THE WEIL-PETERSSON GEODESIC FLOW IS ERGODIC

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Abstract. We prove that the geodesic flow for the moduli space of Riemann surfaces is ergodic and has finite, positive metric entropy.

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Introduction

This paper is about the dynamical properties of the Weil-Petersson geodesic flow for the moduli space of Riemann surfaces. Our main result is that this flow is ergodic: any invariant set must have volume zero or full volume. Ergodicity implies that a randomly chosen, unit speed Weil-Petersson geodesic in moduli space becomes equidistributed over time. What is more, the tangent vectors to such a geodesic also become equidistributed in the space of all unit tangent vectors to moduli space.

To state our result more precisely and to put it in context, we first review the basic setup from Teichmüller theory. Let $S$ be a surface of genus $g \geq 0$ with $n \geq 0$ punctures, and let $\mathcal{M}(S)$ be the moduli space of conformal structures on $S$, up to conformal equivalence. Assume that $3g+n \geq 4$, which implies that in each conformal class there is complete hyperbolic metric. Then $\mathcal{M}(S)$ has the alternate description of the moduli space of hyperbolic structures on $S$, up to isometry. The orbifold universal cover of $\mathcal{M}(S)$ is the Teichmüller space $\text{Teich}(S)$ of marked conformal structures on $S$.

There are several natural ways to define a $\text{MCG}(S)$-invariant distance on $\text{Teich}(S)$. One that has been particularly well-studied is the Teichmüller metric $d_T$, which measures the distance between two marked conformal...
structures in terms of the minimal conformal distortion (“quasiconformal
dilatation”) of maps between the structures preserving the marking. The
Teichmüller metric is a Finsler metric (induced by a norm), which admits
a finite volume on the quotient. This metric induces a volume-preserving
geodesic flow, which embeds in an $SL(2, \mathbb{R})$-action on the unit cotangent
bundle of $Teich(S)$. The dynamics of this Teichmüller geodesic flow have
been studied extensively: in particular, it is nonuniformly hyperbolic and
ergodic, and these properties turn out to have close connections with the er-
godic and dynamical properties of directional flows on translation surfaces.
See [42] for a survey of the properties of the Teichmüller geodesic flow.

Another naturally defined and well-studied metric on Teichmüller space,
and the focus of this paper, is the Weil-Petersson metric $g_{WP}$, which is the
Kähler metric induced by the Weil-Petersson symplectic form $\omega_{WP}$ and the
almost complex structure $J$ on $Teich(S)$:

$$g_{WP}(v, w) = \omega_{WP}(v, Jw).$$

We refer to the Weil-Petersson metric as the WP metric, for short. As with
the Teichmüller metric, the WP metric descends to a metric on $M(S)$ with
finite volume.

A striking feature of the WP metric is its intimate connections with hy-
perbolic geometry, among them:

- the hyperbolic length of a closed geodesic (for a fixed free homotopy
class on $S$) is a convex function along WP geodesics in $Teich(S)$ [38];
- in Fenchel-Nielsen coordinates $(l_i, \tau_i)_{i=1}^{3g-3+n}$ on $Teich(S)$, the WP
symplectic form $\omega_{WP}$ has the simple expression $\omega_{WP} = \frac{1}{2} \sum d l_i \wedge d \tau_i$
[34]
- the growth of the hyperbolic lengths of simple closed curves on $S$ is
related to the WP volume of $M(S)$ [19]; and
- the WP metric has a formulation in terms of dynamical invariants
of the geodesic flow on hyperbolic surfaces [5, 22].

The Weil-Petersson metric has several notable features that make it an
interesting geometric object of study in its own right. The WP metric is
negatively curved, but incomplete. The sectional curvatures are neither
bounded away from 0 (except in the simplest cases of $(g, n) = (1, 1)$ and
$(0, 4)$), nor bounded away from $-\infty$. The WP geodesic flow thus presents
a naturally-occurring example of a singular hyperbolic dynamical system, for
which one might hope to reproduce the known properties of the geodesic flow
for a compact, negatively curved manifold, such as: ergodicity, equidistri-
bution of closed orbits, exponentially fast mixing and decay of correlations,
and central limit theorem.

We summarize the previous literature on the WP geodesic flow. Wolpert
[36] showed that the geodesic flow is defined for all time on a full volume sub-
set of the the unit tangent bundle $T^1Teich(S)$ and thus descends to a volume-
preserving flow on the finite volume quotient $M^1(S) := T^1Teich(S)/\text{MCG}(S)$. 
Pollicott, Weiss and Wolpert [27] proved in the case \((g,n) = (1,1)\) that the geodesic flow is transitive on \(\mathcal{M}^1(S)\) and periodic orbits are dense in \(\mathcal{M}^1(S)\) [27]. Brock, Masur and Minsky [6] proved transitivity and denseness of periodic orbits for arbitrary \((g,n)\) and also showed that the topological entropy of the geodesic flow is infinite (that is, unbounded on compact invariant sets). Hamenstädt [12] proved a measure-theoretic version of density of closed orbits: the set of invariant Borel probability measures for the Weil-Petersson flow that are supported on a closed orbit is dense in the space of all ergodic invariant probability measures.

In this paper, we prove:

**Theorem 1.** Let \(S\) be a Riemann surface of genus \(g \geq 0\), with \(n \geq 0\) punctures. Assume that \(3g + n \geq 4\). The Weil-Petersson geodesic flow on \(\mathcal{M}^1(S)\) is ergodic with respect to WP volume and has finite measure-theoretic entropy.

Our basic approach to the problem is as follows. The WP geodesic flow \(\varphi_t\) preserves a finite volume \(m\) on \(\mathcal{M}^1(S)\), and one can show using properties of the WP metric that \(\log \|D\varphi_1\|\) is integrable with respect to \(m\). The Multiplicative Ergodic Theorem of Oseledec then implies that there is a full volume subset \(\Omega \subset \mathcal{M}^1(S)\) such that for every \(v \in \Omega\) and every nonzero tangent vector \(\xi \in T_v\mathcal{M}^1(S)\), the limit

\[
\lambda(\xi) := \lim_{t \to \pm \infty} \frac{1}{t} \log \|D_\nu \varphi_t(\xi)\|
\]

exists and is finite. The real number \(\lambda(\xi)\) is called the Lyapunov exponent of \(\varphi_t\) at \(\xi\). Observe that if \(\xi\) is in the line bundle \(\mathbb{R}\hat{\varphi}(v)\) tangent to the orbits of the flow, then \(\lambda(\xi) = 0\). We say that \(\varphi_t\) is *nonuniformly hyperbolic* if for almost every \(v \in \Omega\) and every \(\xi \in T_v\mathcal{M}^1(S) \setminus \mathbb{R}\hat{\varphi}(v)\), the Lyapunov exponent \(\lambda(\xi)\) is nonzero.

Using the fact that the WP sectional curvatures are negative, we prove that the WP geodesic flow is nonuniformly hyperbolic. Nonuniform hyperbolicity is the starting point for a rich ergodic theory of volume-preserving diffeomorphisms and flows, developed first by Pesin for closed manifolds, and expanded by Sinai, Katok-Strelcyn, Chernov and others to systems with singularities, such as the WP geodesic flow. The basic argument for establishing ergodicity of such systems originates with Eberhard Hopf and his proof of ergodicity for geodesic flows for closed, negatively curved surfaces [13]. His method was to study the Birkhoff averages of continuous functions along leaves of the stable and unstable foliations of the flow. This type of argument has been used since then in increasingly general contexts, and has come to be known as the Hopf Argument.

The core of the Hopf Argument is very simple. Suppose that \(\psi_t\) is a \(C^\infty\) flow defined on a full measure subset \(\Omega\) of an open Riemannian manifold \(V\), preserving a finite volume on \(V\). For any \(x \in \Omega\) one defines the stable set:

\[
\mathcal{W}^s(x) = \{x' \in \Omega : \lim_{t \to -\infty} d(\psi_t(x), \psi_t(x')) = 0\},
\]
and the unstable set
\[ W^u(x) = \{ x' \in \Omega : \lim_{t \to -\infty} d(\psi_t(x), \psi_t(x')) = 0 \}. \]
The stable (respectively unstable) sets partition \( \Omega \) into measurable subsets.

The first step in the Hopf Argument is to observe that for any continuous function \( f : V \to \mathbb{R} \) with compact support, the forward upper Birkhoff average
\[ f^+ = \limsup_{T \to +\infty} \frac{1}{T} \int_0^T f \circ \psi_t \, dt \]
is constant on any stable set \( W^s(x) \), and the the backward upper Birkhoff average
\[ f^- = \limsup_{T \to -\infty} \frac{1}{T} \int_0^T f \circ \psi_t \, dt \]
is constant on any unstable set \( W^u(x) \). Both functions \( f^+ \) and \( f^- \) are evidently invariant under the flow \( \psi_t \), and the Birkhoff and von Neumann Ergodic Theorems imply that \( f^+ = f^- \) almost everywhere. To show that \( \psi_t \) is ergodic it suffices to show that \( f^+ \) is constant almost everywhere, for every continuous \( f \) with compact support. The fundamental idea is to use the properties of the equivalence relation generated by the stable sets, the unstable sets, and the flow to conclude that \( f^+ = f^- \) must be constant.

In the next step in the Hopf Argument, one assumes some form of hyperbolicity of the flow, which will imply that the stable and unstable sets are in fact smooth manifolds. In the original context of Hopf’s argument, \( V = \Omega = T^1S \) is the unit tangent bundle of a compact, negatively curved surface \( S \) and \( \psi_t \) is the geodesic flow. In this setting, the stable and unstable sets have a particularly nice description as level sets of the gradient of Busemann functions. They are \( C^\infty \), globally defined, and for \( \ast \in \{ s, u \} \), the collection
\[ W^\ast := \{ W^\ast(v) : v \in T^1S \} \]
defines a \( C^1 \) foliation of \( T^1S \). At each point \( v \in T^1S \), the tangent space \( T_vT^1S \) is spanned by the tangents to \( W^s(v), W^u(v) \) and the direction \( \dot{\psi}(v) \) of the flow. A local argument in \( C^1 \) charts using Fubini’s theorem shows that any \( \psi_t \)-invariant function that is almost everywhere constant along leaves of \( W^s \) and \( W^u \) must be locally constant, and hence globally constant, since \( T^1S \) is connected. In particular the function \( f^+ \) is constant for any continuous, compactly supported \( f \), and so \( \psi_t \) is ergodic.

Hopf’s original argument does not generalize immediately to geodesic flows for higher dimensional compact, negatively curved manifolds. In this higher-dimensional setting, the stable and unstable foliations \( W^s \) and \( W^u \) exist, are level sets of gradients of Busemann functions, and have \( C^\infty \) leaves. In general, however they fail to be \( C^1 \) foliations (except when the curvature is \( 1/4 \)-pinched) and so the argument using Fubini’s theorem in local \( C^1 \) charts fails. In the late 1960’s Anosov [1] overcame this obstacle by proving that for any compact, negatively curved manifold, the foliations \( W^s \) and
$W^u$ are absolutely continuous. Absolute continuity, a strictly weaker property than $C^1$, is sufficient to carry out a Fubini-type argument to show that any $\psi_t$-invariant function almost everywhere constant along leaves of $W^s$ and $W^u$ is locally constant. See Section 6 for a more detailed discussion of absolute continuity. Anosov thereby proved that the geodesic flow for any compact manifold of negative sectional curvatures is ergodic.

There is an extensive literature devoted to extending the Hopf Argument beyond the uniformly hyperbolic setting of geodesic flows on compact negatively curved manifolds. For smooth flows defined everywhere on compact manifolds, Pesin [26] introduced an ergodic theory of nonuniformly hyperbolic systems. In short, Pesin theory shows that if $\psi_t: V \to V$ preserves a finite volume and is nonuniformly hyperbolic, then almost everywhere the stable and unstable sets are smooth manifolds. The family of stable manifolds is measurable and absolutely continuous in a suitable sense. From Pesin theory, one deduces that a nonuniformly hyperbolic diffeomorphism of a compact manifold has countably many ergodic components of positive measure. More information about the flow can be used in some contexts to deduce ergodicity. The obstruction to using the full Hopf Argument in this setting is that stable manifolds are defined only almost everywhere, and they may be arbitrarily small in diameter, with poorly controlled curvatures, etc.

In a somewhat different direction than Pesin theory, Sinai [32] introduced methods for proving ergodicity of hyperbolic flows with singularities and applied them in his study of the $n$-body problem of celestial mechanics. Here the flow $\psi_t$ locally resembles the geodesic flow for a compact, negatively curved manifold, but globally encounters discontinuities and places where the norms of the derivatives $\|D\psi_t\|$ and $\|D^2\psi_t\|$ become unbounded.

