FINITE TOPOLOGICAL SPACES AS A PEDAGOGICAL TOOL

Randall D. Helmstutler
Department of Mathematics
University of Mary Washington
1301 College Ave
Fredericksburg, VA 22401
rhelmstu@umw.edu

Ryan S. Higginbottom
Department of Mathematics
Washington & Jefferson College
60 S Lincoln St
Washington, PA 15301
ryan.higginbottom@washjeff.edu

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Abstract: We propose the use of finite topological spaces as examples in a point-set topology class especially suited to help students transition into abstract mathematics. We describe how carefully-chosen examples involving finite spaces may be used to reinforce concepts, highlight pathologies, and develop students' non-Euclidean intuition. We end with a project in which finite spaces are featured.

Keywords: topology, finite spaces, abstraction

1 INTRODUCTION

For a majority of undergraduate mathematics majors, point-set topology is the most abstract class they encounter in the mathematics curriculum. This presents tremendous opportunities and challenges to the professor. We have the privilege of watching our students grow in mathematical maturity before our eyes, which can lead to exciting conversations and satisfying exercises and projects. On the other hand, the subject matter may feel farther away from—or perhaps higher above—students' previous mathematical experience. In this paper we describe a tool that helps students adapt to this new environment.

To transition into the abstract world of topology, we favor beginning with familiar definitions and examples, temporarily delaying the introduction of more exotic topological spaces. In this context students are encouraged initially to think of open sets within the real line as open intervals and to think of a continuous function from $\mathbb{R}$ to $\mathbb{R}$ as essentially one that can be drawn without picking up one's pencil from the paper.

This begins the process of enculturation: we begin where the students are and slowly inject rigor into their naïve mathematical notions.

After discussing the properties of open intervals within $\mathbb{R}$ and proving the continuity of a few real-valued functions via open sets and pre-images, we give the proper definitions of a topological space and a continuous function. Examples that follow may include the real line with nonstandard topologies. This gradually pushes students into more interesting waters—keep the set, change the topology, examine the effects.

Shortly thereafter students reach a point where they should grapple with these initial definitions without the aid of their intuition. They need to encounter some strange or sublime examples, wholly new to their experience. At this moment, we hand our students problems on finite topological spaces.

Finite spaces are a great fit in those first weeks of class because the student can (and should) *write down everything.* Is this a topology? Check all of the intersections and unions! Is this a continuous function? Write down all of the relevant pre-images! If the underlying sets for these finite spaces are small enough, these problems can be used as class activities or homework exercises.

In considering examples like this, we break students of their reliance on the real numbers. Their experience with $\mathbb{R}$ is helpful in motivating topological definitions and putting the generalizations in context, but Euclidean space alone can not demonstrate all of the subtleties of abstract spaces. In particular, students must realize that topological spaces need not have an ordering or a metric. For this reason, we suggest using symbols or letters—never numbers—to denote the points within finite spaces.

We have found that finite topological spaces are useful long after we move beyond continuity. As we show in Section 2, finite spaces can force students to confront definitions, theorems, and counterexamples at virtually any point in the course. We will demonstrate how these spaces can help students formulate conjectures and solutions to open-ended questions in Section 3. This particular class of examples can also be used in the context of an individual or group project; we outline such a project in Section 4.

The examples and exercises that follow are gathered from the authors' personal experiences in teaching undergraduate courses and directing undergraduate research projects in topology over the past five years. The suggestions on the usage and benefits of these concepts are based on our observations, hence the conclusions we draw should be regarded as mostly anecdotal.

2 BRIEF EXAMPLES

Examine any introductory textbook in topology and you will likely see finite spaces vanish shortly after their first appearance. (The textbook [1] may provide an exception, as finite spaces arise throughout the first half of the book.) Most texts use finite spaces to illustrate a topology on a set, continuity, and interior and closure matters. We see no reason to stop here; save for compactness, any notion of point-set topology—both the pathologies and the good behavior—can be highlighted with finite spaces. In this section we provide some examples for use with connectedness and universal constructions.

Students may assume that they understand connectedness if they only draw pictures in the Euclidean plane. "This figure is in one piece, so it is connected. This one is not, so it is disconnected." Confronting finite spaces forces them to use the definition in a precise manner.

