i in minimal form, $x_i \otimes (S_i, [n_i])$. Then $x_1 \otimes ([n_1]) \sim x_2 \otimes (S_2) \sim S_2^* x_2 \otimes ([m_2])$, so by uniqueness of minimal form, $S_2^* x_2 = \sigma^* x_1$ for some surjective σ. Let β be a right inverse for σ. Let α_2 be the injective map to $[n_2]$ with image S_2. Then $x_2 \otimes (\alpha_2 \circ \beta([n_1]), S_2) \sim S_2^* x_2 \otimes (\beta([n_1]), [m_2]) \sim x_1 \otimes ([n_1], [n_1])$, a degenerate simplex. Considering 2-simplex $x_2 \otimes (\alpha_2 \circ \beta([n_1]), S_2, [n_2])$, we see that as morphisms, $x_2 \otimes (S_2, [n_2]) \approx x_2 \otimes (\alpha_2 \circ \beta([n_1]), [n_2])$.

Next, observe that $x_1 \otimes (S_1, [n_1]) \sim x_2 \otimes (\alpha_2 \circ \beta(S_1), \alpha_2 \circ \beta([n_1]))$. Then considering 2-simplex $x_2 \otimes (\alpha_2 \circ \beta(S_1), \alpha_2 \circ \beta([n_1]), [n_2])$, we see that $x_2 \otimes (S_2, [n_2]) \star x_1 \otimes (S_1, [n_1]) \approx x_2 \otimes (\alpha_2 \circ \beta(S_1), [n_2])$.

Therefore we have that our original formal string of length q is equivalent to a formal string of length $q-1$. By induction, it must be equivalent to a string of length 1, and can therefore be represented by a 1-simplex.
$\square$.

Corollary 5.2: $sdK \cong N \circ \tau_1 (sdK)$ iff K has property A.

Proof: By claim 5.1, if K has property A, then sdK is the nerve of a category for some category $\mathcal{C}$. Then $N \circ \tau_1 (sdK) = N(\tau_1 \circ N(\mathcal{C})) \cong N(\mathcal{C}) = sdK$.

Conversely, if K does not have property A, then by Claim 4.1, sdK is not a quasicategory. Therefore it cannot be isomorphic to quasicategory $N \circ \tau_1 (sdK)$.
$\square$.

I will say that a category $\mathcal{C}$ has property A (or B, or C) iff simplicial set $N(\mathcal{C})$ has property A (or B, or C).

Claim 5.3: For $\mathcal{C} \in \mathcal{C}at$, $\mathcal{C}$ has property A iff for any $A \xrightarrow{f} B$, $B \xrightarrow{g} A$ in $\mathcal{C}$ such that $g \circ f = id_A$, $A = B$ and $f = g = id$.

Proof: Suppose $N(\mathcal{C})$ and $g \circ f = id_A$. Then 2-simplex $A \xrightarrow{f} B \xrightarrow{g} A$ has degenerate face $A \xrightarrow{id_A} A$ and must itself be degenerate, and therefore $f = id_A$ or $g = id_A$, therefore $f = g = id_A$.

To prove the converse, suppose property A does not hold for $N(\mathcal{C})$. Then by definition of the nerve of $\mathcal{C}$ we have that for some A, some non-identity maps $f_1, \ldots f_n$, $id_A = f_q \circ \ldots \circ f_1 = (f_n \circ \ldots \circ f_2) \circ f_1$. But f_1 is not an identity map; then this contradicts the second statement as desired.
□.

Corollary 5.3.1: If $\mathcal{C}$ is a category with a left- or right-invertible non-identity arrow, then $N(\mathcal{C})$ is not isomorphic to the subdivision of any simplicial set.

Proof: Claims 0.1 and 4.1.
□.

Claim 5.4: For $\mathcal{C} \in \mathcal{C}at$, $\mathcal{C}$ has property B if and only if for any $A \xrightarrow{f} B$, $B \xrightarrow{g} A$ in $\mathcal{C}$, $A = B$ and $f = g = id$.

Proof: Suppose $N(\mathcal{C})$ has property B and that we have $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$. Then 2-simplex $A \xrightarrow{f} B \xrightarrow{g} A$ does not have 3 distinct vertices, and so it must be degenerate. Therefore $A = B$ (and either $f = id_a$ or $g = id_a$). Furthermore 1-simplices $A \xrightarrow{f} A$ and $A \xrightarrow{g} A$ do not have 2 distinct vertices and so they must both be denegerate; therefore $f = g = id_A$.