Introducing new techniques in the Hopf argument, Sinai was able to show that for several important classes of systems, including some billiards and flows connected to the $n$-body system, ergodicity holds. These arguments have since been generalized to much larger classes of singular hyperbolic systems and singular nonuniformly hyperbolic systems.

In the singular nonuniformly hyperbolic setting, all aspects of Hopf’s argument require careful revisiting. The mere existence of local stable manifolds is a delicate matter and depends in a strong way on the growth of the derivative of $\psi_t$ near the singularities. To give a sense of how delicate these issues can be, we remark that:

- for compact surfaces of nonpositive curvature and genus $g \geq 2$, it is unknown whether the geodesic flow is always ergodic (even though it is always transitive)
- there exist complete, finite volume surfaces of pinched negative curvature (but unbounded derivative of curvature) whose stable foliations are not Hölder continuous [3];
- for $C^1$ nonuniformly hyperbolic systems that are not $C^2$, stable sets can fail to be manifolds [29];
• nonuniformly hyperbolic systems on compact manifolds can fail to be ergodic and can even have infinitely many ergodic components with positive measure [10].

A general result providing for the existence and absolute continuity of local stable and unstable manifolds for singular, nonuniformly hyperbolic systems was proved by Katok-Strelcyn [17]. We will use this work in an important way in this paper.

Returning to the context of the present paper, the WP geodesic flow is a singular, nonuniformly hyperbolic system. To prove that it is ergodic, the first step is to verify the Katok-Strelcyn conditions to establish existence and absolute continuity of local stable and unstable manifolds. In particular, one needs to control the norm of the first two derivatives of the geodesic flow in a neighborhood of the boundary of $M^1(S)$. To control the first derivative, we use the asymptotic expansions of Wolpert for the WP curvature and covariant derivative found in [36, 35, 37], combined with a careful analysis of the solutions to the WP Jacobi equations. This is the content of Theorem 3.1. The precise estimates obtained by Wolpert appear to be essential for these calculations.

Since Wolpert’s expansions of the WP metric are only to second order, and we need third order control to estimate the second derivative of the flow, we borrow ideas of McMullen in [21]. There is a non holomorphic (in fact totally real) embedding of Teich$(S)$ into quasifuchsian space $QF(S)$, under which the WP symplectic form has a holomorphic extension. This holomorphic form is the derivative of a one-form that is bounded in the Teichmüller metric. Using the Cauchy Integral Formula and a comparison formula between Teichmüller and WP metrics, one can then obtain bounds on all derivatives of the WP metric. This is the content of Proposition 5.1. These bounds are adequate to control the second derivative of the geodesic flow, using the bounds on the first derivative already obtained.

Once the conditions of [17] have been verified, we are guaranteed the almost everywhere existence of absolutely continuous families $W^s$ and $W^u$ of stable and unstable manifolds. These families are only measurable and a priori their local geometry can be highly nonuniform. At this point, we use negative curvature and another key property of the WP metric called geodesic convexity to show that in fact $W^s$ and $W^u$ have well-controlled local geometry. As a by-product of our arguments, we obtain that the WP Busemann function for almost every tangent direction to Teich$(S)$ is $C^\infty$ (see Proposition 6.10). The local geometry of $W^s$ and $W^u$ is sufficiently nice that Hopf’s original argument can be used with small modifications. In particular, none of the more complicated local ergodicity arguments, such as the “Hopf chains” developed by Sinai, are necessary. We also obtain positive, finite entropy of the WP flow using results of Katok-Strelcyn and Ledrappier-Strelcyn in [17].
The paper does not follow the structure of this outline exactly. Rather than restricting to the special case of the WP metric, we instead develop an abstract criterion for ergodicity of the geodesic flow for an incomplete, negatively curved manifold. This is carried out in Section 6, which may be read independently of the rest of the paper. The remainder of this paper is devoted to setting up and verifying the conditions in Section 6.

0.1. The case of the punctured torus. Several interesting features of the WP metric are already present in the simplest cases \((g, n) = (1, 1)\) and \((0, 4)\). When \(S\) is the once-punctured torus, \(\text{Teich}(S)\) is the upper half space \(\mathbb{H}\), the Teichmüller metric \(d_T\) is the hyperbolic metric, and \(\mathcal{M}(S)\) is the classical moduli space of elliptic curves \(\mathbb{H}/\text{PSL}(2, \mathbb{Z})\), which is a sphere with one puncture and two cone singularities of order 2 and 3.

The mapping class group \(\text{MCG}(S)\) is the modular group \(\text{SL}(2, \mathbb{Z})\). Due to the presence of torsion elements in \(\text{PSL}(2, \mathbb{Z})\), the space \(\mathcal{M}(S)\) is not a manifold, but the finite branched cover \(\mathbb{H}/\Gamma[k]\), for \(k \geq 3\) is a manifold, where \(\Gamma[k]\) is the level-\(k\) congruence subgroup

\[
\Gamma[k] = \{ A \in \text{PSL}(2, \mathbb{Z}) \mid A \equiv I \mod k \}.
\]

The tangent bundle to \(\text{Teich}(S)\) is canonically identified with \(\text{PGL}(2, \mathbb{R})\), and the unit tangent bundle in the Teichmüller metric with \(\text{PSL}(2, \mathbb{R})\).

There are global coordinates \((\ell, \tau)\) in \(\text{Teich}(S)\), the so-called Fenchel-Nielsen coordinates, which have the asymptotic (first-order) expansions

\[
\ell(z) \sim \frac{1}{\text{Im}(z)}, \quad \text{and} \quad \tau(z) \sim \frac{\text{Re}(z)}{\text{Im}(z)}, \quad \text{as} \quad \text{Im}(z) \to \infty,
\]

and the WP form has the first-order asymptotic expansion

\[
\omega_{WP} = \frac{1}{2} d\ell \wedge d\tau \sim \frac{1}{\text{Im}(z)^3} dz \wedge d\bar{z}, \quad \text{as} \quad \text{Im}(z) \to \infty.
\]

Since the complex structure on \(\text{Teich}(S)\) is the standard one on \(\mathbb{H}\), we obtain the expansion

\[
g_{WP}^2 \sim \frac{|dz|^2}{\text{Im}(z)^3}.
\]

By contrast,

\[
g_{T}^2 = \frac{|dz|^2}{\text{Im}(z)^2}.
\]

A neighborhood of the cusp in \(\mathcal{M}(S)\) is formed by taking the quotient of the points above the line \(\text{Im}(z) = \text{Im}(z_0)\), for \(\text{Im}(z_0)\) sufficiently large, by the mapping class element \(z \mapsto z + 1\). A model for this neighborhood is the surface of revolution \(\{ y = x^3 : x > 0 \}\). From the form of the metric one can see the incompleteness: a vertical ray to the cusp at infinity has finite length. Nonetheless the metric is convex. Moreover the curvature goes to \(-\infty\) as \(\text{Im}(z) \to \infty\).
Pollicott and Weiss [28] studied the model case of a negatively curved surface whose singularities coincide with a surface of revolution and proved ergodicity of the geodesic flow in this case.

0.2. A word about notation. Throughout the paper, we have strived to minimize confusion in notation, but because we refer heavily in places to the work of different authors and are combining techniques from disparate areas, inevitable conflicts arise. Among the most notable are:

- the use of $\sigma$ to denote a section of the bundle of projective structures (see §1.3) a simplex in the curve complex (see §1.6), and a geodesic (see §4);
- the use of $\text{Teich}(\mathcal{S})$ to denote the Teichmüller space of the conjugate Riemann surface $\overline{\mathcal{S}}$ (see §1.4) and of $\overline{\text{Teich}}(\mathcal{S})$ to denote augmented Teichmüller space (see §1.6)
- the use of $\alpha, \beta, \kappa$ and other Greek letters to denote, variously, a simple closed curve, a variation of geodesics, a real number, a bundle, a connection, a differential form and a map.

The notation is locally consistent, so it should be clear from the context which use is intended; the reader should remain alert to these distinctions in turning from one part of the paper to another.

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1. Background on Teichmüller theory, Quasifuchsian space, and Weil–Petersson geometry

Much of the discussion in this section is based on McMullen’s paper [21]. Useful background can be found in [25] and the course notes [20].

1.1. Riemann surfaces and tensors of type $(r, s)$. We begin with some preliminary facts about Riemann surfaces. A Riemann surface is a topological surface equipped with an atlas of charts into $\mathbb{C}$ with holomorphic transition maps. Suppose that $X$ is a Riemann surface of genus $g$ with $n$ punctures. Uniformization implies that $X$ is conformally equivalent to a quotient $\mathbb{H}/\Gamma$, where $\mathbb{H}$ denotes the upper half plane, and $\Gamma$ is a discrete subgroup of $PSL(2, \mathbb{R})$. The hyperbolic metric $\rho$ on $\mathbb{H}$ given by

$$\rho(z) = \frac{|dz|}{\text{Im} z}$$

descends to a metric $\rho$ on $\mathbb{H}/\Gamma$ of finite area, which is the unique Riemannian metric of constant curvature $-1$ on $X$ that induces the same conformal structure.

Denote by $\kappa$ the holomorphic cotangent bundle and by $\kappa^{-1}$ the holomorphic tangent bundle of $X$, both of which are holomorphic complex line
bundles over $X$. For $r$ an integer, we denote by $\kappa^r$ the $|r|$-fold complex tensor product $\otimes^{|r|} \kappa$, if $r \geq 0$, and $\otimes^{|r|} \kappa^{-1}$ if $r < 0$.

A tensor of type $(r,s)$ on $X$ is a section of the complex line bundle $\kappa^r \otimes \bar{\kappa}^s$ over $X$. Equivalently, a tensor of type $(r,s)$ is the assignment of a function $\beta(z)$ to each local holomorphic chart $z$ on $X$ such that $\beta(z) dz^r d\bar{z}^s$ is invariant under local holomorphic transition maps. Thus, if $w$ is another holomorphic coordinate, and $\hat{\beta}(w)$ is the representing function, one has:

$$\hat{\beta}(w) = \beta(z(w)) \left( \frac{dz}{dw} \right)^r \left( \frac{d\bar{z}}{dw} \right)^s.$$  

Since $\kappa^r \otimes \bar{\kappa}^s$ has a real analytic structure, it is natural to speak of measurable or smooth sections; in the special case where $s = 0$, the bundle is complex analytic and so the notion of holomorphic or meromorphic sections of $\kappa^r$ is also well-defined.

The sum, product, conjugate and modulus of $(r,s)$ tensors are defined in the obvious way. Observe that the square $\rho^2$ of the hyperbolic metric on $X$ is a tensor of type $(1,1)$ and that the modulus of any measurable tensor $\psi$ on $X$ of type $(1,1)$ has a well-defined integral $\int_X |\psi|$, which can be infinite. More generally if $\psi$ is an $(r,s)$-tensor and $p \geq 1$, then $\rho^{2-p(r+s)} |\psi|^p$ is a $(1,1)$-tensor. These observations lead to the construction of $L^p$ norms on the space of measurable $(r,s)$ tensors, defined as follows; for $\psi$ an $(r,s)$ tensor, and $p \geq 1$ we define:

$$\|\psi\|_p := \left( \int_X \rho^{2-p(r+s)} |\psi|^p \right)^{1/p} \quad \|\psi\|_\infty := \text{ess sup}_X \rho^{-(r+s)} |\psi|.$$  

These norms will give rise to the Teichmüller and WP metrics on Teichmüller space, which we now define.

1.2. **Teichmüller and Moduli spaces.** A marked complex structure is a Riemann surface $X$ together with a homeomorphism $f: S \to X$, where $S$ is a fixed Riemann surface. Given a marking surface $S$ of genus $g$ with $n$ punctures, we define the Teichmüller space $\text{Teich}(S)$ to be the set of equivalence classes of marked complex structures $f: S \to X$ and $f_2: S \to X_2$ are equivalent if there is a holomorphic map $h: X_1 \to X_2$ isotopic to $f_2 f_1^{-1}$.

The Teichmüller distance $d_T$ between equivalence classes $[f_1: S \to X_1]$ and $[f_2: S \to X_2]$ is defined to be $\frac{1}{2} \log K$, where $K$ is the minimum quasiconformal dilatation of a quasiconformal homeomorphism between $X_1$ and $X_2$ isotopic to $f_2 f_1^{-1}$:

$$K = \inf_{h \simeq f_1^{-1}} \sup_{X_1} \frac{|\partial h/\partial z| + |\partial h/\partial \bar{z}|}{|\partial h/\partial z| - |\partial h/\partial \bar{z}|}.$$  

1 We remark that every equivalence class $[f: S \to X]$ can itself be represented by a quasiconformal homeomorphism $f: S \to X$ of minimal dilatation.
With respect to the the Teichmüller distance, Teich($S$) is a complete metric space, homeomorphic to $\mathbb{R}^{6g-6+2n}$. Uniformization gives an identification of Teich($S$) with an open component of the representation variety of homomorphisms from $\pi_1(S)$ into the real Lie group $PSL(2,\mathbb{R})$, modulo conjugacy; this identification gives Teich($S$) a real analytic structure. Teich($S$) also carries a compatible complex analytic structure, which we shall describe a little later.

