LOW DIMENSIONAL EXAMPLES

NOTES FOR REU BY J.P. MAY

Drawing posets, and thinking about them, leads to lots of eliminations from the list of finite T_0-spaces that might not be contractible or weakly contractible. We see the difference between contractible and weakly contractible when we reach five point spaces.

Lemma 0.1. *If X has a unique maximal element or a unique minimal element, then X is contractible.*

Proof. If X has a unique maximal element, then the only open set containing that point is X and therefore X is contractible. Replacing open by closed gives the conclusion when X has a unique minimal point. □

In the following, we regard homeomorphic spaces as the same.

Proposition 0.2. *The only connected minimal finite space X with at most four points that is not contractible is the four point circle $\mathbb{S}S^0$.*

Proof. Since X is not contractible, it has at least two minimal points and at least two maximal points, so it must have exactly four points. Unless both minimal points are less than both maximal points, X contains an upbeat or downbeat point and is thus not minimal. □

Proposition 0.3. *There are two connected minimal five point spaces X that are not contractible. One is the opposite of the other and both are weakly contractible.*

Proof. Since X is not contractible, it must have at least two minimal and two maximal points. If it has exactly two minimal and two maximal points, then it has one intermediate point y. But then a point connected to y must be either upbeat or downbeat, contradicting minimality. By antisymmetry, we can assume that there are exactly two minimal and three maximal elements. By the minimality of X again, each maximal element must be connected to both minimal elements. □

Remark 0.4. The space $|\mathscr{K}(X)|$ associated to either X is a graph, and it looks like a W or an M.

Proposition 0.5. *There are eight connected minimal six point spaces X, and none of them are weakly contractible. One is the six point two sphere $\mathbb{S}^2S^0$ and the five are graphs X such that $|\mathscr{K}(X)|$ is homotopy equivalent to a wedge of one, two, three, or four circles.*

Proof. We must have at least two minimal and at least two maximal points. If we have just one intermediate point y, any point greater or less than it is upbeat or downbeat. If we have two intermediate points, they cannot be comparable without again contradicting minimality, and if they are incomparable we arrive by minimality at $\mathbb{S}^2S^0$, which is homeomorphic to its opposite. The only remaining cases have all points either minimal or maximal. By the minimality of X, each minimal point must be less than at least two maximal points and each maximal point must be greater than at least two minimal points. There is only one example with two minimal points, and its opposite is the only example with four minimal points. We are left with the case when there are three minimal and three maximal points, which means that we have a bipartite graph. Here each minimal point must be less than at least two maximal points and one, two, or all three of them can be less than all three maximal points. In all four cases, the resulting space is homeomorphic to its opposite. $\square$

The height $h(X)$ of a poset X is the maximal length h of a chain $x_1 < \cdots < x_h$ in X. It is one more than the dimension $d(X)$ of the space $|\mathscr{K}(X)|$. In the analysis just given, we noticed that if X has six elements then $h(X)$ is 2 or 3. Barmak and Minian [1] observed the following related inequality.

Proposition 0.6. *Let $X \neq *$ be a minimal finite space. Then X has at least $2h(X)$ points. It has exactly $2h(X)$ points if and only if it is homeomorphic to $\mathbb{S}^{h(X)-1}S^0$.*

Proof. Let $x_1 < \cdots < x_h$ be a maximal chain in X. Since X cannot have a minimimum point, there is a y_1 which is not greater than x_1. Since no x_i is an upbeat point, $1 \leq i < h$, there must be some $y_{i+1} > x_i$ such that y_{i+1} is not greater than x_{i+1}. The points y_i are easily checked to be distinct from each other and from the x_j. Now suppose that X has exactly these $2h$ points. By the maximality of our chain, the x_i and y_j are incomparable. For $i < j$, we started with $x_i < x_j$, and we check by cases from the absence of upbeat and downbeat points that $y_i < x_j$, $y_i < y_j$, and $x_i < y_j$. Comparing with the interated suspension, we see that this implies that X is homeomorphic to $\mathbb{S}^{h-1}S^0$. $\square$

Corollary 0.7. *If $|\mathscr{K}(X)|$ is homotopy equivalent to a sphere S^n, then X has at least $2n+2$ points, and if it has exactly $2n+2$ points it is homeomorphic to $\mathbb{S}^n S^0$.*

Proof. The dimension $h(X)-1$ of $|\mathscr{K}(X)|$ must be at least n, so $h(X) \geq n+1$. The conclusion is immediate from the previous result. $\square$

References

[1] J.A. Barmak and E.G. Minian. Minimal finite models. J. Homotopy Relat. Struct. 2(2007), 127–140.