Superrigidity and mapping class groups

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1 Introduction

Margulis's superrigidity theorem (see [Zi1]) says that if Γ is an irreducible lattice in a real semisimple[1] Lie group G of rank at least two, then any homomorphism $\Gamma \to H$ into a semisimple Lie group H has bounded image or extends to a homomorphism $G \to H$, and the image is always bounded when H is a p-adic group. In this note we give two examples of nonlinear superrigidity: one for mapping class groups and Teichmüller space, another for diffeomorphism groups of surfaces.

Let S be a compact, orientable, connected surface, possibly with boundary. The *mapping class group* $\mathrm{Mod}(S)$ is the group $\mathrm{Homeo}^+(S)/\mathrm{Homeo}_0(S)$ of isotopy classes of orientation-preserving homeomorphisms of S. The proof of our first result will come directly from theorems of Kazhdan-Margulis and Kaimanovich-Masur, together with some structure theory of $\mathrm{Mod}(S)$.

Theorem 1.1 (Superrigidity with $Mod(S)$ target) *Let Γ be an irreducible lattice in a semisimple lie group G of $\mathbf{R}$-rank at least two. Then any homomorphism $\phi : \Gamma \to Mod(S)$ has finite image.*

Remarks.

1. Theorem 1.1 was conjectured by Ivanov [Iv1]. He proved [Iv2] that ϕ cannot be an isomorphism.

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[1] By *semisimple* we mean connected (in the real case) semisimple with finite center and no nontrivial compact factors.

2. Theorem 1.1 is false for $G = SO(n,1), SU(n,1)$ since, for example, there are lattices Γ in such G which admit surjective homomorphisms to $\mathbf{Z}$. We don't know if the theorem holds for lattices Γ in $SP(n,1)$ and F_4, although Corlette [Co] has proven the usual superrigidity for these lattices.

Let Teich(S) denote the Teichmüller space of the surface S. The isometry group[2] of Teich(S) is Mod(S) [Ro]. Kerckhoff [Ke] proved that any finite subgroup of Mod(S) can be realized as a group of isometries of some Riemann surface $X \in$ Teich(S). Hence Theorem 1.1 gives:

Corollary 1.2 (Global fixed point) *Let Γ be an irreducible lattice in a semisimple Lie group G of $\mathbf{R}$-rank at least two. Then any isometric action of Γ on a Teichmüller space Teich(S) has a global fixed point.*

Actions on surfaces. The Zimmer program studies representations with target $H = \mathrm{Diff}(M)_\omega$; that is, actions of lattices and Lie groups on compact manifolds M preserving various geometric structures ω (see [Zi3] for a survey, and [Zi2] for the case of M a surface). The only known examples of superrigidity with target Diff(M) are due to Witte [Wi] for $M = S^1$ (and Γ higher $\mathbf{Q}$-rank) and Weinberger [We] for $M = (S^1)^m$ (and $\Gamma = SL_n(\mathbf{Z}), 3 < n > m$). The following theorem provides many more examples.

Theorem 1.3 ($SL_n(\mathbf{Z})$ actions on surfaces) *Let Σ_g denote the closed surface of genus g. Then for every $g \geq 2$ there exists $N > 0$ so that, for all $n \geq N$, every homomorphism $SL_n(\mathbf{Z}) \to Homeo^+(\Sigma_g)$ is trivial. More precisely, we can choose N to be any number satisfying*

$$2^{N(N-1)/2} \prod_{i=2}^{N-1} (2^i - 1) > 84(g-1)$$

An analogous theorem for $SL_n(\mathbf{Z})$ actions on higher-dimensional, nonpositively curved manifolds with no torus factors will be given in [Fa].

Remarks.

1. In case the action preserves a connection and volume density, Theorem 1.3 follows from the results of [Zi4].

[2] in either the Teichmüller or Weil-Peterson metrics

2. Nontrivial actions are not precluded *a priori* by algebra: $SL_n(\mathbf{Z})$ even acts faithfully by automorphisms on $H_1(\Sigma_g) \approx \mathbf{Z}^{2g}$ for $n \leq g/2$. One might guess (incorrectly) that some such action might come from an action on Σ_g.