The mapping class group $MCG(S)$ is the set of equivalence classes of orientation preserving diffeomorphisms of $S$ modulo isotopy, which forms a discrete group under composition. $MCG(S)$ acts properly by diffeomorphisms of Teich($S$) via precomposition with the marking homeomorphisms $f: S \to X$; the quotient $\mathcal{M}(S) = \text{Teich}(S)/MCG(S)$ is easily seen to be the moduli space of Riemann surfaces homeomorphic to $S$, modulo conformal equivalence. The $MCG(S)$-stabilizer of any point in Teich($S$) is finite. In denoting an element of Teich($S$), we will often omit the marking given by the equivalence class of maps $f: S \to X$ and refer only to the target Riemann surface $X$.

Fix an element $X$ of Teich($S$); we review the definition of the Teichmüller and Weil-Petersson norms on the tangent and cotangent spaces $T^*_X\text{Teich}(S)$ and $T_X\text{Teich}(S)$. These definitions are made by identifying the cotangent and tangent spaces with appropriate spaces of differentials on $X$, as we now explain.

A holomorphic quadratic differential on $X$ is a holomorphic tensor of type $(2, 0)$, which has a local representation of the form $q(z)dz^2$. We define $Q(X)$ to be the vector space of holomorphic quadratic differentials $\phi$ on $X$ having at most simple poles at the punctures of $X$. The Riemann-Roch theorem implies that $Q(X)$ is a vector space of complex dimension $3g - 3 + n$. Three equivalent characterizations of the holomorphic quadratic differentials $\phi$ that are in $Q(X)$ are:

- $\|\phi\|_p = \left(\int_X \rho^{-2p}|\phi|^p\right)^{1/p} < \infty$ for some $p \geq 1$;
- $\|\phi\|_p = \left(\int_X \rho^{-2p}|\phi|^p\right)^{1/p} < \infty$ for all $p \geq 1$;
- $\|\phi\|_{\infty} = \text{ess sup}_X \rho^{-2}|\phi| < \infty$.

Moreover, since $Q(X)$ is finite dimensional, all of these norms are equivalent, but there is no uniform comparison between these norms that holds for all $X$. This means the appropriate norm to consider on $Q(X)$ may depend on the context. We will be particularly interested in the cases $p = 1$, which will arise in defining the Teichmüller norm, $p = 2$, which will arise in defining the WP norm and $p = \infty$, which will arise in the statement of Nehari’s theorem. The space $Q(X)$ is naturally paired with the space of essentially bounded Beltrami differentials, which we next define.

\[\text{We remark that if } X \text{ is a surface with infinite hyperbolic area, then } Q(X) \text{ is not finite dimensional, and these characterizations are not equivalent.}\]
A Beltrami differential on $X$ is a measurable tensor of type $(-1, 1)$, which has a local representation of the form $b(z) d\bar{z}/dz$. Note that the product of a Beltrami differential with a quadratic differential is a $(1, 1)$-tensor. Let $M(X)$ be the vector space of all measurable Beltrami differentials $\mu$ on $X$ with the property that $\int_X |\phi\mu| < \infty$, for every $\phi \in Q(X)$. We then have a natural complex pairing of the space $M(X)$ with $Q(X)$ given by

$$\langle \phi, \mu \rangle = \int_X \phi \mu \text{ for } \phi \in Q(X), \mu \in M(X).$$

In view of the fact that elements of $Q(X)$ have finite $L^p$ norm for every $1 \leq p \leq \infty$, it follows that elements of $M(X)$ are precisely those Beltrami differentials $\mu$ on $X$ of finite $L^q$ norm, for $1 \leq q \leq \infty$.

The Measurable Riemann Mapping Theorem states that any measurable Beltrami differential $\mu \in M(X)$ with $\|\mu\|_\infty < 1$, there exist a Riemann surface $X_\mu$ and a quasiconformal homeomorphism $h_\mu: X \to X_\mu$ such that, with $\mu$ expressed in local coordinates as $b(z) d\bar{z}/dz$, we have:

$$\frac{\partial h_\mu}{\partial \bar{z}} \frac{\partial h_\mu}{\partial z} = b(z).$$

The map $h_\mu$ is called a solution to the Beltrami equation and is unique up to composition with a conformal isomorphism. Moreover there is analytic dependence on parameters, so that if $f: S \to X$ is a quasiconformal marking of $X$ in $\text{Teich}(S)$, and $\|\mu\|_\infty = 1$, then $t \mapsto (h_{t\mu} \circ f: S \to X_{t\mu})$ describes an analytic path in $\text{Teich}(S)$ through $X$ at $t = 0$. One can show that two Beltrami differentials $\mu, \nu \in M(X)$ of unit $L^\infty$ norm describe paths that are tangent in $\text{Teich}(S)$ at $t = 0$ if and only if $\langle \mu, \phi \rangle = \langle \nu, \phi \rangle$, for every $\phi \in Q(X)$. This gives rise to the fundamental isomorphisms of vector spaces:

$$T_X \text{Teich}(S) \cong M(X)/Q(X)^\perp \quad \text{and} \quad T^*_X \text{Teich}(S) \cong Q(X),$$

where $Q(X)^\perp = \{\mu \in M(X) : \langle \mu, \phi \rangle = 0, \forall \phi \in Q(X)\}$.

Having described these identifications, we now can define the Teichmüller and WP norms. The Teichmüller distance $d_T$ on $\text{Teich}(S)$ described above is the path metric for a Finsler structure $\|\cdot\|_T$ whose restriction to $T^*_X \text{Teich}(S)$ is the $L^1$ norm on $Q(X)$:

$$\|\phi\|_T = \|\phi\|_1 = \int_X |\phi|.$$

The Teichmüller norm $\|\phi\|_T$ encodes information about variations of the conformal structure $X$ induced by the covector $\phi$. The Weil-Petersson metric is defined by the $L^2$ norm:

$$\|\phi\|_{WP} = \|\phi\|_2 = \left(\int_X \rho^{-2}|\phi|^2\right)^{1/2}.$$

Note that the definition of the WP metric involves both conformal and hyperbolic data from $X$; this feature makes the WP metric somewhat tricky to work with. On the other hand, the hyperbolic input from the metric $\rho$
leads to the delicate and beautiful connections between the WP metric and hyperbolic geometry and dynamics discussed in the introduction.

The Teichmüller and WP norms on the tangent space $T_X\text{Teich}(S)$ are then induced by this pairing via the formulae:

$$
\|v\|_T = \sup_{\phi \in Q(X), \|\phi\|_T=1} \text{Re}(\langle \phi, \mu \rangle); \quad \|v\|_{WP} = \sup_{\phi \in Q(X), \|\phi\|_{WP}=1} \text{Re}(\langle \phi, \mu \rangle),
$$

for any $\mu \in M(X)$ representing the tangent vector $v \in T_X\text{Teich}(S)$. By duality of the appropriate $L^p$ norms, it follows that the Teichmüller norm is induced by the $L^\infty$ norm on $M(X)$ and the WP norm is induced by the $L^2$ norm:

$$
\|v\|_T = \|\mu\|_\infty; \quad \|v\|_{WP} = \|\mu\|_2 = \left(\int_X \rho^2|\mu|^2\right)^{1/2},
$$

for any $\mu \in M(X)$ representing $v \in T_X\text{Teich}(S)$. The Teichmüller and WP norms are both invariant under the derivative action of $\text{MCG}(S)$.

A simple comparison between the WP and Teichmüller norms follows from the Cauchy-Schwarz and Gauss-Bonnet Theorems (see Prop 2.4 in [21]):

$$
\|v\|_{WP} \leq |2\pi \chi(S)|^{1/2}\|v\|_T,
$$

for any $X \in \text{Teich}(S)$ and $v \in T_X\text{Teich}(S)$. We prove in Section 5.1 a complementary bound: there exists $C > 0$ such that

$$
\|v\|_{WP} \geq \ell(X)^{-1/2}\|v\|_T,
$$

for any $v \in T_X\text{Teich}(S)$, where $\ell(X)$ denotes the hyperbolic length of the shortest closed geodesic on $X$, also known as the systole of $X$. This lower bound will be used to estimate from above the derivatives of the WP metric.

1.3. The bundle of projective structures on $S$. A projective structure on a surface $X$ is an atlas of charts into $\mathbb{C}$ whose overlaps are Möbius transformations (elements of $\text{PSL}(2, \mathbb{C})$); note that a projective structure determines a unique complex structure. Fix as above a Riemann surface $S$ of genus $g$ with $n$ punctures. A marked projective structure is a homeomorphism $f: S \to X$, where $X$ is endowed with a projective structure; we say that two marked structures $f_1: S \to X_1$ and $f_2: S \to X_2$ are equivalent if there is a projective isomorphism from $X_1$ to $X_2$ homotopic to $f_2f_1^{-1}$. Denote by $\text{Proj}(S)$ the space of equivalence classes of projective structures marked by $S$.

It is a classical fact that $\text{Proj}(S)$ has the structure of a complex manifold that arises from its embedding into the representation variety of homomorphisms from $\pi_1(S)$ into $\text{PSL}(2, \mathbb{C})$, modulo conjugacy (see [14]). The map that assigns to each marked projective structure the compatible marked conformal structure defines a fibration $\pi: \text{Proj}(S) \to \text{Teich}(S)$. The fiber $\text{Proj}_X(S)$ over $X$ is an affine space modelled on $Q(X)$. That is, given $X_0 \in \text{Proj}_X(S)$ and $\phi \in Q(X)$ there is a unique $X_1 \in \text{Proj}_X(S)$ and a
conformal map \( f : X_0 \to X_1 \) respecting markings such that \( Sf = \phi \). Here \( Sf \) is the Schwarzian derivative

\[
Sf(z) = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 dz^2.
\]

In particular, there is a well-defined difference \( \beta_1 - \beta_2 \in Q(X) \), for \( \beta_1, \beta_2 \in \text{Proj}_X(S) \) that defines a holomorphic map from \( \text{Proj}_X(S) \times \text{Proj}_X(S) \) to \( Q(X) \).

In the next subsection we describe how to define a complex structure on \( \text{Teich}(S) \) that gives \( \text{Proj}(S) \) the structure of a holomorphic fiber bundle, and a holomorphic identification of \( \text{Proj}(S) \) with the cotangent bundle \( T^* \text{Teich}(S) \). We shall also describe how to construct a holomorphic section \( \sigma_{QF} \) of this bundle that arises from a process called quasifuchsian uniformization. For now we note that the Fuchsian uniformization \( X \cong \mathbb{H}/\Gamma \) defines a section \( \sigma_F : \text{Teich}(S) \to \text{Proj}(S) \). This section is real analytic but not holomorphic.

### 1.4. Quasifuchsian space.

Let \( S = \mathbb{H}/\Gamma \) be a hyperbolic Riemann surface with \( \Gamma < \text{PSL}(2, \mathbb{R}) \), and denote by \( \overline{S} \) the hyperbolic Riemann surface \( \mathbb{L}/\Gamma \), where \( \mathbb{L} \) is the lower half plane. Since \( \Gamma \) is a Fuchsian group, it acts on the Riemann sphere \( \hat{\mathbb{C}} \) fixing \( \mathbb{H}, \mathbb{L} \) and the real axis/circle at infinity \( \mathbb{R}_\infty = \hat{\mathbb{C}} \setminus (\mathbb{H} \cup \mathbb{L}) \). Following McMullen [21], we define quasifuchsian space \( QF(S) \) to be the product:

\[
QF(S) = \text{Teich}(S) \times \text{Teich}(\overline{S}).
\]

Then \( QF(S) \) parametrizes marked quasifuchsian groups equivalent to \( \Gamma(S) \); more precisely we have:

**Theorem 1.1** (Bers Simultaneous Uniformization). For each pair \([f : S \to X], [g : \overline{S} \to Y]\) \( \in QF(S) \), there exist a quasiconformal homeomorphism \( \phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \), a Kleinian group \( \Gamma(X,Y) < \text{PSL}(2, \mathbb{C}) \), and a homomorphism \( h : \Gamma \to \Gamma(X,Y) \) (all unique up to the action of \( \text{PSL}(2, \mathbb{C}) \)) with the following properties:

