AN INTRODUCTION TO SIMPLICIAL SETS

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Simplicial sets, and more generally simplicial objects in a given category, are central to modern mathematics. While I am not a mathematical historian, I thought I would describe in conceptual outline how naturally simplicial sets arise from the classical study of simplicial complexes. I suspect that something like this recapitulates the historical development.

We have described simplicial complexes in several different forms: abstract simplicial complexes, ordered simplicial complexes, geometric simplicial complexes, ordered geometric simplicial complexes and realizations of geometric simplicial complexes. It is possible to go directly from abstract simplicial complexes to realizations without passing through geometric simplicial complexes, although the construction is perhaps less intuitive. I may describe it later. (I did briefly in class.)

An abstract simplicial complex is equivalent to a geometric simplicial complex, and neither of these notions involves anything about ordering the vertices. If one has a simplicial complex of either type, one can choose a partial ordering of the vertices that restricts to a linear ordering of the vertices of each simplex, and this gives the notion of an ordered simplicial complex. This can be done most simply, but not most generally, just by choosing a total ordering of the set of all vertices and restricting that ordering to simplices.

We have seen in studying products of simplicial complexes that geometric realization behaves especially nicely only in the ordered setting. Both the category $\mathscr{S}\mathscr{C}$ of simplicial complexes and the category $\mathscr{O}\mathscr{S}\mathscr{C}$ of ordered simplicial complexes have categorical products. Geometric realization preserves products when defined on $\mathscr{O}\mathscr{S}\mathscr{C}$, but it does not preserve products when defined on $\mathscr{S}\mathscr{C}$. The functor $\mathscr{K}$ is best viewed as a functor from the category $\mathscr{P}$ of partially ordered sets to the category $\mathscr{O}\mathscr{S}\mathscr{C}$ rather than just to the category $\mathscr{S}\mathscr{C}$. The functor $\mathscr{X}$, on the other hand, starts in $\mathscr{S}\mathscr{C}$ and lands in $\mathscr{P}$, which can be identified with the category of Alexandroff T_0-spaces. The composite $\mathscr{K}\mathscr{X}$ is the barycentric subdivision functor $Sd\colon \mathscr{S}\mathscr{C} \longrightarrow \mathscr{O}\mathscr{S}\mathscr{C}$, and since the geometric realization functor gives a space $|SdK|$ that can be identified with $|K|$ there is no loss of topological generality working in $\mathscr{O}\mathscr{S}\mathscr{C}$ instead of $\mathscr{S}\mathscr{C}$.

The most important motivation for working with ordered rather than unordered simplicial complexes is that the ordering leads to the definition of an associated chain complex and thus to a quick definition of homology. I'll explain that in the talks and add it to the notes if I have time.

In the early literature of algebraic topology, a topological space X is called a polyhedron if it is homeomorphic to $|K|$ for a (given) simplicial complex K. Such a homeomorphism $|K| \longrightarrow X$ is called a triangulation of X, and X is said to be triangulable if it admits a triangulation. Then we can define the homology of X to be the homology of K. This is a quick definition, and useful where it applies, but

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it raises many questions and is quite unsatisfactory conceptually. Not every space is triangulable, and triangulable spaces can admit many different triangulations. It is far from obvious that the homology is independent of the choice of triangulation.

Simplicial sets abstract the notion of ordered simplicial complexes, retaining enough of the combinatorial structure that homology can be defined with equal ease. The generalization allow myriads of examples that do not come from simplicial complexes. The original motivating example gives a functor from topological spaces to simplicial sets. Composing with the functor from simplicial sets to homology groups gives the quickest way of defining the homology groups of a space and leads to the proof that these groups depend only on the weak homotopy type of the space, not on any triangulation, and to the proofs that different triangulations, when they exist, give canonically isomorphic homology groups.

Perhaps the quickest and most intuitive way to motivate the definition of simplicial sets is to start from structure clearly visible in the case of ordered simplicial complexes. Let X denote the partially ordered set $V(K)$ of vertices of an ordered simplicial complex K. The reader might prefer to start with an ordered simplicial complex of the form $\mathscr{K}(X)$, where X is a poset. The reader may also want to insist that X is finite, but that is not necessary to the construction, and we later want to allow infinite sets.