3. Theorem 1.3 implies that, for genus $g = 2, 3, 4, \ldots, 13$, $SL_n(\mathbf{Z})$ doesn't act for $n > 3$, and the group $SL_5(\mathbf{Z})$ doesn't act on any surface of genus less than 3200.

4. The conclusion of Theorem 1.3 is false without some bound on N in terms of g: $SL_n(\mathbf{Z})$ maps onto $SL_n(\mathbf{Z}/2\mathbf{Z})$, and any finite group acts faithfully by automorphisms of some (necessarily high genus) Reimann surface (see Proposition 3.1 below).

The key to our proof of Theorem 1.3 is, following the method of Weinberger [We], to see how finite subgroups act. We also use the following.

Theorem 1.4 (Surface actions are virtually isotopically trivial) *Let Γ be an irreducible lattice in a semisimple lie group G of $\mathbf{R}$-rank at least two. If Γ acts by homeomorphisms on the compact surface S, then there is a finite index subgroup $\Gamma' < \Gamma$ such that every $g \in \Gamma'$ acts by a homeomorphism isotopic to the identity. In particular Γ' acts homologically trivially on S.*

One simply lets Γ' be the kernel of the composition $\Gamma \to \mathrm{Homeo}^+(S) \to \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0(S) = \mathrm{Mod}(S)$ and applies Theorem 1.1.

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2 Proof of Theorem 1.1

Unless otherwise specified, by *surface* we will mean a compact, orientable, connected 2-manifold with (perhaps empty) boundary. For background, definitions, and references on $Mod(S)$ and Thurston's theory of surface diffeomorphisms we refer the reader to [Th, FLP, Iv3].

We emphasize that when S has boundary, an element of $\mathrm{Mod}(S)$ is an isotopy class of homeomorphisms which need not fix an invariant boundary component pointwise, and where the isotopy need not fix such a boundary component pointwise. Hence if S has boundary components $T_1, \ldots, T_r$ then

$\mathrm{Mod}(S)$ is isomorphic to the mapping class group $\mathrm{Mod}(T)$ of the punctured surface T obtained by gluing a once-punctured disc D_i to each T_i along ∂D_i.

To prove the theorem we induct on $\tau(S) = 3g - 3 + n$, where g is the genus of S and n is the number of boundary components of S. We may also define $\tau(S)$ as the maximal number of isotopy classes of disjoint, simple, homotopically nontrivial, nonperipheral (i.e. not isotopic to ∂S), closed curves in S. In particular, if γ is a simple closed curve in S, and if R is the surface-with-boundary obtained from S by cutting along γ, then R is orientable and $\tau(R) < \tau(S)$.

To begin with, note that the theorem is true for all surfaces S with $\tau(S) \leq 0$: if S is an annulus, sphere, pair of pants, or disc then $Mod(S)$ is trivial; if S is a torus then $\mathrm{Mod}(S) \approx SL_2(\mathbf{Z})$ and the theorem follows from Margulis's superrigidity theorem (see [Zi1], Theorem 5.1.2).

Now let S be any surface with $\tau(S) > 0$, and suppose we know the theorem is true for all surfaces T with $\tau(T) < \tau(S)$. Let Γ be any irreducible lattice in a semisimple Lie group G of rank at least two, and let $\phi : \Gamma \to \mathrm{Mod}(S)$. We suppose that $\phi(\gamma)$ is infinite, and will get a contradiction.

The Kazhdan-Margulis Theorem (see [Zi1], Theorem 8.1.2) states that a normal subgroup of Γ is either contained in the center $Z(G)$ of the semisimple Lie group G (in particular is finite) or is finite index in Γ. Since $\phi(\Gamma)$ is not finite then the kernel of ϕ must be finite. By the Selberg Lemma, Γ has a finite index subgroup Γ' which is torsion-free. Replacing Γ by Γ', we may assume that the kernel of ϕ is trivial, so that $\phi(\Gamma)$ is a subgroup of $\mathrm{Mod}(S)$. Henceforth we will denote this subgroup by Γ.