Exercise 1 Let X be the set $X = \{a, b, c, d\}$. Let $\mathcal{T}$ be the following topology on X:

$$\mathcal{T} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}.$$

Is $(X, \mathcal{T})$ a connected space?

A direct inspection yields an answer of "yes." It is fairly simple to concoct examples of finite disconnected spaces, but a finite connected space makes a stronger point. Student tendency is to depend on the visual: a figure is connected if it may be drawn in one stroke of a pencil. However, there is no way to draw the connected four-point space of Exercise 1 in such a way.

The advantage of working with a finite space here is that students can check for potential separations of the space X algorithmically, and in small examples this doesn't take long. The natural thought process is this: if open set A is to be one of the separating sets, the other open set must be $X - A$. If students experiment with such examples, they should discover for themselves the following theorem: a space is disconnected if and only if it has a proper, non-empty subset that is simultaneously open and closed.

Finite disconnected spaces also play an important pedagogical role.

Exercise 2 Let X be the set $\{a, b, c, d\}$. Let $\mathcal{T}$ be the following topology on X:

$$\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, X\}.$$

Show that $(X, \mathcal{T})$ is disconnected and determine its connected components.

Solution The separation $X = \{a, b\} \cup \{c, d\}$ is easy to spot, proving that $(X, \mathcal{T})$ is disconnected.

Since there are only 16 subsets of X, it is possible to write them all out and examine which ones are connected. After doing this, we find that there are only 5 subsets of X which are connected: the four singleton subsets and $\{a, b\}$. Since the connected component of a point x is simply the union of the connected sets which contain x, it is easy to see that the connected components of X are $\{a, b\}$, $\{c\}$, and $\{d\}$.

The point here is nearly the same: understanding decomposition by components is relatively simple if one relies solely on Euclidean-informed intuition. There are several ways of defining connected components, but they all boil down to searching for connected subsets of maximal size. Hence the search for connected subsets will reinforce the definition of the subspace topology in addition to basic connectedness concepts.

For the most part, the basic universal constructions in topology are not as straightforward as their algebraic analogues. For example, the definition of the topological product is not as intuitive and natural as the direct product of groups. The basic idea is clear, but why must one form all those unions of products of open sets? The Euclidean plane is a great example of why this is correct (the union of two rectangles need not be a rectangle), but a finite example allows the students to write down *every* open set and see that the result truly is a topology.[1] The same can be said for disjoint unions (coproducts) and quotients.

Exercise 3 Let $X = \{a, b, c\}$ and let $Y = \{d, e, f, g\}$. Define topologies $\mathcal{T}_1$ on X and $\mathcal{T}_2$ on Y by

$$\mathcal{T}_1 = \{\varnothing, \{a\}, \{a, c\}, X\} \qquad \mathcal{T}_2 = \{\varnothing, \{d, e\}, \{f, g\}, Y\}.$$

Write down the set $X \times Y$ and the product topology on $X \times Y$. Do the same for the disjoint union $X \amalg Y$.

Quotient constructions are possibly the most mystifying topic for students in any introductory course. Points are now entire subsets, and subsets are families of subsets. Students must learn to deal with these abstractions and become comfortable working with the array of braces and brackets in quotient topologies. With finite spaces, we can keep track of the structure easily, allowing the student to see the relation of $X/{\sim}$ to X more clearly. Again, there is no mystery to the open sets in a finite space, as the student can write them all down and see them all at once.

Exercise 4 Let X be the set $X = \{a, b, c, d, e, f\}$ and consider these two topologies on X:

$$\mathcal{T}_1 = \{\varnothing, \{a, b\}, \{a, b, d, f\}, \{d, f\}, X\}$$
$$\mathcal{T}_2 = \{\varnothing, \{f\}, \{a, c\}, \{a, c, f\}, \{a, c, d\}, \{a, c, d, f\}, \{a, b, c, d, f\}, X\}.$$

Additionally, consider the equivalence relation on X generated by the following relations: $a \sim c$, $b \sim d$, $d \sim f$. Write down the elements of $(X, \mathcal{T}_1)/{\sim}$ along with the quotient topology. Do the same for $(X, \mathcal{T}_2)/{\sim}$.