1. for each \( \gamma \in \Gamma \), we have \( \phi \circ \gamma = h(\gamma) \circ \phi \);
2. \( \phi \) sends \( (\mathbb{H} \cup \mathbb{L}, \mathbb{R}_\infty) \) to \( (\Omega(X,Y), \Lambda(X,Y)) \), where \( \Lambda(X,Y) \subset \hat{\mathbb{C}} \) is a quasicircle;
3. there is a conformal isomorphism \( \Omega(X,Y)/\Gamma(X,Y) \cong (X \cup Y) \) making the following diagram commute:

\[
\begin{array}{ccc}
\mathbb{H} \cup \mathbb{L} & \xrightarrow{\phi} & \Omega(X,Y) \\
\mod \Gamma \downarrow & & \downarrow \mod \Gamma(X,Y) \\
S \cup \overline{S} & \xrightarrow{f \cup g} & (X \cup Y)
\end{array}
\]
We define the Fuchsian locus $F(S)$ to be the image of $\text{Teich}(S)$ under the antidiagonal embedding $\tilde{\alpha}(X) = (X, \overline{X}) \in QF(S)$. Theorem 1.1 gives a natural “quasifuchsian uniformization” map

$$
\sigma: \text{Teich}(S) \times \text{Teich}(\overline{S}) \to \text{Proj}(S) \times \text{Proj}(\overline{S})
$$

that sends $(X, Y)$ to the projective structures on $X$ and $Y$ inherited from $\Omega(X,Y)$. We write:

$$
\sigma(X,Y) = (\sigma_{QF}(X,Y), \sigma_{QF}(X,Y)).
$$

Observe that $\sigma_{QF}(X,X) = \sigma_{F}(X)$. The complex structure on $\text{Teich}(S)$ is then defined via the Bers embedding: fixing $X \in \text{Teich}(S)$, we define

$$
\beta_{X}: \text{Teich}(S) \to Q(X) \text{ by } \beta_{X}(Y) = \sigma_{QF}(X,Y) - \sigma_{F}(X).
$$

The map $\beta_{X}$ is an embedding, and the pullback of the complex structure on $Q(X)$ gives a complex structure on $\text{Teich}(S)$ that is independent of $X$ (that is, two different $X$s give isomorphic structures). Recall that $Q(X)$ is a Banach space when endowed with any $L^{p}$ norm. A fundamental result of Nehari implies that the image of $\text{Teich}(S)$ under $\beta_{X}$ is a bounded subset of $Q(X)$ in the $L^\infty$ norm, where the bound is independent of $X$. This fact is used in a crucial way in McMullen’s paper and in ours.

We have defined a complex structure on $\text{Teich}(S)$, which induces a conjugate complex structure on $\text{Teich}(\overline{S})$. The complex structure on $QF(S)$ is defined to be the product complex structure. The Fuchsian locus $F(S)$ is then a totally real submanifold of $QF(S)$. It can be checked that the fibration $\text{Proj}(S) \to \text{Teich}(S)$ is holomorphic with respect to these structures. Furthermore, the naturally-defined section $\sigma: QF(S) \to \text{Proj}(S) \times \text{Proj}(\overline{S})$ is holomorphic. Hence, for a fixed $Y \in \text{Teich}(S)$, the map $X \mapsto \sigma_{QF}(X,Y)$ gives a holomorphic section of $\text{Proj}(S)$ over $\text{Teich}(S)$; this section gives an isomorphism between $\text{Proj}(S)$ and the cotangent bundle $T^{*}\text{Teich}(S)$.

We will use the section $\sigma$ in a crucial way to estimate higher derivatives of the $WP$ metric. Some important properties of $\sigma$, which we record at this point are:

- $\sigma$ is holomorphic;
- $\sigma_{QF}(X, \overline{X}) = \sigma_{F}(X)$; where $\sigma_{F}$ is the Fuchsian uniformization section defined above;
- for any $Y, Z \in \text{Teich}(\overline{S})$, the map $X \mapsto \sigma_{QF}(X,Y) - \sigma_{QF}(X,Z)$ defines a holomorphic 1-form on $\text{Teich}(S)$;
- there exists $C \geq 1$ such that for any $X \in \text{Teich}(S)$ and $Y, Z \in \text{Teich}(\overline{S})$, we have

$$
C^{-1} \leq \|\sigma_{QF}(X,Y) - \sigma_{QF}(X,Z)\|_{T} \leq C,
$$

where $\| \cdot \|_{T}$ is the Teichmüller norm on $T^{*}_{X}\text{Teich}(S)$.

The final item on this list follows from Nehari’s bound: see Theorems 2.2 in [21].
The main result from McMullen we will use relates the Bers embedding to the Kähler form $\omega_{WP}$ of the WP metric.

**Theorem 1.2** ([21], Theorem 7.1). Fix $Z \in \text{Teich}(S)$, and let $\theta_{WP}$ be the $(1,0)$-form on $\text{Teich}(S)$ given by

$$\begin{align*}
\theta_{WP}(X) &= \sigma_F(X) - \sigma_{QF}(X, Z) \\
 &= -\beta_X(Z).
\end{align*}$$

Then $\theta_{WP}$ is a primitive for the Weil-Petersson Kähler form; that is,

$$d(i\theta_{WP}) = \omega_{WP}.$$

1.5. **Fenchel-Nielsen coordinates.** Continue to denote by $S$ a marking Riemann surface of genus $g$ with $n$ punctures. We define here a natural system of global coordinates on $\text{Teich}(S)$, called Fenchel-Nielsen coordinates, in which the Kähler form $\omega_{WP}$ takes a simple form.

Recall that a curve in $S$ is nonperipheral if it is not homotopic to a loop surrounding a single puncture. A pants decomposition of $S$ is a collection $P$ of $3g - 3 + n$ pairwise disjoint, homotopically nontrivial, nonperipheral and homotopically distinct simple closed curves. The complement of these curves is a collection of surfaces called pairs of pants. Topologically, a pair of pants is a three-times punctured sphere. A pair of pants has one of three types of conformal structure depending on whether each puncture is locally modelled on the punctured plane or on the complement of a disk, in which case we say that the boundary component is a circle. A pair of pants with $j$ boundary circles has a $j$-dimensional space of hyperbolic structures, parametrized by the hyperbolic lengths of the boundary circles.

We introduce notation that will be used throughout the paper. If $f: S \to X$ is a marked Riemann surface and $\alpha$ is a homotopically nontrivial, nonperipheral, simple closed curve in $S$, we denote by $\ell_\alpha(X)$ the hyperbolic length in $X$ of the unique geodesic in the homotopy class of $f_*[\alpha]$. This geodesic length function is intimately connected with the WP metric and is used to define Fenchel-Nielsen coordinates.

Fix a pair of pants decomposition $P = \{\alpha_1, \ldots, \alpha_{3g-3+n}\}$ of $S$. The Fenchel-Nielsen coordinates

$$\ell_\alpha, \tau_\alpha)_{\alpha \in P}: \text{Teich}(S) \to (\mathbb{R}_{>0} \times \mathbb{R})^{3g-3+n}$$

determined by $P$ are defined as follows. For $f: S \to X$ a marked Riemann surface and $\alpha \in \sigma$, we define $\ell_\alpha(X)$ to be the geodesic length as above and $\tau_\alpha(X)$ to be the twist parameter, which records the relative displacement in how the pairs of pants are glued together along $\alpha$ to obtain the hyperbolic metric on $X$; more precisely, a full Dehn twist about the curve $\alpha$ changes $\tau_\alpha$ by the amount $\ell_\alpha$. One must adopt a convention for how this relative displacement $\tau$ is measured, as it is intrinsically only well-defined up to a constant, but this does not introduce any serious issues. These give global coordinates on $\text{Teich}(S)$. 
The Fenchel-Nielsen coordinates are natural with respect to the WP metric. Wolpert [34] proved that for any pants decomposition $P$, we have

$$\omega_{WP} = \frac{1}{2} \sum_{\alpha \in P} d\ell_\alpha \wedge d\tau_\alpha.$$  

A component of this formula is the important fact that the vector field $\partial/\partial \tau_\alpha$, which generates the Dehn twist flow about $\alpha$, is the symplectic gradient of the Hamiltonian function $\frac{1}{2}\ell_\alpha$:

$$\frac{1}{2} d\ell_\alpha = \omega_{WP}(\partial/\partial \tau_\alpha);$$

put another way:

$$\text{grad} \ell_\alpha = -2J \frac{\partial}{\partial \tau_\alpha}.$$  

This fundamental relationship is the starting point for many of Wolpert’s deep asymptotic expansions for the WP metric, which we discuss in more detail in Section 3.

1.6. The Deligne-Mumford compactification of moduli space. As mentioned earlier, $\text{Teich}(S)$ is incomplete with respect to the WP distance [33]. In particular, it is possible to shrink a simple closed curve $\alpha$ to a point and leave Teichmüller space along a WP geodesic in finite time — indeed, the time it takes is on the order of $\ell_\alpha^{1/2}$. It turns out that pinching one or more closed geodesics to a point is the only way to leave every compact set of $\text{Teich}(S)$ in finite time along a WP geodesic; this fact allows us to define a completion of $\text{Teich}(S)$ called augmented Teichmüller space and denoted $\overline{\text{Teich}}(S)$. The mapping class group $\text{MCG}(S)$ acts on $\overline{\text{Teich}}(S)$ and the quotient $\overline{\mathcal{M}}(S)$ is the Deligne-Mumford compactification of the moduli space $\mathcal{M}(S)$.

Augmented Teichmüller space $\overline{\text{Teich}}(S)$ is obtained by adjoining lower-dimensional Teichmüller spaces of noded Riemann surfaces, which gives it the structure of a stratified space. The combinatorics of this stratification are encoded by a simplicial complex $C(S)$ called the curve complex. We review this construction here.

We first define the curve complex $C(S)$, which is a $3g - 4 + n$ dimensional simplicial complex. The vertices of $C(S)$ are homotopy classes of homotopically nontrivial, nonperipheral simple closed curves on $S$. We join two vertices by an edge if the corresponding pair of curves has disjoint representatives. More generally, a $k$ simplex $\sigma \in C(S)$ consists of $k + 1$ distinct vertices that have disjoint representatives. We note that in the sporadic cases of the punctured torus $(g, n) = (1, 1)$ and 4-times punctured sphere $(g, n) = (0, 4)$, $C(S)$ is just an infinite discrete set of vertices, since there do not exist disjoint homotopically distinct curves on the underlying surface $S$. Except in these sporadic cases, $C(S)$ is a connected locally infinite complex.\footnote{In the sporadic cases there is more than one possible definition of $C(S)$; in another, very standard definition in these cases, one adds edges joining curves that intersect minimally (once in the case of the torus and twice in the case of the sphere). The resulting 1-complex is the Farey graph in both cases.}
Note that a maximal simplex in $\mathcal{C}(S)$ defines a pants decomposition of $S$. The mapping class group $\text{MCG}(S)$ acts on $\mathcal{C}(S)$.

Masur gave an interpretation of points in the completion of $\text{Teich}(S)$ as marked noded Riemann surfaces [23]. A **noded Riemann surface** is a complex space with at most isolated singularities, called nodes, each possessing a neighborhood biholomorphic to a neighborhood of $(0,0)$ in the curve

$$\{(z, w) \in \mathbb{C}^2 : zw = 0\}.$$  

Removing the nodes of a noded Riemann surface $Y$ yields a (possibly disconnected) Riemann surface which we will usually denote by $\hat{Y}$. The components of $\hat{Y}$ are called the **pieces** of $Y$.

Given a simplex $\sigma \in \mathcal{C}(S)$, a **marked noded Riemann surface** with nodes corresponding to $\sigma$ is a noded Riemann surface $X_\sigma$ equipped with a continuous mapping $f: S \to X_\sigma$ so that $f|_{S\setminus \sigma}$ is a homeomorphism to $\hat{X}_\sigma$. Two marked noded Riemann surfaces $[f_1: S \to X_{\sigma_1}]$ and $[f_2: S \to X_{\sigma_2}]$ are equivalent if there is a map $h: X_{\sigma_1} \to X_{\sigma_2}$ that preserves nodes, is isotopic to $f_2 \circ f_1^{-1}$ rel the nodes, and induces a biholomorphic map from $\hat{X}_{\sigma_1}$ to $\hat{X}_{\sigma_2}$.

We then define $\text{Teich}(S)^4$ to be the set of all noded and unnoded Riemann surfaces marked by $S$, modulo this equivalence, where the nodes $\sigma$ range over all simplices of $\mathcal{C}(S)$. Each element of $\text{Teich}(S)$ has a representative $[f: S \to X_\sigma]$, where either $\sigma \in \mathcal{C}(S)$ or $\sigma = \emptyset$, and we adopt the convention that $X_\emptyset$ is an unnoded Riemann surface. We will usually denote $X_\emptyset$ simply by $X$.

**Notational convention.** As with the elements of $\text{Teich}(S)$, we will frequently abuse notation and omit the marking when referring to an element of $\text{Teich}(S)$; hence "let $X_\sigma \in \text{Teich}(S)$" is shorthand for "let $[f: S \to X_\sigma]$ be an element of $\text{Teich}(S)$."