An n-simplex σ of K is a totally ordered $n+1$-tuple of elements of X. Write such a tuple as $(x_0, \cdots, x_n)$. When studying products, we saw that it can become essential to consider tuples $(x_0, \cdots, x_n)$, where $x_0 \leq x_1 \leq \cdots x_n$. Of course, $(x_0, \cdots, x_n)$ is no longer a simplex, but one can obtain a simplex from it by deleting repeated entries. When there are repeated entries, we think of $(x_0, \cdots, x_n)$ as a "degenerate" n-simplex. Let K_n denote the set of such generalized n-simplices, degenerate or not. For $0 \leq i \leq n$, define functions

$$d_i \colon K_n \longrightarrow K_{n-1} \quad \text{and} \quad s_i \colon K_n \longrightarrow K_{n+1},$$

called face and degeneracy operators, by

$$d_i(x_0, \cdots, x_n) = (x_0, \cdots x_{i-1}, x_{i+1}, \cdots, x_n)$$

and

$$s_i(x_0, \cdots, x_n) = (x_0, \cdots x_i, x_i, \cdots, x_n).$$

Of course, the d_i and s_i just defined also depend on n, but it is standard not to indicate that in the notation. In words, d_i deletes the i^{th} entry and s_i repeats the i^{th} entry. If $i < j$ and we first delete the j^{th} entry and then the i^{th} entry, we get the same thing as if we first delete the i^{th} entry and then delete the (new) $(j-1)^{\text{st}}$ entry. Similarly, elementary inspections give commutation relations between the d_i and s_j and between the s_i. Here is a list of all such relations:

$$d_i \circ d_j = d_{j-1} \circ d_i \quad \text{if} \quad i < j$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if} \quad i < j \\ \text{id} & \text{if} \quad i = j \quad \text{or} \quad i = j+1 \\ s_j \circ d_{i-1} & \text{if} \quad i > j+1. \end{cases}$$

$$s_i \circ s_j = s_{j+1} \circ s_i \quad \text{if} \quad i \leq j.$$

The reader can easily check that these identities really do follow immediately from the definition of the K_n, d_i, and s_i above.

The K_n are defined in terms of the partially ordered vertex set $V(K)$ of K, but there are many examples of precisely similar structure that arise differently. This motivates our first definition of simplicial sets.

Definition 0.1. A simplicial set K_* is a sequence of sets K_n, $n \geq 0$, and functions $d_i \colon K_n \longrightarrow K_{n-1}$ and $s_i \colon K_n \longrightarrow K_{n+1}$ for $0 \leq i \leq n$ that satisfy the displayed identities. A map $f_* \colon K_* \longrightarrow L_*$ of simplicial sets is a sequence of functions $f_n \colon K_n \longrightarrow L_n$ such that $f_{n-1} \circ d_i = d_i \circ f_n$ and $f_{n+1} \circ s_i = s_i \circ f_n$. With these objects and morphisms, we have the category $s\mathscr{S}et$ of simplicial sets.

Now our motivating example can be recapitulated in the following statement.

Proposition 0.2. *There is a canonical functor $\mathscr{O}\mathscr{S}\mathscr{C} \longrightarrow s\mathscr{S}et$ from the category of ordered simplicial complexes to the category of simplicial sets. It assigns to an ordered simplicial set K the simplicial set K_* given by the sequence of sets K_n and functions d_i and s_i defined above. It assigns to a map $f \colon K \longrightarrow L$ of ordered simplicial complexes the map $f_* \colon K_* \longrightarrow L_*$ induced by its map of vertex sets:*

$$f_n(x_0, \cdots, x_n) = (f(x_0), \cdots f(x_n)).$$

The identities listed above are hard to remember and do not appear to be very conceptual. The definition admits a conceptual reformulation that may or may not make things clearer, depending on personal taste, but definitely allows many arguments and constructions to be described more clearly and conceptually than would be possible without it. We define the category Δ of finite ordered sets.