Conversely suppose that for any $A \xrightarrow{f} B$, $B \xrightarrow{g} A$ in $\mathcal{C}$, $A = B$ and $f = g = id$. Now take an n-simplex $A_0 \xrightarrow{f_1} A_1 \ldots A_n$ in $N(\mathcal{C})$. If this simplex is non-degenerate, then by the assumptions on $\mathcal{C}$, the A_i's are distinct. Then there are clearly $n + 1$ distinct vertices of this simplex, and so property B holds.
□.

Claim 5.5: For any K, $\tau_1(sdK)$ has property B.

Proof: By Claim 5.4, this is equivalent to showing that for any K, for any $A \xrightarrow{f} B$, $B \xrightarrow{g} A$ in $\tau_1(sdK)$, $A = B$ and $f = g = id$.

Take any such A, B, f, g. By definition of τ_1, A and B are 0-simplices in sdK. Write $A = a \otimes ([m])$ and $B = b \otimes ([n])$, in minimal form.

By Lemma 5.1.1, morphisms in $\tau_1(sdK)$ are equivalence classes of 1-simplices, so we can represent f and g by 1-simplices X and Y in sdK written in minimal form. By uniqueness of minimal form, since $d_1^* X \sim b \otimes ([n])$ and $d_1^* Y \sim a \otimes ([m])$, we can write $X = b \otimes (S, [n])$ and $Y = a \otimes (T, [m])$ for some S, T. Find injective $[s] \xrightarrow{\gamma} [n]$ and $[t] \xrightarrow{\delta} [m]$ with images S and T. Then we have $\gamma^* b \otimes ([s]) \sim a \otimes ([m])$ and $\delta^* a \otimes ([t]) \sim b \otimes ([n])$.

Write $\gamma^* b = \sigma^* c$ for unique surjective σ and nondegenerate $c \in K_r$. Then $\gamma^* b \otimes ([s]) \sim c \otimes ([r]) \sim a \otimes ([m])$, and so $r = m$ and therefore $s \geq m$. Similarly $t \geq n$. But by definition, $s \leq n$ and $t \leq m$. So we have $m \leq s \leq n \leq t \leq m$, and so $m = n = s = t$.

Therefore X and Y are degeneracies, and so $A = B$ and $f = g = id_A$ as desired.

$\square$.

8 Relationship to posets

Claim 6.1: For $\mathcal{C} \in \mathcal{C}at$, $\mathcal{C}$ has properties B and C if and only if $\mathcal{C}$ is a poset.

Proof: Suppose $\mathcal{C}$ is a poset. Then for a q-simplex $A_0 \xrightarrow{f_1} A_1 \ldots A_q$ in $N(\mathcal{C})$, if it is non-degenerate then the A_i's are distinct by definition of a poset, so property B holds. Furthermore, for a set $\{A_0, \ldots A_q\}$ of distinct 0-simplices (objects in $\mathcal{C}$), a q-simplex with this vertex set gives a total ordering on these objects, agreeing with the partial order on $\mathcal{C}$. This total ordering, if it exists, must be unique. Therefore property C holds.

Conversely suppose $\mathcal{C}$ is a small category and $N(\mathcal{C})$ has properties B and C.

Then for any x and y objects in $\mathcal{C}$, there is at most one 1-simplex in $N(\mathcal{C})$ with vertices x and y, and therefore there is at most one morphism $x \to y$ for any $x, y \in \mathcal{C}$, and morphisms $x \to y$ and $y \to x$ imply $x = y$. Therefore $\mathcal{C}$ is a poset. $\square$.

Claim 6.2: Any quasicategory K with properties B and C is the nerve of a poset, i.e. the full subcategory of $\mathcal{S}$ consisting of all quasicategories with properties B and C is isomorphic to $\mathcal{P}$ via N (since N is a full embedding).