To describe a neighborhood of a point $[f: S \to X_\sigma]$ in $\text{Teich}(S)$, we give coordinates adapted to the simplex $\sigma$. For any such $\sigma$, let $P$ be maximal simplex in $\mathcal{C}(S)$ (pants decomposition) containing $\sigma$, and let $(\ell_\alpha, \tau_\alpha)_{\alpha \in P}$ be the corresponding Fenchel-Nielsen coordinates on $\text{Teich}(S)$. Then the **extended Fenchel-Nielsen coordinates** for $P$ are obtained by allowing lengths $\ell_\alpha$ to range in $\mathbb{R}_{\geq 0}$ and taking the quotient by identifying $(0, t)$ with $(0, t')$ in each $\mathbb{R}$ factor corresponding to the curves in $\sigma$.

This also defines a topology on $\overline{\text{Teich}(S)}$. We note that the space is not locally compact. A neighborhood of a noded surface allows for the twists $\tau_\alpha$ corresponding to the curves $\alpha \in \sigma$ to be arbitrary real numbers.

We introduce here a shorthand notation that will be used in various parts of the paper. If the topological type of the surface $S$ is fixed, we will sometimes write $\mathcal{T}$ for $\text{Teich}(S)$. In this notation, $\overline{\mathcal{T}}$ will then denote the augmented space $\overline{\text{Teich}(S)}$. For each $\sigma \in \mathcal{C}(S)$, we denote by $\mathcal{T}_\sigma$ the set of

---

4Not to be confused with $\text{Teich}(S)$, which was introduced in §1.4.
equivalence classes \([f: S \to X_\sigma]\) with nodes at \(\sigma\). Then

\[
\overline{T} = T \cup \bigcup_{\sigma \in \mathcal{C}(S)} T_\sigma,
\]

and we refer to \(T_\sigma\) as a stratum of \(\overline{T}\). The main stratum, \(T = T_\emptyset\), is simply the full Teichmüller space \(T\). Each boundary stratum \(T_\sigma\) is naturally identified with the product of the Teichmüller spaces of the pieces of any element \(X_\sigma\); equivalently, \(T_\sigma\) is the Teichmüller space of the Riemann surface \(\hat{X}_\sigma\).

### 2. Background on the geodesic flow

Let \(M\) be a Riemannian manifold. As usual \(\langle v, w \rangle\) denotes the inner product of two vectors and \(\nabla\) is the Levi-Civita connection defined by the Riemannian metric. The covariant derivative along a curve \(t \mapsto c(t)\) in \(M\) is denoted by \(D_{c,t}\) or simply \(\cdot\) if it is not necessary to specify the curve; if \(V(t)\) is a vector field along \(c\) that extends to a vector field \(\hat{V}\) on \(M\), we have

\[
V'(t) = \nabla_{\hat{c}(t)}\hat{V}.
\]

Given a smooth map \((s,t) \mapsto \alpha(s,t)\), we let \(\frac{D}{ds}\) denote covariant differentiation along a curve of the form \(s \mapsto \alpha(s,t)\) for a fixed \(t\). Similarly \(\frac{D}{dt}\) denotes covariant differentiation along a curve of the form \(t \mapsto \alpha(s,t)\) for a fixed \(s\). The symmetry of the Levi-Civita connection means that

\[
\frac{D}{ds} \frac{\partial \alpha}{\partial t}(s,t) = \frac{D}{dt} \frac{\partial \alpha}{\partial s}(s,t)
\]

for all \(s\) and \(t\).

The curve \(c\) is a geodesic if it satisfies the equation \(D_{c,\dot{c}}(t) = 0\). Since this equation is a first order ODE, a geodesic is uniquely determined by its initial tangent vector. Geodesics have constant speed, since we have

\[
\frac{d}{dt} \langle \dot{c}(t), \dot{c}(t) \rangle = 2 \langle \dot{c}'(t), \dot{c}(t) \rangle = 0 \text{ if } c \text{ is a geodesic}.
\]

The Riemannian curvature tensor \(R\) is defined by

\[
R(A, B)C = (\nabla_A \nabla_B - \nabla_B \nabla_A - [A, B])C.
\]

The sectional curvature of the 2-plane spanned by vectors \(A\) and \(B\) is defined by

\[
K(A, B) = \frac{\langle R(A, B)B, A \rangle}{\|A \wedge B\|^2}.
\]

### 2.1. Vertical and horizontal subspaces and the Sasaki metric

The tangent bundle \(TTM\) to \(TM\) may be viewed as a bundle over \(M\) in three
natural ways shown in the following commutative diagram:

\[
\begin{array}{ccc}
TTM & \xrightarrow{D\pi_M} & TM \\
\downarrow{\pi_{TM}} & & \downarrow{\pi_M} \\
TM & \xrightarrow{\pi_M} & M
\end{array}
\]

The first is via the composition of the natural bundle projections \(\pi_{TM} : TTM \to TM\) and \(\pi_M : TM \to M\). The second is via the composition of the derivative map \(D\pi_M : TTM \to TM\) with \(\pi_M\). The third involves a map \(\kappa : TTM \to TM\), often called the connector map, that is determined by the Levi-Civita connection. If \(\xi \in TTM\) is tangent at \(t = 0\) to a curve \(t \mapsto V(t)\) in \(TM\) \(c(t) = \pi_M(V(t))\) is the curve of footpoints of the vectors \(V(t)\), then

\[\kappa(\xi) = DcV(0)\].

The subbundle \(\ker(D\pi_M)\) is the vertical subbundle. It is naturally identified with \(TM\) via the map \(\pi_{TM}\). The subbundle \(\ker(\kappa)\) is the horizontal subbundle. It is naturally identified with \(TM\) via the map \(\kappa\) and is transverse to the vertical subbundle. If \(v \in T_pM\), we may identify \(T_vTM\) with \(T_pM \times T_pM\) via the map \(D\pi_M \times \kappa : TTM \to TM \times TM\).

Each element of \(T_vTM\) can thus be represented uniquely by a pair \((v_1, v_2)\) with \(v_1 \in T_pM\) and \(v_2 \in T_pM\). Put another way, every element \(\xi\) of \(T_vTM\) is tangent to a curve \(V : (-1, 1) \to TM\) with \(V(0) = v\). Let \(c = \pi_M \circ V : (-1, 1) \to M\) be the curve of basepoints of \(V\) in \(M\). Then \(\xi\) is represented by the pair

\[(c(0), D_cV(0)) \in T_pM \times T_pM.\]

These coordinates on the fibers of \(TTM\) restrict to coordinates on \(TT^1M\).

Regarding \(TTM\) as a bundle over \(M\) in this way gives rise to a natural Riemannian metric on \(TM\), called the Sasaki metric. In this metric, the inner product of two elements \((v_1, w_1)\) and \((v_2, w_2)\) of \(T_vTM\) is defined:

\[\langle (v_1, w_1), (v_2, w_2) \rangle = \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle.\]

This metric is induced by a symplectic form \(\omega\) on \(TTM\); for vectors \((v_1, w_1)\) and \((v_2, w_2)\) in \(T_vTM\), we have:

\[\omega((v_1, w_1), (v_2, w_2)) = \langle v_1, w_2 \rangle - \langle w_1, v_2 \rangle.\]

This symplectic form is the pull back of the canonical symplectic form on the cotangent bundle \(T^*M\) by the map from \(TM\) to \(T^*M\) induced by identifying a vector \(v \in T_pM\) with the linear function \(\langle v, \cdot \rangle\) on \(T_pM\).

Sasaki [31] showed that the fibers of the tangent bundle are totally geodesic submanifolds of \(TTM\) with the Sasaki metric. A parallel vector field along a geodesic of \(M\) (viewed as a curve in \(TM\)) is a geodesic of the Sasaki metric. Such a geodesic is orthogonal to the fibers of \(TM\). If \(v \in T_pM\) and \(v' \in T_{p'}M\), we can join them by first parallel translating \(v\) along a geodesic from \(p\) to \(p'\) to obtain \(w \in T_{p'}M\) and then moving from \(w\) to \(v'\) along a line
in $T_p'M$. If $v'$ is close to $v$, we can choose the geodesic so that its length is $d(p, p')$. Then $d_{Sas}(v, v')$ is the length of the hypotenuse of a small right angled triangle whose other two sides have lengths $d(p, p')$ and $\|w - v'\|$. Toponogov’s comparison theorem [8, Theorem 2.2] tells us that if $-k^2$ is a lower bound for the curvature of the Sasaki metric in a neighborhood of $v$, then $d_{Sas}(v, v')$ is less than the length of the hypotenuse of a right triangle in a space form of constant curvature $-k^2$ with sides of length $d(p, p')$ and $\|w - v'\|$ meeting at the right angle. It follows easily that

$$d_{Sas}(v, v') \approx d(p, p') + \|w - v'\|,$$

as $v' \to v$. The notation $a \approx b$, here and in the rest of the paper, means that the ratios $a/b$ and $b/a$ are bounded above by a constant. In this case the constant is 2.

2.2. The geodesic flow and Jacobi fields. For $v \in TM$ let $\gamma_v$ denote the unique geodesic $\gamma_v$ such that $\dot{\gamma}_v(0) = v$. The geodesic flow $\phi_t : TM \to TM$ is defined by

$$\phi_t(v) = \dot{\gamma}_v(t),$$

wherever this is well-defined. If $M$ is an open Riemannian manifold, the geodesic flow is always defined locally. Since the geodesic flow is a Hamiltonian flow, it preserves a natural volume form on $T^1M$ called the Liouville volume form. When the integral of this form is finite, it induces a unique probability measure on $T^1M$ called the Liouville measure or Liouville volume.

Consider now a one-parameter family of geodesics, that is a map $\alpha : (-1, 1)^2 \to M$ with the property that $\alpha(s, \cdot)$ is a geodesic for each $s \in (-1, 1)$. Denote by $J(t)$ the vector field

$$J(t) = \frac{\partial \alpha}{\partial s}(0, t)$$

along the geodesic $\gamma(t) = \alpha(0, t)$. Then $J$ satisfies the Jacobi equation:

$$J'' + R(J, \dot{\gamma})\dot{\gamma} = 0,$$

in which $'$ denotes covariant differentiation along $\gamma$. Since this is a second order linear ODE, the pair of vectors $(J(0), J'(0)) \in T_{\gamma(0)}M \times T_{\gamma(0)}M$ uniquely determines the vectors $J(t)$ and $J'(t)$ along $\gamma(t)$. A vector field $J$ along a geodesic $\gamma$ satisfying the Jacobi equation is called a Jacobi field.

The pair $(J(0), J'(0))$ corresponds in the manner described above to the tangent vector at $s = 0$ to the curve $V(s) = \frac{\partial \alpha}{\partial t}(s, 0)$. To see this, note that $V(s)$ is a vector field along the curve $c(s) = \alpha(s, 0)$, so $V'(0)$ corresponds to the pair

$$(\dot{c}(0), D_c \frac{\partial \alpha}{\partial t}(s, 0)) = (J(0), D \frac{\partial \alpha}{\partial s} \frac{\partial \alpha}{\partial t}(s, 0)) = (J(0), \frac{\partial \alpha}{\partial t} \frac{\partial \alpha}{\partial s}(s, 0)) = (J(0), J'(0)).$$

In the same way one sees that $(J(t), J'(t))$ corresponds to the tangent vector at $s = 0$ to the curve $s \mapsto \frac{\partial \alpha}{\partial t}(s, t) = \phi_t \circ V(s)$, which is $D\phi_t(V'(0))$. 


To summarize the preceding discussion, there is a one-one correspondence between elements of $T_v TM$ and Jacobi fields along the geodesic $\gamma$ with $\dot{\gamma}(0) = v$.

Note that the pair $(J(t), J'(t))$ defines a section of $TTM$ over $\gamma(t)$. We have the following key proposition:

**Proposition 2.1.** The image of the tangent vector $(v_1, v_2) \in T_v TM$ under the derivative of the geodesic flow $D_v \phi_t$ is the tangent vector $(J(t), J'(t)) \in T_{\phi_t(v)} TM$, where $J$ is the unique Jacobi field along $\gamma$ satisfying $J(0) = v_1$ and $J'(0) = v_2$.

2.3. The Jacobi and Riccati equations. Choose an orthonormal basis $e_1 = \dot{\gamma}(0), \ldots, e_n$ at $0$ for the tangent space at $\gamma(0)$ and parallel transport the basis along $\gamma(t)$. Any Jacobi field can be written in terms of the basis as $J(t) = \sum y^k e_k(t)$ and so the Jacobi equation can be written as

\[
\frac{d^2 y^k}{dt^2} = \sum_j y^j(t) \langle R(e_j(t), e_1(t)) e_1(t), e_k(t) \rangle.
\]

A choice of initial condition $y^k(0), y^k'(0)$ determines a solution.