Let $\mathcal{PMF}(S)$ denote the space of projectived measured foliations on S (see [Th]). Recall that a subgroup $\Gamma < \mathrm{Mod}(S)$ is *nonelementary* if either of the following two equivalent (by [KM], Lemma 1.2.1) conditions holds:

- Γ fixes no finite subset of $\mathcal{PMF}(S)$.
- Γ contains a pair of *independent* pseudo-Anosov homeomorphisms h_1, h_2; that is, the sets of fixed points of h_1 and h_2 in $\mathcal{PMF}(S)$ are disjoint.

Theorem 2.4.1 of [KM] states that an irreducible lattice in a semisimple Lie group of rank at least two is not isomorphic to a nonelementary subgroup of $\mathrm{Mod}(S)$. Hence Γ is elementary. We note that, although the Theorem 2.4.1 of [KM] is stated only for closed surfaces, all of the arguments apply verbatim to prove the same result for punctured surfaces. But as noted above, if S has nonempty boundary then $\mathrm{Mod}(S)$ is isomorphic to the mapping class group of a punctured surface.

Note. Unlike in the case of discrete groups of isometries of hyperbolic space, there do exist examples of elementary groups of mapping classes which are nevertheless *nonamenable*. For example, the subgroup generated by Dehn twists about two intersecting curves that are disjoint from a third curve is elementary and nonamenable.

McCarthy and Papadapoulos (see Theorem 4.6 of [MP], as well as §4 below), based on earlier work of Ivanov (see [Iv4, Iv3]), have classified subgroups of $\mathrm{Mod}(S)$ into four classes:

1. subgroups containing a pair of independent pseudo-Anosov elements (called *sufficiently large* subgroups in [MP]);
2. subgroups fixing the pair $\{F_-(g), F_+(g)\}$ of fixed points in $\mathcal{PMF}(S)$ for a certain pseudo-Anosov element $g \in \mathrm{Mod}(S)$ (such subgroups are virtually cyclic);
3. finite subgroups;
4. infinite subgroups leaving invariant a finite, nonempty, system of disjoint, nonperipheral, simple closed curves on S (such subgroups are called *reducible*).

Since Γ is elementary option 1 cannot occur. Since Γ is a lattice in a semisimple Lie group, options 2 and 3 clearly cannot occur (e.g., Γ is nonamenable). Hence Γ must be reducible, say leaving a finite collection C of disjoint simple closed curves invariant.

Since Γ leaves C invariant, each element of Γ gives a permutation of the connected components $\{S_1, \ldots, S_n\}, n > 0$ of $S \setminus C$. Hence there is a homomorphism $\Gamma \to H_n$ where H_n is the permutation group on n letters.

Let L be the kernel of this homomorphism. Then L, being finite index in Γ, is an irreducible, higher rank lattice. But L leaves each subsurface S_i invariant (here we are thinking of each S_i as a compact surface with boundary). Hence there is a homomorphism

$$\psi : L \to \mathrm{Mod}(S_1) \times \cdots \times \mathrm{Mod}(S_n)$$

with abelian kernel (see [BLM], Lemma 2.1), namely $\psi(g) = 1$ if and only if ψ is a product of Dehn twists about curves in C.

Applying the Kazhdan-Margulis theorem to L, we see that the kernel of ψ must be finite, for otherwise it would be finite index and so L would be virtually abelian, which cannot happen since L is nonamenable. Again applying the Selberg Lemma we may, perhaps after passing to a finite index subgroup of L, assume that ψ has trivial kernel. In particular $\psi(L)$ is infinite.