Proof: Take such a K. First I will show that K is the nerve of a category. Suppose we have a map $\Lambda_n^k \xrightarrow{f} K$ and two extensions, $\Delta_n \overset{g_1,g_2}{\to} K$. Each extension is uniquely determined by the image of the single non-degenerate n-simplex in Δ_n, call them x_1 and x_2. But the vertex sets of x_1 and x_2 are identical, because f determines the images of the 0-simplices in Δ_n (assuming here that $n \geq 2$ for non-triviality). Then by condition C, $x_1 = x_2$ and therefore the extension is unique. Therefore $K = N(\mathcal{C})$ for some $\mathcal{C}$.

But by Claim 6.1, $\mathcal{C}$ must be a poset.
$\square$.

Summary so far:

For any simplicial set K, if K has property A, sd^2K is the nerve of a poset5. If K does not have property A, sdK is not a quasicategory and does not have property A, and therefore by induction, no iterated Kan subdivision of K yields a poset.

9 Subdivision of categories

Next I will describe Anderson's category subdivision functor $\mathcal{C}at \overset{sd_{\mathcal{C}at}}{\to} \mathcal{C}at$. I will also show that this functor is isomorphic to $\tau_1 \circ sd \circ N$, but first I will rework the results in terms of categories.

For $\mathcal{C} \in \mathcal{C}at$, let $\mathbf{\Delta}\backslash\mathcal{C}$ be the category of chains $\underline{A} = A_0 \xrightarrow{f_1} A_1 \ldots A_q$ over any $q \geq 0$, i.e. maps from objects of $\mathbf{\Delta}$ to $\mathcal{C}$. For $\underline{B} = B_0 \xrightarrow{g_1} B_1 \ldots B_q$, morphisms in $\mathbf{\Delta}\backslash\mathcal{C}\,(\underline{A},\underline{B})$ are maps $[q] \xrightarrow{\gamma} [p]$ such that $\gamma^*\underline{B} = \underline{A}$.

Now localize $\mathbf{\Delta}\backslash\mathcal{C}$ at all surjections; in other words, construct the equivalence relation generated by $\beta_1 \sim \beta_2 : \underline{A} \to \sigma^*\underline{A}$ where σ is a surjection $[q+1] \to [q]$ and β_1, β_2 are two right inverses for σ.

Let $sd_{\mathcal{C}at}\mathcal{C}$ be the full subcategory of this localization of $\mathbf{\Delta}\backslash\mathcal{C}$ with objects all nondegenerate chains in $\mathcal{C}$.

An equivalent description is as follows. Let $\mathcal{N}_\mathcal{C}$ be the category with the objects consisting of chains (of any length $n \geq 0$) of composable non-identity arrows, e.g $\underline{A} = A_0 \xrightarrow{f_1} A_1 \ldots A_q$. For $\underline{B} = B_0 \xrightarrow{g_1} B_1 \ldots B_p$, set $\mathcal{N}_\mathcal{C}\,(\underline{A},\underline{B})$ to equal the set of maps $[q] \overset{\gamma}{\hookrightarrow} [p]$ in $\mathbf{\Delta}$ such that $\gamma^*\underline{B} = \underline{A}$, where $\gamma^*\underline{B}$ is taken by composing adjacent maps or deleting first or last arrows, as in the nerve of a category.

Note that any such γ must be injective, since $\underline{A}$ is nondegenerate.

Now let $sd_{\mathcal{C}at}\mathcal{C}$ be the quotient of $\mathcal{N}_\mathcal{C}$ taken via the equivalence relation on morphisms generated by the following relations. (By quotient of a category, I mean one in which an equivalence relation is imposed on morphism sets, but the object set remains the same).

For any surjective σ and injective α such that $\alpha^*\underline{B} = \sigma^*\underline{A}$, if β_1 and β_2 are right inverses of σ, set $\alpha \circ \beta_1 \sim \alpha \circ \beta_2 : \underline{A} \to \underline{B}$.

Claim 7.2: For any $\mathcal{C}$, $\mathcal{N}_\mathcal{C}$ has property B.

