EXERCISES/PROBLEMS ON FINITE TOPOLOGICAL SPACES

Exercises are meant to test your understanding. Problems are meant to be less obvious and some could make for papers. I may add to them later.

Exercises

1. Show that a finite T_0 space has at least one point which is a closed subset.

2. Prove that all coarse finite spaces C_n are homotopically equivalent to the discrete space $C_1 = D_1$ with one element. Is this true for all coarse spaces, whether or not they are finite?

3. Prove that a locally compact Hausdorff space is smally compact, in the sense of Definition 5.3.

4. Prove Proposition 5.4 (assuming the tube lemma, Munkres' 3.5.8).

5. Prove that all spaces $P_{m,n}$ of Definition 3.1 are contractible. Deduce that all spaces in the table of 3-point spaces on page 6 are disjoint unions of contractible spaces.

6. Do Problem 3.3. Which of these spaces with at most 4 points are connected minimal finite spaces? Later, describe explicitly the simplicial complexes associated to each of them. Are there any weakly homotopy equivalent 4 point spaces that are not homotopy equivalent?

7. Give an explicit description of two finite spaces that are weakly homotopy equivalent but not homotopy equivalent.

Papers/Problems

1. Determine all weak homotopy types of spaces with at most n points for $n \leq 7$ (definitely within reach) or, better, $n \leq 8$ or higher. There is a possibility of interesting programming to help solve this (and also problems 5 and 6).

2. What is the smallest n for which there is an n point space X such that $|\mathscr{K}(X)|$ is not a wedge or product of spheres?

3. Describe all (or at least many) surfaces as $|\mathscr{K}(X)|$ for finite spaces X (up to homotopy equivalence). This has been done for the real projective plane and the torus, but not for any other examples as far as I know.

4. How can one tell whether or not a given simplicial complex is of the form $\mathscr{K}(X)$ for a finite space X? What are some necessary conditions?

5. How can one enumerate weak homotopy types of spaces with at most n points? What can one say about the asymptotics of the number of weak homotopy types of n-point spaces as n increases?

6. What more can one say about the explicit modelling of well-known maps $f\colon |\mathscr{K}(X)| \longrightarrow \mathscr{K}(Y)|$? For some n, there is a map $g\colon X^{(n)} \longrightarrow Y$ such that $|\mathscr{K}(g)| \simeq f$. Explicit models g for the product $S^n \times S^n \longrightarrow S^n$ are known for $n = 1$ and $n = 3$, but not for $n = 7$. Explicit models for some related "Hopf maps" are also known. However, little more is known.

7. Explore the relationships among nerves of categories, subdivisions of categories, and subdivisions of simplicial sets.

8. Finite spaces are used in Computer Science. Search the web to find out more. Expository papers describing some such applications or surveying the area in general would be welcome.

8. See if other applications, for example to RNA and evolution, seem interesting. Describe the ideas behind some such application, making clear why finite spaces are relevant.