Let $\mathcal{J}(t)$ denote any matrix of solutions to the Jacobi equation. Define the Wronskian of $\mathcal{J}$ to be the matrix

\[
W(\mathcal{J}, \mathcal{J}) = (\mathcal{J}^*)^t \mathcal{J} - \mathcal{J}^* \mathcal{J}',
\]

where $*$ denotes transpose. Note that by the Jacobi equation,

\[
W'(\mathcal{J}, \mathcal{J}) = (\mathcal{J}^*)^t \mathcal{J} - \mathcal{J}^* \mathcal{J}'' = (R\mathcal{J})^* \mathcal{J} - \mathcal{J}^* R\mathcal{J} = 0,
\]

since $R$ is symmetric.

Let

\[
U = \mathcal{J}' \mathcal{J}^{-1}.
\]

It satisfies the matrix Riccati equation:

\[
U' = -(R(\mathcal{J}) \dot{\mathcal{J}}, \mathcal{J}) - U^2.
\]

**Lemma 2.2.** The operator $U = \mathcal{J}' \mathcal{J}^{-1}$ is symmetric if and only if

\[
W(\mathcal{J}(t), \mathcal{J}(t)) = 0,
\]

and this is equivalent to the statement that for any two columns $J_i, J_j$ of $\mathcal{J}$, we have

\[
\omega((J_i, J'_i), (J_j, J'_j)) = 0,
\]

where $\omega$ is the symplectic form on $T^1 M$.

**Proof.** The first statement follows from (1) and the second follows from the definition of $\omega$. $\diamondsuit$
2.4. **Perpendicular Jacobi fields.** There are two natural subbundles of $TTM$ that are invariant under the derivative $D\varphi_1$ of the geodesic flow, the first containing the second. The first is the tangent bundle $TT^1M$ to the unit tangent bundle of $M$. Under the natural identification $T_{v}TM \cong (T_{x}M)^2$, for $v \in T^1_x M$, the subspace $T_{v}T^1M$ is the set of all pairs $(w_0, w_1)$ such that $\langle v, w_1 \rangle = 0$. To see this, note that if $\alpha(s, t)$ is a variation of geodesics generating the Jacobi field $J$ along $\gamma$, and $\|\frac{d}{ds}\alpha(s, t)\|^2 = 1$ for all $s, t$, then

$$0 = \frac{d}{ds}\left[ \frac{d}{dt} \alpha \right] \left. \right|_{(0,0)} = 2\left( \frac{d^2}{ds dt} \alpha, \frac{d}{dt} \alpha \right)_{(0,0)} = 2\langle J'(0), \dot{\gamma}(0) \rangle.$$ 

The $D\varphi_1$-invariance of $TT^1M$ follows from the $\varphi_1$-invariance of $T^1M$.

The second natural invariant subbundle is the orthogonal complement $\varphi_1^\perp$ in $TT^1M$ to the vector field $\varphi$ generating the geodesic flow. Under the natural identification $T_{x}TM \cong (T_{x}M)^2$, for $v \in T^1_x M$, the subspace $\varphi_1^\perp(v)$ is the set of all pairs $(w_0, w_1)$ such that $\langle v, w_0 \rangle = \langle v, w_1 \rangle = 0$. To see this, note that in these coordinates, $\dot{\varphi}(v) = (v, 0)$, which generates the tangent Jacobi field $J(t) = \dot{\gamma}(t)$ along the geodesic with $\dot{\gamma}(0) = v$.

To see that $\varphi_1^\perp$ is $D\varphi_1$-invariant, observe that if $\langle J(0), \dot{\gamma}(0) \rangle = 0$ and $\langle J'(0), \dot{\gamma}(0) \rangle = 0$, then the Jacobi equation implies that

$$\langle J(t), \dot{\gamma}(t) \rangle = 0 \text{ and } \langle J'(t), \dot{\gamma}(t) \rangle = 0, \text{ for all } t.$$ 

A Jacobi field $J$ along $\gamma$ satisfying (2) is called a **perpendicular Jacobi field**.

To summarize, the space of all perpendicular Jacobi fields along $\gamma$ corresponds to the orthogonal complement to the direction of the geodesic flow $\varphi(v)$ at the point $v = \dot{\gamma}(0) \in T^1M$. To estimate the norm of the derivative $D\varphi_1$ on $TT^1M$, it suffices to restrict attention to vectors in the invariant subspace $\varphi_1^\perp$; that is, it suffices to estimate the growth of perpendicular Jacobi fields along geodesics.

2.5. **Consequences of negative curvature.** If the sectional curvatures of the Riemannian metric are negative along $\gamma$, then it follows from the Jacobi equation that $\langle J'', J \rangle > 0$, for any Jacobi field with the property that $J(t)$ and $\dot{\gamma}(t)$ are linearly independent. This has the following consequence.

**Lemma 2.3.** If the sectional curvatures are negative along $\gamma$, then the functions $\|J(t)\|$ and $\|J(t)\|^2$ are strictly convex, for any nontrivial perpendicular Jacobi field $J$ along $\gamma$.

**Proof.** It follows from the Jacobi equation and negativity of the sectional curvatures along $\gamma$ that

$$\left( \frac{\|J\|^2}{2} \right)'' = \langle J, J' \rangle' = \langle J', J' \rangle + \langle J'', J \rangle = \langle J', J' \rangle - \langle R(J, \dot{\gamma}) \dot{\gamma}, J \rangle \geq 0,$$

for any Jacobi field $J$. We have $\langle J'(t), J'(t) \rangle > 0$ unless $J'(t) = 0$; and, since $J$ is perpendicular, we have $-\langle R(J, \dot{\gamma}(t)) \dot{\gamma}(t), J \rangle > 0$ unless $J(t) = 0$. But $J(t)$ and $J'(t)$ cannot both vanish because $J$ is nontrivial. Thus $\langle \|J\|^2 \rangle'' > 0$, which implies that $\|J(t)\|^2$ is a strictly convex function of $t$. 

To show convexity of $\|J(t)\|$, first note that if $J(t_0) = 0$ we have

$$\lim_{t \to t_0} \frac{\|J(t)\|}{|t - t_0|\|J'(t_0)\|} = 1.$$  

When $J \neq 0$ we have

$$\|J\|' = \left(\langle J, J \rangle \right)^{\frac{1}{2}}' = \frac{\langle J', J \rangle}{\langle J, J \rangle^{\frac{1}{2}}}$$

and

$$\langle J, J \rangle'' = \frac{\langle J, J \rangle^{\frac{1}{2}} \left[ \langle J', J' \rangle + \langle J'', J \rangle \right] - \langle J', J \rangle^2 / \langle J, J \rangle^{\frac{1}{2}}}{\langle J, J \rangle}$$

$$= \frac{\langle J, J \rangle \langle J', J' \rangle - \langle J', J \rangle^2 + \langle J, J \rangle \langle J'', J \rangle}{\langle J, J \rangle^{\frac{3}{2}}}$$

$$\geq \frac{\langle J, J \rangle \langle J'', J \rangle}{\langle J, J \rangle^{\frac{3}{2}}},$$

by the Cauchy Schwarz inequality. Thus $\|J\|'' \geq \langle J'', J \rangle / \|J\|$ when $J \neq 0$, and as we saw above, $\langle J'', J \rangle = -\langle R(J, \dot{\gamma}) \dot{\gamma}, J \rangle > 0$ in this case. □

2.6. Unstable Jacobi fields. We continue to assume that the sectional curvatures of $M$ are negative. We have the following result from [11, Section 1.10].

**Proposition 2.4.** If the sectional curvatures of the Riemannian metric are negative along the infinite geodesic ray $\gamma: (-\infty, a] \to M$, then there exists a continuous family of positive definite, symmetric operators

$$\{U_+(t): T_{\gamma(t)}M \to T_{\gamma(t)}M : t \in (-\infty, a]\}$$

such that for every $t \leq a$ and $v \in T_{\gamma(t)}M$, there exists a Jacobi field $J(s)$ defined for all $s \leq t$ and satisfying $J(t) = v$ and $J'(t) = U_+(t)v$.

The family of operators $U_+$ satisfies the Riccati equation:

$$U' = -\langle R(\cdot, \dot{\gamma}) \dot{\gamma}, \cdot \rangle - U^2.$$

This means that for any vector $v$,

$$\langle v, U'(v) \rangle = -\langle R(v, \dot{\gamma}) \dot{\gamma}, v \rangle - \langle v, U(v) \rangle^2.$$

A Jacobi field satisfying $J' = U_+ J$ is called an unstable Jacobi field.

2.7. Bounding the derivative of the geodesic flow. In the next proposition we show how the norm of the derivative of the geodesic flow given by Proposition 2.1 can be estimated from the growth of unstable Jacobi fields.

**Proposition 2.5.** Let $\gamma: (-\infty, \infty) \to M$ be a geodesic in a manifold with negative sectional curvatures. Any perpendicular Jacobi field $J$ along $\gamma$ satisfies

$$\|(J(1), J'(1))\| \leq (1 + m_+)(1 + u_+)^2 \|(J(0), J'(0))\|,$$
where
\[ m_+ = \max \left\| \frac{(J_+(1), J'_+(1))}{(J_+(0), J'_+(0))} \right\|. \]
and
\[ u_+ = \max \left\| \frac{J'_+(0)}{J_+(0)} \right\|, \]
the maxima being taken over all unstable Jacobi fields \( J_+ \).

We remark that the point of the lemma is that the constant in the bound is essentially linear in \( m_+ \).

Proof. We begin by showing that if \( L \) is a perpendicular Jacobi field with \( L(1) = 0 \), then
\[ \left\| L'(0) \right\| \geq \left\| L(0) \right\| \geq \left\| L'(1) \right\|. \]
These quantities are all 0 if \( L(0) = 0 \). If \( L(0) \neq 0 \), both inequalities are strict as we now show. Since \( \left\| L \right\| \) is convex by Lemma 2.3 and decreases from \( \left\| L(0) \right\| \) to 0 across the interval \([0, 1]\), we have
\[ -\left\| L \right\|'(0) > \left\| L(0) \right\| > -\lim_{t \to 1^-} \left\| L \right\|'(t). \]
Since \( L(1) = 0 \) it follows from (3) that
\[ \lim_{t \to 1^-} \left\| L \right\|'(t) = \lim_{t \to 1^-} \left\| L'(t) \right\| = -\left\| L'(1) \right\|. \]
This implies the right hand inequality in (5). From (4) and the Cauchy-Schwarz inequality, we have that
\[ \left\| \left\| L \right\|'(0) \right\| \leq \left\| L'(0) \right\|. \]
This gives the left hand inequality in (5).

Now let \( L' \) be a fundamental solution of the Jacobi equation with \( L(0) = I \) and \( L(1) = 0 \). Thus for any Jacobi field \( L \) as above we have \( L(t) = L(t)L(0) \).
Let
\[ U_1 = L'L^{-1}. \]
Then for any such Jacobi field \( L \) with \( \left\| L(0) \right\| = 1 \), the inequality
\[ \langle U_1(0)L(0), L(0) \rangle = \langle L'(0), L(0) \rangle = \left\| L(0) \right\| \left\| L'(0) \right\| < -1 \]
implies that all of the eigenvalues of \( U_1(0) \) are less than \(-1\).

Now we wish to express a perpendicular Jacobi field \((J, J') with \( \left\| (J(0), J'(0)) \right\| = 1 \) as a sum of a perpendicular Jacobi field of the form \((L, L')\) as studied above and an unstable perpendicular Jacobi field \((J_+, J'_+)\). Then \( \left\| (L(0), L'(0)) \right\| \leq 1 + \left\| (J_+(0), J'_+(0)) \right\| \) and we will obtain
\[
\left\| (J(1), J'(1)) \right\| \leq \left\| (J_+(1), J'_+(1)) \right\| + \left\| (L(1), L'(1)) \right\|
\leq m_+ \left\| (J_+(0), J'_+(0)) \right\| + \left\| (L(0), L'(0)) \right\|
\leq (1 + m_+) \left\| (J_+(0), J'_+(0)) \right\| + 1
\leq u_+(1 + m_+) \left\| J_+(0) \right\| + 1.
\]
Recall from Proposition 2.4 that $J'_+ = U_+ J_+$, where $U_+$ is symmetric and positive definite. Hence $U_+(0) - U_1(0)$ is symmetric and has all eigenvalues greater than 1. Consequently $[U_+(0) - U_1(0)]^{-1}$ is a contraction.

Let $v = J(0)$, $v' = J'(0)$ and $w = [U_+(0) - U_1(0)]^{-1}(v' - U_+(0)v)$. Note that

$$\|w\| \leq \|v'\| + \|U_+(0)v\| \leq 1 + u_+,$$

since $\|(v, v')\| = 1$. Now we obtain

$$\langle v, v' \rangle = \langle v, U_+(0)v \rangle + \langle 0, v' - U_+(0)v \rangle = \langle v + w, U_+(0)(v + w) \rangle - \langle w, U_1(0)w \rangle.$$

This is the desired decomposition of $(J, J')$ at $t = 0$. We have

$$\|v + w\| \leq \|v\| + \|w\| \leq 1 + (1 + u_+) = 2 + u_+.$$

Putting everything together finally gives

$$\|(J(1), J'(1))\| \leq u_+(1 + m_+)(2 + u_+) + 1 \leq (1 + m_+)(1 + u_+)^2.$$

Hence to estimate the norm of the derivative of the geodesic flow, it suffices to estimate the growth of unstable Jacobi fields. This task is simplified by considering the Riccati equation.