[1] The sizes of such topologies grow rather quickly. Alternatively, the students can write down all products of open sets and then use a basis theorem to check that this does in fact generate a topology on the product.

Exercise 5 Let Y be the set $Y = \{l, m, n\}$ and consider the following topology on Y:

$$\mathcal{T}' = \{\varnothing, \{m\}, \{l, m\}, Y\}.$$

Let X be the set $X = \{a, b, c, d, e, f\}$. Find a topology $\mathcal{T}$ on X and an equivalence relation $\sim$ on X such that $(X, \mathcal{T})/\sim$ is homeomorphic to $(Y, \mathcal{T}')$.

Exercise 6 Let $X = \{a, b, c\}$ and consider the following topology on X:

$$\mathcal{T} = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}.$$

Is it possible for $(X, \mathcal{T})$ to be a quotient space of $\mathbb{R}$? If so, provide an equivalence relation $\sim$ on $\mathbb{R}$ which makes this happen and prove it with complete detail. If not, explain why not.

Solution Partition $\mathbb{R}$ into three classes: the positive numbers, the negative numbers, and 0. Then declare that $x \sim y$ if x and y are in the same class. We claim that $\mathbb{R}/\sim$ is homeomorphic to $(X, \mathcal{T})$. Under the equivalence relation $\sim$, we will label the equivalence classes $[1]$, $[-1]$, and $[0]$. Since $(0, \infty)$ and $(-\infty, 0)$ are open in $\mathbb{R}$, we see that $\{[1]\}$, $\{[-1]\}$, and $\{[-1], [1]\}$ are open subsets of $\mathbb{R}/\sim$. Aside from $\varnothing$ and $\mathbb{R}/\sim$, these are the only open subsets of $\mathbb{R}/\sim$, as one can check by examining the pre-images under p of the other subsets ($p : \mathbb{R} \to \mathbb{R}/\sim$ is the quotient map).

We have argued that $\mathbb{R}/\sim = \{[-1], [0], [1]\}$ with topology

$$\{\varnothing, \{[-1]\}, \{[1]\}, \{[-1], [1]\}, \mathbb{R}/\sim\}.$$

We can see that this is homeomorphic to $(X, \mathcal{T})$ through the assignments

$$[-1] \mapsto a, \qquad [0] \mapsto c, \qquad [1] \mapsto b.$$

3 OPEN-ENDED QUESTIONS

We have seen that finite topological spaces are convenient examples to use when new ideas and definitions arise. Because they can check *everything* in a relatively small amount of time, students can internalize these concepts more quickly. In this section we will show how finite spaces serve as a laboratory for testing new conjectures and ideas. Notably, finite spaces provide the easiest access to non-Hausdorff spaces.

As students mature mathematically, they will begin to ask more sophisticated questions. When presented with a theorem, they should ponder the converse. They should ask which hypotheses are necessary and investigate in what other settings propositions are true. Below we present two situations in which finite spaces are well-suited to help students carry out such experiments. A definition is required for the first example.

Definition A subspace A of a topological space $(X, \mathcal{T})$ is called a *retract* of X if there exists a continuous function $r : X \to A$ such that $r(a) = a$ for all $a \in A$.

Retracts are an enjoyable topic for a point-set topology class. In a discussion of Hausdorff spaces, it is an excellent exercise to prove that if X is a Hausdorff space and A is a retract of X, then A is a closed set in X. Students may then consider this open-ended question.

Exercise 7 Can closed retracts arise otherwise? Specifically, can you find a non-Hausdorff space $(X, \mathcal{T})$ having a retract A that is a closed set in X?

Most of our commonly-studied spaces are Hausdorff, so many students find the search for answers to this question difficult. Since a finite Hausdorff space is necessarily discrete, any non-discrete finite space is not Hausdorff, making finite spaces a perfect setting in which to begin an investigation. Students can construct examples and explore the relevant definitions and properties without becoming overwhelmed.

It turns out that the answer to Exercise 7 is "yes," and there is a nice example consisting of a space of only three points. This particular example was discovered by one of our own students.

Example Let $X = \{a, b, c\}$ and declare the following subsets to be open:

$$\varnothing, \{a\}, \{b\}, \{a, b\}, X.$$

Let $A = \{b, c\}$ and define $r : X \to A$ by $r(a) = r(b) = b$ and $r(c) = c$. It is straightforward to check that r is continuous and that A is closed in X, even though X is not Hausdorff.