Definition 0.3. The objects of Δ are the finite ordered sets $[n]$ with $n+1$ elements $0 < 1 < \cdots < n$. Its morphisms are the monotonic functions $\mu \colon [m] \leq [n]$. This means that $i < j$ implies $\mu(i) \leq \mu(j)$. Define particular monotonic functions

$$\delta_i \colon [n-1] \longrightarrow [n] \quad \text{and} \quad \sigma_i \colon [n+1] \longrightarrow [n]$$

for $0 \leq i \leq n$ by

$$\delta_i(j) = j \ \text{ if } \ j < i \quad \text{and} \quad \delta_i(j) = j+1 \ \text{ if } \ j \geq i$$

and

$$\sigma_i(j) = j \ \text{ if } \ j \leq i \quad \text{and} \quad \sigma_i(j) = j-1 \ \text{ if } \ j > i.$$

In words, δ_i skips i and σ_i repeats i.

There are identities for composing the δ_i and σ_i that are "dual" to those for composing the d_i and s_i that appear in the definition of a simplicial set. Precisely, the duality amounts to reversing the direction of arrows. The following pair of commutative diagrams should make clear how to interpret this, where $i < j$.

$$\begin{tikzcd} K_n \arrow[r, "d_j"] \arrow[d, "d_i"'] & K_{n-1} \arrow[d, "d_i"] \\ K_{n-1} \arrow[r, "d_{j-1}"'] & K_{n-2} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {[n]} & {[n-1]} \arrow[l, "\delta_j"'] \\ {[n-1]} \arrow[u, "\delta_i"] & {[n-2]} \arrow[l, "\delta_{j-1}"] \arrow[u, "\delta_i"'] \end{tikzcd}$$

A moment's reflection should convince the reader that every monotonic function $\mu \colon [m] \longrightarrow [n]$ can be written as a composite of monotonic functions δ_i and σ_j for varying i and j. That is, μ can be obtained by omitting some of the i's and repeating some of the j's. Just as a group can be defined by specifying a set of generators and relations, so a category can often be specified by a set of generating morphisms and relations between their composites. The category Δ is generated by the δ_i and σ_i subject to our "dual" relations. This leads to the proof of the following reformulation of the notion of a simplicial set. Recall that a contravariant functor F assigns a morphism $FY \longrightarrow FX$ of the target category to each morphism $X \longrightarrow Y$ of the source category.

Proposition 0.4. *A simplicial set is a contravariant functor $K_*\colon \Delta \longrightarrow \mathscr{S}et$. A map $f_*\colon K_* \longrightarrow L_*$ of simplicial sets is a natural transformation of functors.*

Proof. The correspondence is given by viewing the functions d_i and s_i that define a simplicial set as the morphisms of sets induced by the morphisms δ_i and σ_i of the corresponding functor $\Delta \longrightarrow \mathscr{S}et$. It is convenient to write $\mu^*\colon K_n \longrightarrow K_m$ for the function induced by contravariance from a morphism $\mu\colon [m] \longrightarrow [n]$, and then $d_i = \delta_i^*$ and $s_i = \sigma_i^*$. For a map f_*, the corresponding natural transformation is given on the object $[n]$ by the function f_n. $\square$

While we do not want to emphasize abstraction in the first instance, we nevertheless cannot resist the temptation to generalize the definition of simplicial sets to simplicial objects in a perfectly arbitrary category. The generalization has a huge number of applications throughout mathematics, and we shall use it when defining homology.

Definition 0.5. A simplicial object in a category $\mathscr{C}$ is a contravariant functor $K_*\colon \Delta \longrightarrow \mathscr{C}$. A map $f_*\colon K_* \longrightarrow L_*$ of simplicial objects in $\mathscr{C}$ is a natural tranformation f_*; it is given by morphisms $f_n\colon K_n \longrightarrow L_n$ in $\mathscr{C}$. We have the category $s\mathscr{C}$ of simplicial objects in $\mathscr{C}$. By composition of functors and natural transformations, any functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ induces a functor $sF\colon s\mathscr{C} \longrightarrow s\mathscr{D}$. By duality, a *covariant* functor $\Delta \longrightarrow \mathscr{C}$ is called a cosimplicial object in $\mathscr{C}$.