Proof: Take any $\underline{A} = A_0 \xrightarrow{f_1} A_1 \ldots A_q$ and $\underline{B} = B_0 \xrightarrow{g_1} B_1 \ldots B_p$ in $\mathcal{N}_\mathcal{C}$. Suppose $\mathcal{N}_\mathcal{C}\,(\underline{A},\underline{B}) \neq \emptyset$ and $\mathcal{N}_\mathcal{C}\,(\underline{B},\underline{A}) \neq \emptyset$. Then $q = p$. Since the only injective map $[q] \to [q]$ in $\mathbf{\Delta}$ is the identity map, we have that $\underline{A} = \underline{B}$ and $\mathcal{N}_\mathcal{C}\,(\underline{A},\underline{A}) = \{id_{\underline{A}}\}$. By Claim 5.4, this implies that $\mathcal{N}_\mathcal{C}$ has property B.
$\square$.

Corollary 7.3: For any $\mathcal{C}$, $sd_{\mathcal{C}at}\mathcal{C}$ has property B.

Proof: Take any $\underline{A} \xrightarrow{f} \underline{B}$ and $\underline{B} \xrightarrow{g} \underline{A}$ in $sd_{\mathcal{C}at}\mathcal{C}$. Since $sd_{\mathcal{C}at}\mathcal{C}$ is a quotient of

$\mathcal{N}_{\mathcal{C}}$, we can find representatives γ and δ for f and g, respectively. Now we have $\underline{A} \xrightarrow{\gamma} \underline{B}$ and $\underline{B} \xrightarrow{\delta} \underline{A}$ in $\mathcal{N}_{\mathcal{C}}$. By Claim 7.2, therefore $\underline{A} = \underline{B}$, and $\gamma = \delta = id_{\underline{A}}$ in $\mathcal{N}_{\mathcal{C}}$.

Therefore, $f = g = id_{\underline{A}}$ in $sd_{\mathcal{C}at}\mathcal{C}$, as desired.
$\square$.

Claim 7.4: $sd_{\mathcal{C}at}\mathcal{C}$ is a poset iff $\mathcal{C}$ has property B.
Proof: First suppose $\mathcal{C}$ has property B. Since property B implies property A, by Claim 7.1, $sd_{\mathcal{C}at}\mathcal{C} \cong \mathcal{N}_{\mathcal{C}}$. So, it is sufficient to show that $\mathcal{N}_{\mathcal{C}}$ is a poset.

First, we need to show that if in $\mathcal{N}_{\mathcal{C}}$ there is a morphism $\underline{A} \to \underline{B}$ and a morphism $\underline{B} \to \underline{A}$, then $\underline{A} = \underline{B}$. But this true because $\mathcal{N}_{\mathcal{C}}$ has property B for any $\mathcal{C}$.

Second, we need to show that if in $\mathcal{N}_{\mathcal{C}}$ we have $\underline{A} \xrightarrow{\gamma,\delta} \underline{B}$ then $\gamma = \delta$. Write $\underline{A} = A_0 \xrightarrow{f_1} A_1 \dots A_q$ and $\underline{B} = B_0 \xrightarrow{g_1} B_1 \dots B_p$. Since $\underline{A} = \gamma^*\underline{B} = \delta^*\underline{B}$, we know that $A_i = B_{\gamma(i)} = B_{\delta(i)}$ for all i.

Suppose for some $i < j$, $B_i = B_j$. Then we have $B_i \xrightarrow{g_{j-1} \circ \dots \gamma_{i+1}} B_{i+1}$ and $B_{i+1} \xrightarrow{g_j} B_j = B_i$. Since $\mathcal{C}$ has property B, this implies that $g_j = id_{B_j}$. But this is a contradiction, since $\underline{B}$ must be nondegenerate.

Therefore for $i \neq j$, $B_i \neq B_j$. Therefore $\gamma(i) = \delta(i)$ for all i. Therefore $\gamma = \delta$, as desired.

Therefore $\mathcal{N}_{\mathcal{C}} \cong sd_{\mathcal{C}at}\mathcal{C}$ is a poset.

Conversely, suppose $\mathcal{C}$ does not have property B. Then either there exist $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ in $\mathcal{C}$ with $A \neq B$, or $A \xrightarrow{f} A$ such that $f \neq id_A$. In either case, we have $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ such that f and g are both non-identity maps. (In the second case, set $B := A$ and $g := f$).