**Lemma 2.6.** For any unstable Jacobi field $J$ we have

$$\|(J(1), J'(1))\|_{Sas} \|\| J(0), J'(0) \|\|_{Sas} \leq \sqrt{1 + \|U_+(1)\|^2} \exp \left( \int_0^1 \|U_+(t)\| \, dt \right).$$

**Proof.** For any unstable Jacobi field $J$, we have $J' = U_+ J$. Hence

$$\|(J(1), J'(1))\|_{Sas} = \sqrt{J(1)^2 + J'(1)^2} = \|J(1)\| \sqrt{1 + \|U_+(1)\|^2}.$$

Since $\|J(t)^t\| \leq \|J'(t)\|$ by (4), we have

$$\frac{\|J(1)\|}{\|J(0)\|} \leq \exp \left( \int_0^1 \frac{\|J'(t)\|}{\|J(t)\|} \, dt \right) = \exp \left( \int_0^1 \frac{\|U_+(t)\|}{\|J(t)\|} \, dt \right).$$

Putting these last two inequalities together gives the desired estimate.

To estimate the quantities $\int_0^1 \|U_+(s)\| ds$ and $\|U_+(1)\|$, and hence the derivative of the time 1 map of the geodesic flow, we have the following theorem.

**Theorem 2.7.** Let $M$ be a negatively curved manifold, and let $\gamma : [-1, 1] \to M$ be a geodesic. Let $P$ be an arbitrary parallel unit vector field along $\gamma$, and let

$$u_P = \langle U_+ P, P \rangle,$$

and $k_P^2 = -\langle R(P, \gamma) \gamma, P \rangle$.

Then

(1) $u_P' \leq k^2 - u_P^2$, and
(2) for any $0 \leq t \leq 1$,
\[ \|D\phi_t(0)\| \leq (1 + u(0))^2(1 + \sqrt{1 + u(t)^2}) \exp\left(\int_0^t u(s)ds\right), \]
where $\overline{k}(t) = \sup_{v \in T_{\gamma(t),M}} -\langle R(v, \dot{\gamma}(t)), \dot{\gamma}(t), v \rangle$ and $u = \sup_P u_P$.

**Proof.** Since $U_+ +$ is positive definite, we have that $u_P > 0$ and $\|U_+\| \leq \sup_P u_P$. Since $P$ is parallel and has unit length, we have
\[ u' \parallel P = \langle U'_+ P, P \rangle = k_0^2 - \langle U_+^2 P, P \rangle = k_0^2 - \langle U_+ P, U_+ P \rangle \]
\[ \leq \overline{k}^2 - \langle U_+ P, P \rangle^2 = \overline{k}^2 - u_P^2. \]
This proves the first conclusion.

The second conclusion follows from Proposition 2.1 using the bounds given by Proposition 2.5, Lemma 2.6 and the bound $\|U_+\| \leq u$.

In the next section, we find bounds for solutions $u_P$ of the differential inequality $u'_P \leq k_0^2 - u_P^2$ for the WP metric.

### 3. Bounds on the derivative of $\varphi_1$ in the WP metric

The WP metric is negatively curved, so the analysis in the previous section applies in this setting. In this section, we fix a Riemann surface $S$ and denote by $\mathcal{T}$ the Teichmüller space $\text{Teich}(S)$, by $\overline{\mathcal{T}}$ the augmented Teichmüller space $\overline{\text{Teich}}(S)$, and by $\partial \mathcal{T}$ the boundary $\overline{\mathcal{T}} \setminus \mathcal{T}$. We will omit the dependence on $S$ in the mapping class group $\text{MCG} = \text{MCG}(S)$ and curve complex $\mathcal{C} = \mathcal{C}(S)$. For $\sigma \in \mathcal{C}$, the boundary stratum corresponding to the noded surface $S_{\sigma}$ is denoted by $T_{\sigma}$. We denote by $\pi: \mathcal{T} \rightarrow \mathcal{T}$ the natural projection and by $\varphi_t$ the geodesic flow on $T^1\mathcal{T}$.

For each unit WP tangent vector $v \in T^1\mathcal{T}$ and $t \geq 0$, we denote by $\rho_t(v)$ the minimum WP distance from the geodesic segment $\pi(\varphi_{[-t,t]}(v))$ in $\mathcal{T}$ to the singular locus $\partial \mathcal{T}$. The main result of this section is:

**Theorem 3.1.** There exist constants $\beta > 0$, $\epsilon_0 > 0$ and $C > 1$ with the following property. For every $t \in [0, 1]$ and $v \in T^1\mathcal{T}$ with $\rho_t(v) \in (0, \epsilon_0)$, we have:
\[ \|D\varphi_t\|_{WP} \leq C(\rho_t(v))^{-\beta}. \]

Since we will be dealing only with the WP metric in this section and the next, we will omit the subscript “WP” from the notation for inner product, norm and distance functions. Since Section 5 is largely concerned with comparisons between the WP and Teichmüller metric, the subscripts used in notation will return at that point.
3.1. Combined length bases. The next two sections summarize work of Wolpert in a constellation of papers [35, 36, 37, 38]. If $\chi$ is an arbitrary finite collection of vertices in $C$ and $X \in T$ we define:

$$\ell_{\chi}(X) = \min_{\beta \in \chi} \ell_{\beta}(X), \quad \overline{\ell}_{\chi}(X) = \max_{\beta \in \chi} \ell_{\beta}(X).$$

For $X \in T$, we continue to denote by $\ell(X)$ the systole of $X$, which is the hyperbolic length of the shortest closed geodesic in $X$. By a “simple closed curve in $S$” we will always mean a vertex of $C$; that is, the homotopy class of a nonperipheral, homotopically nontrivial, simple closed curve in $S$. Two curves are disjoint if they are connected by an edge in $C$. Let $B$ be the set of pairs $(\sigma, \chi)$, where $\sigma \in C$ and $\chi$ is a collection of simple closed curves in $S$ such that each $\beta \in \chi$ is disjoint from every $\alpha \in \sigma$ (we allow for the possibility that $\chi = \emptyset$).

For each simple closed curve $\alpha$ in $S$, the root length function

$$\ell^{1/2}_{\alpha}: T \to \mathbb{R}_{>0}$$

plays an important role in various asymptotic expansions of the WP metric. Wolpert proved that the functions $\ell_{\alpha}$ and $\ell^{1/2}_{\alpha}$ are convex along WP geodesics in $T$ (see Corollary 3.4 and Example 3.5 of [38] and Corollary 8.2 of [39]). The WP gradient of $\ell^{1/2}_{\alpha}$ defines a vector field

$$\lambda_{\alpha} = \operatorname{grad} \ell^{1/2}_{\alpha}.$$ 

Following Wolpert, we say that $(\sigma, \chi) \in B$ is a combined (short and relative) length basis at $X \in T$ if the collection

$$\{\lambda_{\alpha}(X), J\lambda_{\alpha}(X), \operatorname{grad} \ell_{\beta}(X)\}_{\alpha \in \sigma, \beta \in \chi}$$

is a basis for $T_X T$.

For each $\eta > 0$, let

$$U(\eta) = \{X \in T \mid \ell(X) < \eta\},$$

which is a deleted open neighborhood of $\partial T$ in $T$.

**Proposition 3.2.** There exist constants $c > 1$, $\eta, \delta > 0$ and a countable collection $\mathcal{U}$ of open sets in $T$ with the following properties.

1. For each $U \in \mathcal{U}$, there exists a combined length basis $(\sigma, \chi) \in B$ such that, for every $X \in U$:

$$1/c < \ell_{\chi}(X) \leq \overline{\ell}_{\chi}(X) < c.$$ 

2. For each $X \in U(\eta)$, there exists $U \in \mathcal{U}$ such that for any $Y \in T$,

$$d(X, Y) < \delta \implies Y \in U;$$

in particular, the sets in $\mathcal{U}$ cover $U(\eta)$.

Before proving this proposition, we discuss further the properties of the WP metric in a neighborhood of the boundary strata of $T$. Let $\sigma \in \mathcal{C}$ be a simplex, and consider a marked noded Riemann surface $f: S \to X_\sigma$
representing an element of the boundary stratum $\mathcal{T}_\sigma$. Recall that the hyperbolic surface $\hat{X}_\sigma$ is obtained from $X_\sigma$ by deleting its nodes. If $\beta$ is a simple closed curve in $S$ that is disjoint from the curves in $\sigma$, then $f_*[\beta]$ is uniquely represented as a closed geodesic on $\hat{X}_\sigma$. In this way, the definition of $\ell_\beta$ extends continuously to the boundary stratum $\mathcal{T}_\sigma$; for such $\beta$, we define $\ell_\beta([f : S \rightarrow X_\sigma])$ to be the hyperbolic length of the geodesic representative of $f_*[\beta]$ on $\hat{X}_\sigma$. For $X_\sigma \in \mathcal{T}_\sigma$, we can also define a relative systole $\ell(X_\sigma)$ to be the infimum of $\ell_\beta(\hat{X}_\sigma)$, taken over all curves $\beta$ disjoint from the curves in $\sigma$.

Recall that the boundary stratum $\mathcal{T}_\sigma$ is isomorphic to the product of Teichmüller spaces. In particular, $\mathcal{T}_\sigma$ itself carries a WP metric, which is the product of the WP metrics on the pieces of $X_\sigma$, for any $X_\sigma \in \mathcal{T}_\sigma$. Let $X_\sigma$ be a marked noded surface in $\mathcal{T}_\sigma$. We say that $\chi$ is a relative length basis at $X_\sigma$ if $(\sigma, \chi) \in \mathcal{B}$ and the functions $\{\ell_\beta\}_{\beta \in \chi}$ give local coordinates for $\mathcal{T}_\sigma$ at $X_\sigma$. Equivalently, $\chi$ is a relative length basis at $X_\sigma$ if the vectors $\{\text{grad } \ell_\beta(X_\sigma)\}_{\beta \in \chi}$ in the induced WP metric on $\mathcal{T}_\sigma$ span the tangent space $T_{X_\sigma} \mathcal{T}_\sigma$.

**Proposition 3.3** (Existence of relative length bases). For each $\sigma \in \mathcal{C}$, and each marked noded Riemann surface $X_\sigma \in \mathcal{T}_\sigma$, there exists $(\sigma, \chi) \in \mathcal{B}$ such that $\chi$ is a relative length basis at $X_\sigma$.

We remark that, unlike Fenchel-Nielsen coordinates, the local coordinates $\{\ell_\beta\}_{\beta \in \chi}$ never extend to a global coordinate system on $\mathcal{T}_\sigma$; the reason is that there are points in $\mathcal{T}$ where the geodesic representatives of the curves in $\chi$ cross each other orthogonally. At these points, the coordinate system hits a singularity. Proposition 3.3 ensures, however, that if one works locally these issues can be ignored. Wolpert proves:

**Theorem 3.4** ([38], Corollary 4.5.). The WP metric is comparable to a sum of differentials of geodesic-length functions for a simplex $\sigma$ of short geodesics and corresponding relative length basis $\chi$ as follows

$$ \langle \cdot, \cdot \rangle \asymp \sum_{\alpha \in \sigma} (d\ell_\alpha^{1/2})^2 + (d\ell_\alpha^{1/2} \circ J)^2 + \sum_{\beta \in \chi} (d\ell_\beta)^2, $$

where, given $X_\sigma \in \mathcal{T}_\sigma$ and $\chi$ there is a neighborhood $U$ of $X_\sigma$ in $\overline{\mathcal{T}}$ in which the comparison holds uniformly.

This has the immediate corollary:

**Corollary 3.5.** If $\chi$ is a relative length basis at $X_\sigma \in \mathcal{T}_\sigma$, then there is a neighborhood $V$ of $X_\sigma$ in $\overline{\mathcal{T}}$ such that for every $X \in V \cap \mathcal{T}$, $(\sigma, \chi)$ is a combined length basis at $X$.

We introduce more notation that will be used in the proof of Proposition 3.2. For $\sigma \in \mathcal{C}$ and $\eta > 0$, we set

$$ U(\sigma, \eta) = \{X \in \mathcal{T} \mid \bar{\ell}_\sigma(X) < \eta\}, $$
so that 
\[ U(\eta) = \bigcup_{\sigma \in C} U(\sigma, \eta). \]

Let \( P: \mathcal{T} \to \overline{\mathcal{M}} \) be the quotient map from \( \mathcal{T} \) to the Deligne-Mumford compactification \( \overline{\mathcal{M}} \) under the action of the mapping class group \( \text{MCG} \). Note that \( P(U(\eta)) \) is a deleted open neighborhood of \( \partial \mathcal{M} \) in \( \overline{\mathcal{M}} \).