We next explain a general conceptual way to construct functors from a given category, $\mathscr{V}$ say, to the category $s\mathscr{S}et$ of simplicial sets. For that purpose suppose that we are given a comsimplicial object in $\mathscr{V}$, that is a covariant functor $\Delta[*]\colon \Delta \longrightarrow \mathscr{V}$. We write it on objects as $[n] \mapsto \Delta[n]$, but we agree to write μ rather than $\Delta[\mu]$ for the map $\Delta[m] \longrightarrow \Delta[n]$ in $\mathscr{V}$ obtained by applying our functor to morphism μ in Δ. For each object V of $\mathscr{V}$ we obtain a *contravariant* functor, denoted $S_*V\Delta \longrightarrow \mathscr{S}et$, by letting S_nV be the set $\mathscr{V}(\Delta[n], V)$ of morphisms $\Delta[n] \longrightarrow V$ in the category $\mathscr{V}$. The faces and degeneracies are induced by precomposition with the maps

$$\delta_i\colon \Delta[n-1] \longrightarrow \Delta[n] \quad \text{and} \quad \sigma_i\colon \Delta[n+1] \longrightarrow \Delta[n]$$

obtained by applying the functor $\Delta[*]$ to the generating morphisms δ_i and σ_i of Δ. That is, for a morphism $\nu\colon \Delta[n] \longrightarrow V$ in $\mathscr{V}$,

$$d_i(\nu) = \nu \circ \delta_i \quad \text{and} \quad s_i(\nu) = \nu \circ \sigma_i.$$

Before turning to the motivating examples, in which $\mathscr{V}$ is the category $\mathscr{U}$ of topological spaces or the category $\mathscr{C}at$ of small categories, we apply this construction to the case $\mathscr{V} = s\mathscr{S}et$.

Definition 0.6. We define a covariant functor $\Delta[*]$ from Δ to the category $s\mathscr{S}et$ of simplicial sets. On objects, the functor sends $[n]$ to the standard simplicial n-simplex $\Delta[n]$, which is the contravariant functor $\Delta \longrightarrow \mathscr{S}et$ represented by $[n]$. This means that the set $\Delta[n]_q$ of q-simplices is the set of all morphisms $\phi\colon [q] \longrightarrow [n]$ in Δ. For a morphism $\nu\colon [p] \longrightarrow [q]$ in Δ, $\nu^*\colon \Delta[n]_q \longrightarrow \Delta[n]_p$ is the function specified by $\nu^*(\phi) = \phi \circ \nu \colon [p] \longrightarrow [q]$. For morphisms $\mu \colon [m] \longrightarrow [n]$ in Δ, $\mu \colon \Delta[m]_q \longrightarrow \Delta[n]_q$ is defined by $\mu(\psi) = \mu \circ \psi \colon [q] \longrightarrow [m] \longrightarrow [n]$. Thus the simplicial set $\Delta[n]$ is defined using pre-composition with morphisms of Δ, and then the covariant functoriality is defined using post-composition with morphisms of Δ. The object $\Delta[*]$ is a cosimplicial simplicial set.

The following result is an example of a general categorical observation called the Yoneda lemma. Let $i_n \in \Delta[n]_n$ be the identity map $\mathrm{id} \colon [n] \longrightarrow [n]$.

Proposition 0.7. *Let K_* be a simplicial set. For each $x \in K_n$, there is a unique map of simplicial sets $Y(x) \colon \Delta[n] \longrightarrow K$ such that $Y(x)(i_n) = x$. Therefore K can be identified with the simplicial set whose n-simplices are the maps $\Delta[n] \longrightarrow K$.*

Proof. The map $Y(x)$ is a natural transformation $Y(x)$ from the contravariant functor $\Delta[n]$ to the contravariant functor Y from Δ to $\mathscr{S}et$. Since a q-simplex $\phi \colon [q] \longrightarrow [n]$ is $\phi^*(i_n)$, we can and must specify $Y(x)$ at the object $[q] \in \Delta$ by the function $\Delta[n]_q \longrightarrow Y_q$ that sends ϕ to the q-simplex $\phi^*(x)$. □

We turn to the historical motivating example $\mathscr{V} = \mathscr{U}$ by constructing the total singular complex S_*X of a topological space X. We need a covariant functor $\Delta[*] \colon \Delta \longrightarrow \mathscr{U}$, and that is given by the standard topological simplices $\Delta[n]$.