Now in $sd_{\mathcal{C}at}\mathcal{C}$, take objects $X = A \xrightarrow{f} B \xrightarrow{g} A$ and $Y := A$. Let $\gamma, \delta : [0] \to [2]$ be the maps with images $\{0\}$ and $\{2\}$, respectively. Then $\gamma^*X = \delta^*X = Y$. But no degeneracy of Y is a face of X, therefore $\gamma \not\sim \delta$ in $sd_{\mathcal{C}at}\mathcal{C}(Y, X)$. There-

fore $sd_{\mathcal{C}at}\mathcal{C}$ is not a poset.
$\square$.

Corollary 7.5: For any $\mathcal{C} \in \mathcal{C}at$, $sd^2_{\mathcal{C}at}\mathcal{C}$ is a poset.

(And as above, first subdivision typically does not yield a poset).

Claim 7.6: $sd_{\mathcal{C}at} \cong \tau_1 \circ sd \circ N$.

Proof: I will show that $sd_{\mathcal{C}at}\mathcal{C} \cong \tau_1 \circ sd \circ N(\mathcal{C})$ for all $\mathcal{C}$; naturality will be obvious.

Take any $\mathcal{C} \in \mathcal{C}at$. First, define functor $\mathcal{N}_{\mathcal{C}} \xrightarrow{F} \tau_1 \circ sd \circ N(\mathcal{C})$ as follows.

For $\underline{A} = A_0 \xrightarrow{f_1} A_1 \ldots A_q$ in $\mathcal{N}_{\mathcal{C}}$, let $F(\underline{A}) = \underline{A} \otimes ([q])$ (a simplex in $sd(N(\mathcal{C}))_0$). For morphism $\gamma \in \mathcal{N}_{\mathcal{C}}(\underline{A}, \underline{B})$ (for injective $[q] \xrightarrow{\gamma} [p]$), let $F(\gamma)$ be the morphism represented by $\underline{B} \otimes (\gamma([q]), [p])$ in $sd(N(\mathcal{C}))_1$. Trivially, F agrees with identities and compositions, and is therefore a functor.

Now suppose we have surjective σ and injective α such that $\alpha^*\underline{B} = \sigma^*\underline{A}$, if β_1 and β_2 are right inverses of σ (so $\alpha \circ \beta_1 \sim \alpha \circ \beta_2 : \underline{A} \to \underline{B}$ is a generating relation for the quotient $\mathcal{N}_{\mathcal{C}} \to sd_{\mathcal{C}at}\mathcal{C}$). Then since $\sigma^*\underline{A} \otimes (\beta_i([q]), [r])$ is degenerate for $i = 1, 2$, by the definition of τ_1, we have that $\underline{B} \otimes (\alpha \circ \beta_1([q]), [p]) \approx \underline{B} \otimes (\alpha \circ \beta_2([q]), [p])$ in $\tau_1 \circ sd \circ N(\mathcal{C})$, since each is equivalent to $\underline{B} \otimes (\alpha([r]), [p])$.

Therefore F can be regarded as a well-defined functor $sd_{\mathcal{C}at}\mathcal{C} \to \tau_1 \circ sd \circ N(\mathcal{C})$.

Next, define functor $\tau_1 \circ sd \circ N(\mathcal{C}) \xrightarrow{G} sd_{\mathcal{C}at}\mathcal{C}$ as follows.

An object X in $\tau_1 \circ sd \circ N(\mathcal{C})$ is a 0-simplex in $sd(N(\mathcal{C}))$. Write this 0-simplex in unique minimal form, $\underline{A} \otimes ([q])$, for a nondegenerate $\underline{A} = A_0 \xrightarrow{f_1} A_1 \ldots A_q$. Define $G(X) = \underline{A}$.