This idea that finite spaces can function as a laboratory can be used in many places, but we will mention just one more. When studying limits of sequences in topological spaces, students soon come across the fact that limits are unique within Hausdorff spaces. At this point the students are ready for an exercise like the following.

Exercise 8 We have proven that if $(X, \mathcal{T})$ is Hausdorff, limits of sequences are unique (when they exist). Are there conditions on $(X, \mathcal{T})$ that make the converse true?

The point here is that having unique limits says something strong about the individual points in the space. With a bit of thought one can show that points in such a space must be closed. Hence, if X is assumed to be finite, it must also be discrete (and hence Hausdorff).

4 A LONGER CASE STUDY

As we have seen, finite topological spaces comprise a helpful class of examples that introduce students to important definitions throughout a point-set topology course. In this section we will demonstrate how these same spaces can provide an enlightening end-of-semester project for an individual or a group.

This project came about when the authors were searching for a simpler alternative to the so-called "topologist's sine curve" as a way to differentiate connectedness and path-connectedness. After the search for an example in finite spaces proved fruitless, we stumbled upon the fact that these two conditions are in fact equivalent in the finite case; this is included below as Theorem A. That this fact is not well-known (or at least not well-advertised) attests to the lack of finite examples in the classroom. The argument presented below is simple and elegant and makes clear the special role that finiteness plays as an essential hypothesis.

What follows is not the only approach to the proof of this theorem (see [4], for example). However, our method brings students into contact with a fairly deep result by reasonable increments. Once again, our goal is to help our students thrive in this abstract world; with finite spaces as a central feature of this project, examples are easily generated and accessible.

In our exercises below, we break down Theorem A into manageable chunks, any one of which could be considered a homework problem of medium difficulty. We will provide solutions or hints to some of these exercises. Since this project is designed to appear at the end of an introductory point-set topology course, we intentionally omit the definitions with which students would already be familiar. The reader should assume that all topological spaces in the rest of this section are finite.

Theorem A *A finite topological space is path-connected if and only if it is connected.*

Definition Let $(X, \mathcal{T})$ be a finite topological space. For every $x \in X$, let M_x be the smallest open set containing x. We call M_x the *minimal neighborhood* of x.

Other proofs of Theorem A use a directed graph generated by the finite space. The ordering that produces this graph can be obtained through our minimal neighborhoods ($x \leq y$ if $x \in M_y$). While such proofs through equivalence relations and orderings are not beyond our students' abilities, the direction we take below is native to the point-set experience. (This program also has the advantage of being more straightforward and easier than the alternatives we have seen.) Additionally, as this approach provides an opportunity to collect topological concepts from throughout the course, it is especially appropriate as an end-of-semester project.

Exercise 9 Prove that M_x exists and is unique for every $x \in X$. Pay special attention to the way *finiteness* enters the picture.

Exercise 10 Prove that M_x is equal to the intersection of all neighborhoods of x. Conclude that M_x is contained in every neighborhood of x.

Though our students prove abstract results in the previous exercises, we also want them to undertake specific examples. Within a project, a mixture of proofs and calculations solidify the concepts for our students: the students appear to be more engaged when they can relate the abstract result to a particular example through calculation.

Exercise 11 Let X be the set $X = \{a, b, c, d\}$ and consider the following two topologies on X:

$$\mathcal{T}_1 = \{\varnothing, \{a\}, \{d\}, \{a, c\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}, X\}$$
$$\mathcal{T}_2 = \{\varnothing, \{b\}, \{c, d\}, \{b, c, d\}, X\}.$$

Write down the minimal neighborhood for each element of X under each topology.

Exercise 12 Let $(X, \mathcal{T})$ be a finite topological space. Prove that the collection of subsets $\{M_x \mid x \in X\}$ forms a basis for the topology on X.

Exercise 13 Let $X = \{a, b, c, d, e\}$ and define a non-empty subset of X to be *open* if it contains the point a (check that this gives a topology on X). Letting $\mathbb{R}$ denote the real line under its usual topology, prove that the function $f : \mathbb{R} \to X$ defined by

$$f(x) = \begin{cases} d, & x \leq 1 \\ a, & 1 < x < 5 \\ e, & x \geq 5 \end{cases}$$

is continuous.