Definition 0.8. Let $\Delta[n]$ be the subspace

$$\{(t_0, \cdots, t_n) | 0 \le t_i \le 1 \quad \text{and} \quad \Sigma_i t_i = 1\}$$

of $\mathbb{R}^{n+1}$. It is a topological n-simplex. Define

$$\delta_i \colon \Delta[n-1] \longrightarrow \Delta[n] \quad \text{and} \quad \sigma_i \colon \Delta[n+1] \longrightarrow \Delta[n]$$

by

$$\delta_i(t_0, \cdots, t_{n-1}) = (t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1})$$

and

$$\sigma_i(t_0, \cdots, t_{n+1}) = (t_0, \cdots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \cdots, t_{n-1}).$$

Then the de_i and σ_i satisfy the commutation relations required to specify a covariant functor $\Delta[*] \longrightarrow \mathscr{U}$.

Definition 0.9. The total singular complex S_*X of a space X is the simplicial set such that S_nX is the set of continuous maps $\Delta[n] \longrightarrow X$, with faces d_i and degeneracies s_i induced by precomposition with δ_i and σ_i. By composition of continuous maps, a map $f \colon X \longrightarrow Y$ induces a map $f_* \colon S_*X \longrightarrow S_*Y$, hence we have the total singular complex functor from topological spaces to simplicial sets.

We shall return to this example after giving an analogue that may seem astonishing at first sight. Although it has become a standard and commonplace construction, its importance and utility were only gradually recognized. Recall that a poset can be viewed as a category with at most one arrow between any pair of objects: either $x \le y$, and then there is a unique arrow $x \longrightarrow y$, or $x \not\le y$, and then there is no arrow $x \longrightarrow y$. Composition is defined in the only possible way. By definition $[n]$ is an ordered set, hence of course it is a partially ordered set. We can view it as a category, and then the monotonic functions $\mu \colon [m] \longrightarrow [n]$ are precisely the functors $[m] \longrightarrow [n]$: monotonicity says that if there is an arrow $i \to j$, then there is an arrow $i \le j$, which must be the value of the functor μ on that arrow.

Definition 0.10. Let $\mathscr{C}at$ denote the category whose objects are small categories (categories with a set of objects) and whose morphisms are the functors between them. Define a covariant functor $\Delta[*]\colon \Delta \longrightarrow \mathscr{C}at$ by sending the ordered set $[n]$ to the corresponding category $[n]$ and sending a morphism $\mu\colon [m] \longrightarrow [n]$ to the corresponding functor.

Definition 0.11. Let $\mathscr{C}$ be a small category. Define the nerve of $\mathscr{C}$, denoted $N_*\mathscr{C}$, to be the simplicial set such that $N_n\mathscr{C}$ is the set of covariant functors $\phi\colon [n] \longrightarrow \mathscr{C}$ and such that the function $\mu^*\colon N_n\mathscr{C} \longrightarrow N_m\mathscr{C}$ induced by $\mu\colon [m] \longrightarrow [n]$ is given by $\mu^*(\phi) = \phi \circ \mu$, where μ is viewed as a functor. A functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ induces a map $F_*\colon N_*\mathscr{C} \longrightarrow N_*\mathscr{D}$ of simplicial sets by composition of functors, hence we have the nerve functor N_* from small categories to simplicial sets.

The definition can easily be unravelled. The category $[0]$ has one object and its identity morphism, hence a functor $\phi\colon [0] \longrightarrow \mathscr{C}$ is just a choice of an object of $\mathscr{C}$. That is, if we write $\mathscr{O}\mathscr{C}$ for the set of objects of $\mathscr{C}$, then $N_0\mathscr{C} = \mathscr{O}\mathscr{C}$. For $n \geq 1$, a functor $\phi\colon [n] \longrightarrow \mathscr{C}$ is a choice of n composable morphisms

$$C_0 \xrightarrow{f_1} C_1 \longrightarrow \cdots \longrightarrow C_{n-1} \xrightarrow{f_n} C_n.$$

The O^{th} and n^{th} faces send this n-simplex to the $(n-1)$-simplex obtained by deleting f_1 or f_n; when $n = 1$ this is to be interpreted as giving the object C_1 or C_0. For $0 < i < n$, the i^{th} face composes $f_{i+1} with f_i$. The i^{th} degeneracy operation inserts the identity morphism of C_i. This looks nothing like our original example of the simplicial set associated to an ordered simplicial complex!