Take objects X and Y in $\tau_1 \circ sd \circ N(\mathcal{C})$, associated to 0-simplices $\underline{A} \otimes ([q])$ and $\underline{B} \otimes ([p])$ (in minimal form) in $sd(N(\mathcal{C}))$. A morphism $X \xrightarrow{f} Y$ in $\tau_1 \circ sd \circ N(\mathcal{C})$ is a formal string of 1-simplices in $sd(N(\mathcal{C}))$ with conditions on the vertices as defined above. In minimal form, write this sequence as $\underline{B}_r \otimes (S_r, [n_r]) \star \ldots \star \underline{B}_1 \otimes (S_1, [n_1])$. By uniqueness of minimal form, we see that $\underline{B}_r = \underline{B}$ and $n_r = p$, that $S_1^* \underline{B}_1 = \sigma^* \underline{A}$ for some surjective σ, and that for each $i < r$, for some surjective σ_i, $s_i^* \underline{B}_i = S_{i+1}^* \underline{B}_{i+1}$. For each i choose some right inverse β_i for σ_i, and choose right inverse β for σ. Then $\underline{A} = \beta^* S_1^* \underline{B}_1 = \beta^* S_1^* \beta_1^* S_2^* \underline{B}_2 = \ldots = \beta^* \mathcal{S}_1^* \beta_1^* \mathcal{S}_2^* \beta_2^* \ldots \beta_{r-1}^* S_r^* \underline{B}$. Then define $G(f) = \alpha_r \circ \beta_{r-1} \circ \ldots \beta_2 \circ \alpha_2 \circ \beta_1 \circ \alpha_1 \circ \beta$, where α_i is the injective map to $[n_i]$ with image S_i.

We must check that G is well-defined on morphisms.

First, note that by the equivalence relation on morphisms of $sd_{\mathcal{C}at}\mathcal{C}$, this definition is independent of choices of β or the β_i's.

Second, we must check that G agrees with composition, when defined as formal concatenation of strings. This is true because $\underline{B}_r \otimes (S_r, [n_r]) \star \ldots \star \underline{B}_1 \otimes (S_1, [n_1]) \ \star\ \underline{B}'_s \otimes (S'_s, [n'_s]) \star \ldots \star \underline{B}'_1 \otimes (S'_1, [n'_1])$ would be mapped to $\alpha_r \circ \beta_{r-1} \circ \ldots \beta_2 \circ \alpha_2 \circ \beta_1 \circ \alpha_1 \circ \beta \ \circ\ \alpha'_s \circ \beta'_{s-1} \circ \ldots \beta'_2 \circ \alpha'_2 \circ \beta'_1 \circ \alpha'_1 \circ \beta'$ (where all the maps here are defined analagously).

Finally, we must also check that G is well-defined with respect to the equivalence relation on morphisms given by the definition of τ_1. There are two types of generating relations. First, for a 2-simplex $\underline{B} \otimes (S, T, [n])$, we set $\underline{B} \otimes (T, [n]) \star \underline{B} \otimes (S, T) \approx \underline{B} \otimes (S, [n])$. But by definition of G, both sides map to morphism α which is the injective map to $[n]$ with image S. Second, if $\underline{B} \otimes (S, [n])$ is degenerate then it must map to the identity. This is true trivially since in minimal form, it would be written as $\underline{B}'([n'], [n'])$.

Therefore G is well-defined.

Regarding F as as a functor with source $sd_{\mathcal{C}at}\mathcal{C}$, it is clear that $F \circ G$ and $G \circ F$ are each the identity on objects, and by inspection we can see that this is true on morphisms as well. Therefore the categories are isomorphic.
$\square$.

Corollary 7.7: $sd(N(\mathcal{C})) \cong N(sd_{\mathcal{C}at}\mathcal{C})$ **iff** $\mathcal{C}$ **has property A.**

Proof: If $\mathcal{C}$ has property A, then by Corollary 5.2, $N \circ \tau_1 (sd(N(\mathcal{C}))) \cong$

$sd\left(N\left(\mathcal{C}\right)\right)$. Therefore, using Claim 7.6, $sd_{\mathcal{C}at}\mathcal{C} \cong \tau_1\left(sd\left(N\left(\mathcal{C}\right)\right)\right) \cong \tau_1 \circ N \circ \tau_1\left(sd\left(N\left(\mathcal{C}\right)\right)\right) \cong \tau_1 \circ sd\left(N\left(\mathcal{C}\right)\right)$.

And therefore $N\left(sd_{\mathcal{C}at}\mathcal{C}\right) \cong N \circ \tau_1 \circ sd\left(N\left(\mathcal{C}\right)\right) \cong sd\left(N\left(\mathcal{C}\right)\right)$.

To prove the converse, observe that if $\mathcal{C}$ does not have property A, then $sd\left(N\left(\mathcal{C}\right)\right)$ does not have property A, by Claim 0.1. However $sd_{\mathcal{C}at}\mathcal{C}$ has property A by Claims 1.2 and 7.3, and therefore the congruence cannot hold.
$\square$.