Let $p_i : \mathrm{Mod}(S_1) \times \cdots \times \mathrm{Mod}(S_n) \to \mathrm{Mod}(S_i)$ be the natural projection. Then $p_j \circ \psi(L)$ is infinite for at least one $1 \leq j \leq n$. But by induction, since $\tau(S_j) < \tau(S)$, the homomorphism $p_j \circ \psi : L \to \mathrm{Mod}(S_j)$ must have finite image, a contradiction. $\diamond$

3 Proof of Theorem 1.3

We begin this section by proving that any finite group acts as a group of isometries of some surface. As noted above, this implies that some bound is needed in the statement of Theorem 1.3.

Proposition 3.1 (Finite groups act) *Let G be any finite group. Then there is some compact Riemann surface X so that G is a subgroup of the group of isometries of X.*

The following proof was suggested to us by R. Narasimhan.

Proof. Write $G = F/N$ where F is free on m generators and N is normal in F. Note that N is finitely generated since it is finite index in F. Let Y be S^2 with $m+1$ points deleted. Then the unramified covering of Y with fundamental group N extends to a ramified covering X of S^2 on which G acts as the group of covering transformations. Of course the full group of isometries of X may be larger than G. $\diamond$

To prove Theorem 1.3 we first need an algebraic lemma.

Lemma 3.2 (Minimal $SL_n(\mathbf{Z}/k\mathbf{Z})$ quotients) *Let $n > 2, k > 1$ be integers. Then the smallest nontrivial quotient group of $SL_n(\mathbf{Z}/kZ)$ has order at least $2^{n(n-1)/2} \prod_{i=2}^{n-1}(2^i - 1)$.*

Proof. Let $k = p_1^{r_1} \cdots p_s^{r_s}$ be the prime factorization of k. Then

$$SL_n(\mathbf{Z}/k\mathbf{Z}) = \prod_{i=1}^{s} SL_n(\mathbf{Z}/p_i^{r_i}\mathbf{Z})$$

Hence it clearly suffices to prove the theorem for $k = p^r$ for some prime p and integer $r > 0$. Let K be the kernel of the surjective homomorphism

$\pi : SL_n(\mathbf{Z}/p^r\mathbf{Z}) \to SL_n(\mathbf{Z}/p\mathbf{Z})$. The group K can be thought of as the matrices in $G = SL_n(\mathbf{Z}/p^r\mathbf{Z})$ which are congruent to the identity matrix when the entries are taken mod p. By performing the binomial expansion on the expression $(I + pA)^{p^{r-1}}$, one may check that K is a p-group.

Now let K^+ denote the preimage under π of the center Z of $SL_n(\mathbf{Z}/p\mathbf{Z})$. Then Z is cyclic of order d, where d is the greatest common divisor of n and $p-1$. Now $G/K^+ \approx PSL_n(\mathbf{Z}/p\mathbf{Z})$, which is simple for $n \geq 3$ (see, e.g., [La], Theorem 9.3), so K^+ is a maximal normal subgroup of G.

We claim that K^+ is the unique maximal normal subgroup of $SL_n(\mathbf{Z}/p^r\mathbf{Z})$. If this were true then any nontrivial quotient of $SL_n(\mathbf{Z}/p^r\mathbf{Z})$ would have order at least

$$|PSL_n(\mathbf{Z}/p\mathbf{Z})| \geq |PSL_n(\mathbf{Z}/2\mathbf{Z})| = 2^{n(n-1)/2} \prod_{i=2}^{n-1} (2^i - 1)$$

The final equality is an easy computation, and is in fact Exercise XIII.16 in [La].

To prove the claim, suppose H is a maximal normal subgroup of $G = SL_n(\mathbf{Z}/p^r\mathbf{Z})$, so that G/H is simple. Now $HK^+ \lhd G$, so if $H \neq K^+$ then by maximality $G = HK^+$. A basic isomorphism theorem gives $HK^+/H \approx K^+/K^+ \cap H$, so $K^+/K^+ \cap H \approx G/H$, a simple group.

