ALGEBRAIC TOPOLOGY, FALL 2015, TAKE HOME FINAL

This final exam is posted online as of December 1, and is due by 11:30 am on Monday, December 7. Collaboration is not allowed, nor is the use of outside materials other than my book and your class notes. Graduate students are not given letter grades. I must give undergraduates such grades. Problems 7 and 8 are bonus questions (not doing them will not lower your grade).

1. Show that a retract of a contractible space is contractible. (Observe that this can be viewed as a conceptual generalization of the Brouwer fixed point theorem.)

2. Show that $\pi_2(S^2 \vee S^1)$ is infinitely generated. (Hint: think about covering spaces.)

3. Compute the cup product structure on the cohomology of a Klein bottle, for both integer and mod 2 coefficients.

4. Show that for any $n \geq 1$, the covering projection $S^{2n} \to \mathbb{R}P^{2n}$ induces zero in integral homology and cohomology in degree $2n$ but is not null-homotopic.

5. Let $i\colon A \to X$ be an inclusion. Show that i is null homotopic (i.e. homotopic to the constant map at a point $x \in X$) if and only if X is a retract of the cofiber $X \cup_A CA$, and then $H_n(X, A) = \tilde{H}_n(X) \oplus \tilde{H}_{n-1}(A)$.

6. Let M be a closed orientable manifold of dimension $n = 2m$. Show that if $H_{m-1}(M; \mathbb{Z})$ is torsion free, then so is $H_m(M; \mathbb{Z})$.

7. Give an example of two closed, connected manifolds whose homotopy groups are isomorphic in every dimension, but which are not homotopy equivalent.

8. A connected closed orientable n-manifold is *spherical* if there is some map $f\colon S^n \to M$ such that $f_*\colon H_n(S^n) \to H_n(M)$ is non-zero. Prove that if M is spherical and $n > 1$, then $\pi_1(M)$ is finite.