10 Non-examples

Claim 8.0.1: In $\mathcal{C}at$, not every $\mathcal{C}$ has property A.
Proof: Most common categories do not have property A, for example Abelian groups or topological spaces.
$\square$.

Corollary 8.0.1.1: Not every simplicial set has property A.

Claim 8.0.2: In $\mathcal{C}at$, property A does not imply property B.
Proof: Let $\mathcal{C}$ be the subcategory of Abelian groups with single object $A = \mathbf{Z} \oplus \mathbf{Z}$ and with maps id_A and p_1, projection onto the first coordinate. (Composition law: $p_1^2 = p_1$). Then $\mathcal{C}$ does not have arrows f, g such that $f \circ g$ is an identity map, except for $f = g = id_A$, so property A holds. However property B clearly does not hold.
$\square$.

Corollary 8.0.2.1: For a nerve of a category or a quasicategory or a simplicial set K, property A does not imply property B.

Claim 8.1: In $\mathcal{C}at$, $\mathcal{C}$ has property B does not imply that $\mathcal{C}$ is a poset.
Proof: Let $\mathcal{C}$ be the category with two objects A and B, and with morphisms

id_A, id_B, $A \xrightarrow{f} B$, and $A \xrightarrow{g} B$.
$\square$.

Corollary 8.1.1: For a nerve of a category or a quasicategory or a simplicial set K, neither property A nor property B imply property C.

Claim 8.2: In $\mathcal{C}at$, $\mathcal{C}$ has property C does not imply that $\mathcal{C}$ has property A.

Proof: Let $\mathcal{C}$ be the cyclic group of order 2, with object A and morphisms id_A, f. Then in $N(\mathcal{C})$, for any q there is exactly one nondegenerate q-simplex (namely, q arrows, each labeled f) and so property C holds trivially. However nondegenerate 2-simplex $\underline{A} = A \xrightarrow{f} A \xrightarrow{f} A$ has degenerate face $d_1^* \underline{A} = A \xrightarrow{id_A} A$.
$\square$.

Corollary 8.2.1: For a nerve of a category or a quasicategory or a simplicial set K, property C does not imply property A or property B.

Specifically, for a category, property C does not imply that $\mathcal{C}$ is a poset.

Claim 8.3: In $\mathcal{S}$, properties B and C together do not imply that K is a quasicategory.

Proof: Take the simplicial set $K = \Lambda_2^1$. K is a simplicial set corresponding to a simplicial complex, therefore by Claim 3.1 K has properties B and C. However, the identity map $\Lambda_2^1 \to K$ does not extend to a map $\Delta_2 \to K$, therefore K is not a quasicategory.
$\square$.

Corollary 8.3.1: Properties A or B or C do not imply that K is a poset or a category or a quasicategory.

Claim 8.4: A quasicategory K with property B is not necessarily a category.

Proof: Start with a category that has property B. Choose a nondegenerate 1-simplex in $N(\mathcal{C})$ and construct the corresponding quasicategory that is not a category, as in my notes on this topic. The resulting quasicategory will have property B as well by definition.

$\square$.

Claim 8.5: A quasicategory K with property C is not necessarily a category.

Proof: Start with $\mathcal{C}$ defined as the cyclic group of order 2. Choose nondegenerate 1-simplex $A \xrightarrow{f} A$ in $N(\mathcal{C})$ and construct the corresponding quasicategory that is not a category, as in my notes on this topic. In $N(\mathcal{C})$, there is only one 0-simplex, and so there is only one 0-simplex in the new quasicategory as well. Then property C holds trivially.

$\square$.

Corollary 8.5.1: Properties A or B, or C without B, for a quasicategory or simplicial set K do not imply that K is the nerve of a category.

(However as in Claim 6.2, a quasicategory with properties B and C is a poset, while a simplicial set with properties B and C may still not be a quasicategory as in Claim 8.3).

Claim 8.6: Given a poset $\mathcal{C}$, $N(\mathcal{C})$ is not necessarily isomorphic to the subdivision of some simplicial set.