Now K is solvable since it is a p-group, and $K^+/K = Z$ is cyclic of order d. Hence K^+ is a solvable group of order dp^m for some m. So G/H is simple and solvable, hence is abelian. But since $n \geq 3$, it is not hard to check that G is a perfect group, that is G equals it's own commutator subgroup (by the same proof for fields, as in Theorem 9.2 of [La]), so that G has no nontrivial abelian quotients. Hence $G/H \approx K^+/K^+ \cap H$ is trivial and so $H = K^+$. $\diamond$

The following lemma was probably known to Nielsen, and is implicit in [Ke].

Lemma 3.3 *Let $f \in Homeo(\Sigma_g)$ where $g \geq 2$. If f has finite order and is isotopic to the identity then f is the identity.*

The lemma isn't true for $g = 1$: there are periodic isometries of the flat torus which are isotopic to the identity but are not the identity.

Proof. Following [Ke], let h be any smooth Riemannian metric on Σ_g. Average the pullbacks of h by the powers of f. Then f acts conformally on this smooth, averaged metric. Gauss showed that any conformal map on a smooth surface is complex-analytic, so by uniformization we have that the action of f is an isometric action for some hyperbolic metric on Σ_g. But any isometry on a hyperbolic surface isotopic to the identity is the identity, so we are done. $\diamond$

Now to prove the theorem. Let $\phi : SL_n(\mathbf{Z}) \to \mathrm{Homeo}^+(\Sigma_g)$ be a homomorphism. Composing ϕ with the natural projection $\pi : \mathrm{Homeo}(\Sigma_g) \to \mathrm{Mod}(\Sigma_g)$, we claim that $\psi := \phi \circ \pi$ is trivial. By Theorem 1.1 ψ has finite image. The congruence subgroup property [BMS] implies that ψ factors through the homomorphism $SL_n(\mathbf{Z}) \to SL_n(\mathbf{Z}/k\mathbf{Z})$ for some integer $k > 1$. By Lemma 3.2, if the image of ψ were not trivial, then it would have order at least $2^{n(n-1)/2} \prod_{i=2}^{n-1}(2^i - 1)$.

But the Nielsen Realization Theorem (see [Ke]) shows that any finite subgroup of $\mathrm{Mod}(S)$ can be reailzed as a group of isometries of some hyperbolic metric on S. Any such group has order at most $84(g-1)$ by Hurwitz's Theorem (see, e.g. [FK], Theorem V.1.3). This contradicts the assumption about g and n.

Hence ψ is trivial, so that every element of $\phi(SL_n(\mathbf{Z}))$ is isotopic to the identity. Choose any noncentral element $g \in SL_n(\mathbf{Z})$ of finite order. By Lemma 3.3 $\phi(g)$ must be the identity. Hence the normal subgroup N generated by g must act trivially on Σ_g. Since g is noncentral, by the Kazhdan-Margulis Theorem N must have finite index in $SL_n(\mathbf{Z})$, so that ϕ must have finite image. But Lemma 3.3 then shows that every element must act trivially. $\diamond$

4 Appendix

In [FLP] the theory of measured foliations and pseudo-Anosov diffeomorphisms is given in complete detail for compact surfaces S without boundary. The same is true for subgroups of $Mod(S)$ in [Iv3] and [MP]. For surfaces with boundary the issue of transverse measured foliations is left to the reader. As we have quoted the results of [Iv3, MP] for surfaces with boundary in our proof of Theorem 1.1, we feel obliged to fill in this detail in this appendix.

Thus assume S is a compact surface wtih boundary. In [FLP] and again in [Iv3], one requires the boundary components C of S to be leaves of the measured foliation F. Moreover C should contain at least one singularity. Now the notion of pseudo-Anosov homeomorphism requires the notion of