The point of the above exercise is to give an appreciation for the efficiency of the minimal open sets. The topology on X contains $2^4 + 1 = 17$ open sets, but only five of them are minimal. As the minimal open sets provide a basis for X, we need only check five pre-images instead of 17. If we gave a set of 10 elements an analogous topology, we could verify continuity by examining the pre-images of only 10 open sets as opposed to $2^9 + 1 = 513$ open sets.

In the next example, students prove the key lemma needed for the proof of Theorem A. They deal with path-connectedness directly and see the strength of minimal neighborhoods. Depending on the capabilities of the students in the class, this exercise may require a hint. We have included the entire proof.

Exercise 14 Prove that the set M_x is path-connected for every x.

PROOF: Let $y \in M_x$ be an element distinct from x. We will produce a path connecting x and y. Let $\lambda : [0,1] \to M_x$ be the following function:

$$\lambda(t) = \begin{cases} x, & 0 \le t \le \frac{1}{2} \\ y, & \frac{1}{2} < t \le 1. \end{cases}$$

It is clear that $\lambda(0) = x$ and $\lambda(1) = y$ so we only need to show that λ is continuous. Let V be a non-empty open set in M_x; note that this means V must be open in X as well. By Exercise 10, in addition to elements other than x and y, V must either contain both x and y, just y, or neither x nor y. In these cases, $\lambda^{-1}(V)$ would be $[0,1]$, $(\frac{1}{2}, 1]$, or $\varnothing$, respectively. Since these are all open subsets of $[0,1]$, λ is a continuous function. $\square$

We now introduce the definition that links the concepts of connectedness and path-connectedness for finite spaces. (See [3, p. 161].)

Definition A topological space $(X, \mathcal{T})$ is said to be *locally path-connected at x* if for every neighborhood U of x, there is a path-connected neighborhood V of x contained in U. If $(X, \mathcal{T})$ is locally path-connected at each of its points, we say that $(X, \mathcal{T})$ is *locally path-connected*.

Once the students have completed Exercise 14, the following exercise should come easily.

Exercise 15 Prove that every finite topological space is locally path-connected.

Students may need some hints for the next exercise, which has Theorem A as a corollary. (In our hints we follow the proof in [3, pp. 161–162].)

Exercise 16 Prove that if $(X, \mathcal{T})$ is locally path-connected, the components and path-components of X are the same.

Hint 1 Let x be a point in X. Let C be the component of X containing x, and let P be the path-component of X containing x. Show that $P \subseteq C$.

Hint 2 Show that $P = C$ by contradiction. Assume that $P \subsetneq C$ and form a separation of C. For part of this separation, consider the subset Q, defined as the union of all path-components of X that are different from P and that intersect C.

Exercise 17 Finish the proof of Theorem A.

5 CONCLUSION

The ideas presented in this paper may be pushed further still. Such topics may even be used for undergraduate research projects. For example, any space for which *arbitrary* intersections of open sets remain open will have minimal open sets as defined in Section 4. Are there infinite spaces with this property? How much of our knowledge of finite spaces translates to this more general setting? For students wanting more, finite spaces are relevant to topics beyond point-set topology. Finite spaces and related generalizations even have an accessible homotopy theory, which is nicely described in a set of online notes [2] written by J.P. May.

We have designed our examples to show that finite spaces may be used in a variety of ways to hone students' intuition and their ability to analyze open-ended questions. Finite spaces provide a means to check definitions and constructions by hand, without the need for more elaborate arguments. Once the students are comfortable with the ideas in a hands-on setting, they can construct the more complex arguments required in the further development of the subject.

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BIOGRAPHICAL SKETCHES

Randall Helmstutler obtained a B.S. in mathematics from Radford University and a Ph.D. in mathematics from the University of Virginia. A frequent instructor of topology and director of undergraduate research projects in the field, he has long enjoyed placing topological problems within the reach of students of all levels.

Ryan Higginbottom received a B.S. in mathematics from Bucknell University and a Ph.D. in mathematics from the University of Virginia. He teaches topology, abstract algebra, and most other classes of the undergraduate curriculum.