Proof: Let $\mathcal{C}$ be the integers with the usual ordering. Suppose $N(\mathcal{C}) \cong sdK$ for simplicial set K. Find 0-simplex X in sdK corresponding to the integer 0. Write X in minimal form, $x \otimes ([n])$. Then for any q-simplex Y with qth vertex X, we can write Y in minimal form as $x \otimes (S_0, \ldots S_q = [n])$. Therefore if Y is nondegenerate, $q \leq n$. However, in $N(\mathcal{C})$, we have non-degenerate $(n+1)$-simplex $(-n-1) \to (-n) \to (-n+1) \to \ldots \to (-1) \to (0)$. This contradicts the isomorphism.

$\square$.

Corollary 8.6.1: Properties A or B or C for a category $\mathcal{C}$ do not imply that $N(\mathcal{C})$ is isomorphic to the subdivision of some simplicial set.

(And as in Corollary 5.3.1, if $\mathcal{C}$ does not have property A then $N(\mathcal{C})$ is never isomorphic to the subdivision of any simplicial set).

Claim 8.7: For posets P and Q, $sd_{\mathcal{C}at}P \cong sd_{\mathcal{C}at}Q$ does not imply that $P \cong Q$, or even that the two are isomorphic up to a reversal of the order on Q.

Proof: I will describe a construction for P and Q such that $P \ncong Q$ but $sd_{\mathcal{C}at}P \cong sd_{\mathcal{C}at}Q$. Since N is an embedding it is equivalent to say that $sd(N(P)) \cong sd(N(Q))$.

Take any two posets A and B. Let $P = A \sqcup B$ with the additional relations that $a \le b \; \forall \; a \in A, \; b \in B$. Let $Q = A \sqcup B$ with the additional relations that $a \ge b \; \forall \; a \in A, \; b \in B$. Note that for general A and B, P and Q are not isomorphic, even up to a reversal of the order on Q.

Define $sd(N(P)) \xrightarrow{f} sd(N(Q))$. Note that any element in $N(P)$ can be written as $a_0 \to a_1 \to \dots a_r \to b_0 \to b_1 \to \dots b_s$ for $a_i \in A$, $b_i \in B$.

Take an element $X \in sd(N(P))$, $X = x \otimes (S_0, \dots S_q)$ for $x = a_0 \to a_1 \to \dots a_r \to b_0 \to b_1 \to \dots b_s$ in $N(P)_{r+s+1}$ and $S_i \subset [r+s+1]$. Define $f(X) = Y := y \otimes (p_s^r(S_0), \dots p_s^r(S_q))$, where $y = b_0 \to b_1 \to \dots b_s \to a_0 \to a_1 \to \dots a_r$ and p_s^r is a permutation of $[r+s+1]$ defined by $x \mapsto x + s + 1 \pmod{r+s+2}$, i.e. swaps the first $(r+1)$ elements with the last $(s+1)$ elements.

It is a trivial exercise to show that this map is bijective, well-defined over the equivalence relation in sd, and agrees with face and degeneracy operators.

Therefore $sd_{\mathcal{C}at}P \cong sd_{\mathcal{C}at}Q$ as desired.

Note that an analagous construction could be used for the nerve of any category, and so assuming that a simplicial sets K, L do NOT have property A (or B, or C) will still not guarantee that $sdK \cong sdL \;\Rightarrow\; K \cong L$.
$\square$.

Corollary 8.8.1: For simplicial sets K and L, $sdK \cong sdL$ does not imply $K \cong L$, under any conditions of properties A or B or C.

(But see Claim 0.0.3 to see what can be concluded).

11 Summary.

For any simplicial set, property A implies property B, but no other implications hold between properties A, B, and C.

If K does not have property A, then $sd^n K$ does not have property A or B or C for any $n > 0$ (even if K itself has property C).

Under any of these properties, being a simplicial set does not imply being a quasicategory, and being a quasicategory does not imply being the nerve of a category, except that quasicategories with properties B and C are nerves of categories.

The category of quasicategories with properties B and C (or equivalently, the category of (small) categories with properties B and C) is isomorphic to the category of posets.

The category of simplicial sets with properties B and C is isomorphic to the category of simplicial complexes.

Nerve and subdivision commute (giving composite functors $\mathcal{C}at \to \mathcal{S}$) iff the category in question has property A.

Second subdivision of a simplicial set K with property A is a poset.

Second 'categorical subdivision' ($sd_{\mathcal{C}at}$) of ANY category $\mathcal{C}$ is a poset.