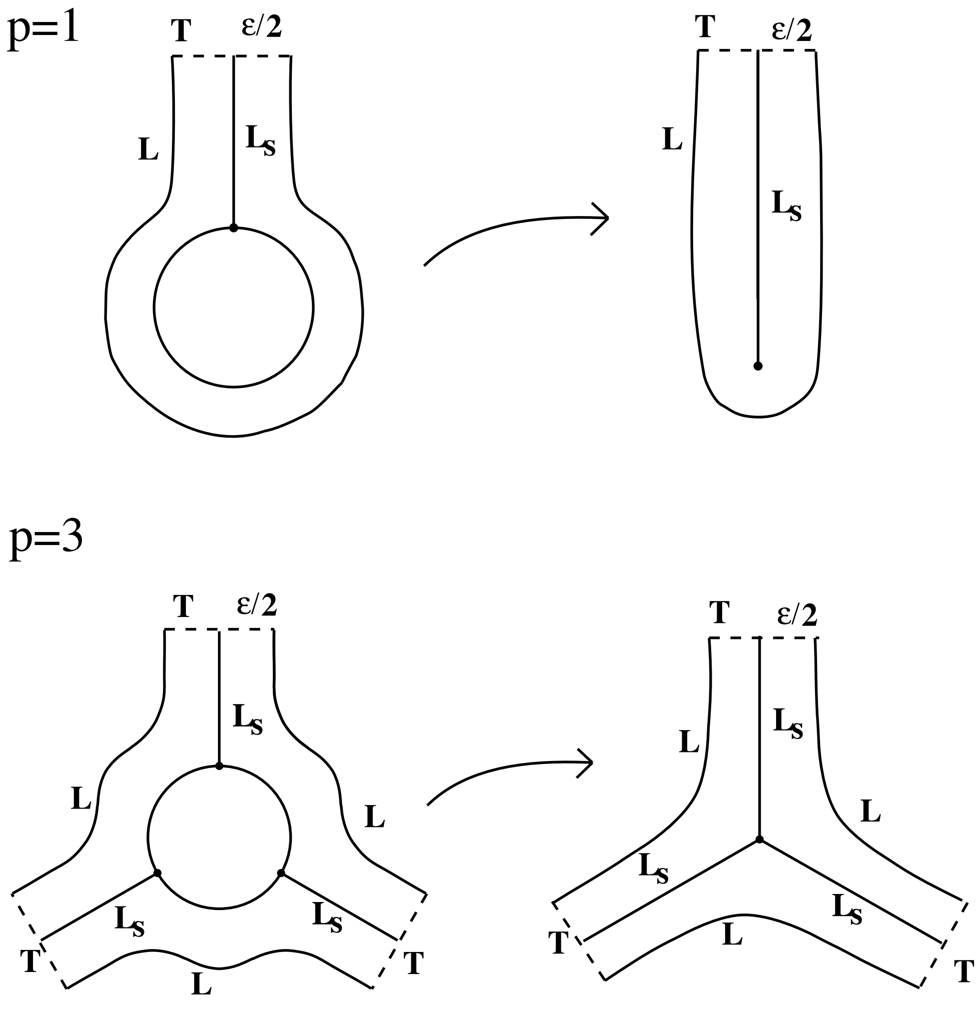

Figure 1: Converting a measured foliation on a surface with boundary to a measured foliation on a punctured surface: the local picture near a p-pronged singularity.

transverse measured foliation, which is clearly impossible if the boundary components are leaves of each foliation. We will show how to canonically associate to F a foliation F' on the punctured surface S', where S' is formed from S by attaching to each boundary component C of S a once-punctured disc along its boundary. Then one defines measured foliations F and G on S to be transverse if the associated measured foliations F' and G' are transverse in the usual sense. The proofs in [Iv3] and [MP] then go through verbatim with this definition.

To define F, let p be the number of singularities on C, counting multiplicity. For each singularity on C let L_s be the singular leaf from that singularity into the interior of S. For each leaf L_s choose a small transversal T of length ϵ such that the midpoint of T lies on L_s. We may choose the transversals so that they are disjoint. Now the endpoints of the transversals are joined to each other by nonsingular leaves L. The union of the L and T is a closed curve made up of p transversals and p leaves of F, and together with C this curve bounds an annulus. Now glue in a punctured disc along C, removing C as a leaf. There is a canonical way of filling in to a measured foliation in the punctured disc, so that the puncture is a p pronged singularity, and such that the leaves L_s extend to the punctures. Figure 4 shows this in the cases of $p = 1$ and $p = 3$.

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