**Proof of Proposition 3.2.** Denote by \( C_k \) the set of all \( k \)-simplices in \( C \), where \( k = 0, \ldots, 3g+n-4 \). The mapping class group has finitely many orbits in \( C_k \).

For each \( k \), let \( \Sigma_k \) be a minimal (necessarily finite) collection of \( k \)-simplices with the property that 
\[ C_k = \bigcup_{g \in \text{MCG}} g_* \left( \Sigma_k \right). \]

For each \( k \), let \( M_k \subset \mathcal{M} \) be the union over \( \sigma \in \Sigma_k \) of \( P(T_{\sigma}) \).

Let \( \sigma \in \Sigma_{3g+n-4} \) be a maximal simplex. Then \( T_{\sigma} \) contains a single point \( X_{\sigma} \) and \( \chi = 0 \) is a relative length basis at \( X_{\sigma} \). Corollary 3.5 implies there exists \( \eta = \eta_{3g+n-4} > 0 \) such that the collection of open sets:
\[ \mathcal{V}_{3g+n-4} = \{ U(\sigma, 2\eta) \mid \sigma \in \Sigma_{3g+n-4} \} \]
has the property that \((\sigma, \emptyset)\) is a combined length basis in \( V \), for every \( V \in \mathcal{V}_{3g+n-4} \). Moreover, the elements of \( \mathcal{V}_{3g+n-4} \) project to an open cover of the compact set 
\[ \bigcup_{\sigma \in \Sigma_{3g+n-4}} P(U(\sigma, \eta_k)) \]
in \( \overline{\mathcal{M}} \).

Working inductively, suppose that there exist constants \( c_k > 1, \eta_k > 0 \), and a finite collection \( \mathcal{V}_k \) of open sets in the augmented space \( \overline{\mathcal{M}} \) such that:
\[(I_k) \text{ each } V \in \mathcal{V}_k \text{ has a combined length basis } (\sigma, \chi) \text{ for some } \sigma \in \Sigma_k, \]
satisfying 
\[ 1/c_k < \ell_\chi(X) \leq \overline{\ell}_\chi(X) < c_k, \]
for all \( X \in V \cap \mathcal{T} \);
\[(II_k) \bigcup_{j \geq k} \bigcup_{V \in \mathcal{V}_j} P(V) \text{ contains the compact neighborhood } \bigcup_{j \geq k} \bigcup_{\sigma \in \Sigma_j} P(U(\sigma, \eta_k)) \]
of \( \bigcup_{j \geq k} \mathcal{M}_j \) in \( \overline{\mathcal{M}} \).

The set \( \mathcal{N} = \mathcal{M}_{k-1} \setminus \bigcup_{j \geq k} \bigcup_{V \in \mathcal{U}_j} P(V) \) is compact in \( \overline{\mathcal{M}} \). Proposition 3.3 and Corollary 3.5 imply that for each \( X \in \mathcal{N} \) there is an open set \( V \) in \( \overline{\mathcal{T}} \) that projects to a neighborhood of \( X \) in \( \overline{\mathcal{M}} \) and a relative length basis \( (\sigma, \chi) \) for \( V \cap \mathcal{T} \), for some \( \sigma \in \Sigma_{k-1} \). The projections of these open sets to \( \overline{\mathcal{M}} \) cover \( \mathcal{N} \); extracting a finite subcover and taking \( \mathcal{V}_{k-1} \) to be the corresponding collection of open sets in \( \mathcal{T} \), we obtain that \( I_{k-1} \) and \( II_{k-1} \) hold for some \( c_{k-1} > 1, \eta_{k-1} > 0 \).
We set $c = \sup_{j \geq 0} c_j$, $\eta = \inf_{j \geq 0} \eta_j$, and let

$$V = \bigcup_{j \geq 0} \mathcal{V}_j.$$ 

Let $\delta > 0$ be the Lebesgue number of the open covering $V$ of the compact set $\overline{P(U(\eta))}$ in $\mathcal{M}$. 

To finish the proof, we define the countable cover $U$ to be the collection 

$$U = \{g(V) \cap T \mid g \in \text{MCG}, \text{ and } V \in \mathcal{V}\}.$$ 

Note that if $(\sigma, \chi)$ is the combined length basis for $V \in \mathcal{V}$ satisfying 

$$1/c < \ell_\chi(X) \leq \overline{\ell}_\chi(X) < c,$$

for all $X \in V \cap T$, then $(g_*^{-1}\sigma, g_*^{-1}\chi)$ is a combined length basis for $g(V)$ satisfying 

$$1/c < \ell_{g_*^{-1}\chi}(X) \leq \overline{\ell}_{g_*^{-1}\chi}(X) < c,$$

for all $X \in g(V) \cap T$. Since MCG acts by WP isometries, the result now follows. ♦

### 3.2. First and second order properties of the WP metric

For each $c > 1$, and $(\sigma, \chi) \in \mathcal{B}$, let 

$$\Omega(\sigma, \chi, c) = \{X \in T \mid \overline{\ell}_{\sigma \cup \chi}(X) < c, \text{ and } 1/c < \ell_\chi(X)\}.$$ 

Wolpert proved key estimates on the WP metric in $\Omega(\sigma, \chi, c)$, which we summarize in the following three propositions.

The first set of estimates expand upon and refine the statement in Theorem 3.4.

**Proposition 3.6** (First order estimates). [35] Fix $c > 1$. For any $(\sigma, \chi) \in \mathcal{B}$, the following estimates hold uniformly on $\Omega(\sigma, \chi, c)$:

1. if $\alpha, \alpha' \in \sigma$, then 
   
   $$\langle J\lambda_\alpha, J\lambda_{\alpha'} \rangle = \langle \lambda_\alpha, \lambda_{\alpha'} \rangle = \frac{1}{2\pi} \delta_{\alpha, \alpha'} + O((\ell_\alpha \ell_{\alpha'})^{3/2});$$

2. if $\alpha, \alpha' \in \sigma$ and $\beta \in \chi$, then
   
   $$\langle \lambda_\alpha, J\lambda_{\alpha'} \rangle = \langle J\lambda_\alpha, \text{grad} \ell_\beta \rangle = 0;$$

3. if $\beta, \beta' \in \chi$, then
   
   $$\langle \text{grad} \ell_\beta, \text{grad} \ell_{\beta'} \rangle \asymp 1;$$

   moreover, $\langle \text{grad} \ell_\beta, \text{grad} \ell_{\beta'} \rangle$ extends continuously to $T_\sigma$;

4. if $\alpha \in \sigma$ and $\beta \in \chi$, then
   
   $$\langle \lambda_\alpha, \text{grad} \ell_\beta \rangle = O(\ell_\beta^{3/2});$$
(5) if $X \in \Omega(\sigma, \chi, c)$, then

$$d(X, T_\sigma) = \left( \frac{2\pi}{\alpha} \sum_{\alpha \in \sigma} \ell_\alpha(X) \right)^{1/2} + O\left( \sum_{\alpha \in \sigma} \ell_\alpha^{5/2}(X) \right).$$

The second set of Wolpert’s estimates are formulae for covariant derivatives, which are described in the next proposition. In each formula in the next proposition, the error term is a vector, and the expression $v = O(a)$ means that the WP length of $v$ is $O(a)$.

**Proposition 3.7** (Second order estimates). [35] Fix $c > 1$. For any $(\sigma, \chi) \in \mathcal{B}$, the following estimates hold uniformly on $\Omega(\sigma, \chi, c)$:

1. for any vector $v \in T\Omega(\sigma, \chi, c)$, and $\alpha \in \sigma$, we have

$$\nabla_v \lambda_\alpha = \frac{3}{2\pi \ell_\alpha^{1/2}} (v, J\lambda_\alpha) J\lambda_\alpha + O(\ell_\alpha^{3/2} \|v\|_{WP});$$

2. for $\beta \in \chi$ and $\alpha \in \sigma$, we have

$$\nabla_{\lambda_\alpha} \text{grad} \ell_\beta = O(\ell_\alpha^{1/2}), \quad \nabla_{J\lambda_\alpha} \text{grad} \ell_\beta = O(\ell_\alpha^{1/2});$$

3. for $\beta, \beta' \in \chi$, $\nabla_{\text{grad} \ell_\beta} \text{grad} \ell_{\beta'}$ extends continuously to $T_\sigma$.

The final set of Wolpert’s estimates we use involve the WP curvature tensor.

**Proposition 3.8** (Bounds on curvature). [35] Fix $c > 1$. For any $(\sigma, \chi) \in \mathcal{B}$, the following estimates hold uniformly on $\Omega(\sigma, \chi, c)$. For all $\alpha \in \sigma$ we have

$$\langle R(\lambda_\alpha, J\lambda_\alpha) J\lambda_\alpha, \lambda_\alpha \rangle = \frac{3}{16\pi^2 \ell_\alpha} + O(\ell_\alpha).$$

Moreover for any quadruple $(v_1, v_2, v_3, v_4) \in \{\lambda_\alpha, J\lambda_\alpha, \text{grad} \ell_\beta\}_{\alpha \in \sigma, \beta \in \chi}$ that is not a curvature-preserving permutation of $(\lambda_\alpha, J\lambda_\alpha, J\lambda_\alpha, \lambda_\alpha)$ for some $\alpha \in \sigma$, we have:

$$\langle R(v_1, v_2)v_3, v_4 \rangle = O(1).$$

### 3.3. Curvature estimates along a geodesic.

Fix a unit speed WP geodesic $\gamma: I \to T$ in Teichmüller space. For each simple closed curve $\alpha$ we define functions $f_\alpha = f_{\alpha, \gamma}: I \to \mathbb{R}_{>0}$ and $r_\alpha = r_{\alpha, \gamma}: I \to \mathbb{R}_{>0}$ by

$$f_\alpha(t) = \ell_\alpha^{1/2}(\gamma(t)), \quad \text{and} \quad r_\alpha^2(t) = \langle \lambda_\alpha, \dot{\gamma}(t) \rangle^2 + \langle J\lambda_\alpha, \dot{\gamma}(t) \rangle^2.$$

Roughly, $r_\alpha$ measures the speed of the geodesic $\gamma$ in the complex line field spanned by $\{\lambda_\alpha, J\lambda_\alpha\}$. Wolpert used the function $r_\alpha$ to study the behavior of geodesics terminating in the boundary strata of $T$. We will use $r_\alpha$ and $f_\alpha$ to bound sectional curvatures along $\gamma$. We summarize in the next few lemmas the key properties of $r_\alpha$ and $f_\alpha$ that will be used in the sequel.

---

5Wolpert actually proves more: each vector $v_i$ appearing in this expression that is of the form $\lambda_\alpha$ or $J\lambda_\alpha$ introduces a multiplicative bound $O(\ell_\alpha^{1/2})$ in the curvature tensor. This means that there are sectional curvatures that are arbitrarily close to 0.
The first property is an immediate consequence of part (5) of Proposition 3.6 and explains the significance of the quantity \( f_\alpha \).

**Lemma 3.9.** Fix \( c > 1 \). For every \( (\sigma, \chi) \in \mathcal{B} \) and any \( \gamma \), if \( \gamma(t) \in \Omega(\sigma, \chi, c) \), then

\[
d(\gamma(t), T_\sigma) = \left( \frac{2\pi}{\alpha \in \sigma} \sum_{\alpha \in \sigma} f_\alpha^2(t) \right)^{1/2} + O(f_\alpha^5(t)).
\]

The next two lemmas will allow us to bound the variations of \( r_\alpha \) and \( f_\alpha \) along a geodesic.

**Lemma 3.10.** Fix \( c > 1 \). For every \( (\sigma, \chi) \in \mathcal{B} \) and any \( \gamma \), if \( \gamma(t) \in \Omega(\sigma, \chi, c) \), then

\[
r_\alpha'(t) = O(f_\alpha^3(t)),
\]

for every \( \alpha \in \sigma \).

**Proof.** Since the WP metric is Kähler, the almost complex structure \( J \) is parallel, and so we have

\[
2r_\alpha(t)r_\alpha'(t) = 2\langle \lambda_\alpha, \dot{\gamma}(t) \rangle \frac{D}{dt} \lambda_\alpha, \dot{\gamma}(t) + 2\langle J\lambda_\alpha, \dot{\gamma}(t) \rangle \langle J\frac{D}{dt} \lambda_\alpha, \dot{\gamma}(t) \rangle.
\]

Now by part (1) of Proposition 3.7, we have

\[
\frac{D}{dt} \lambda_\alpha = \langle \dot{\gamma}, J\lambda_\alpha \rangle \frac{3}{2\pi f_\alpha} J\lambda_\alpha + O(f_\alpha^3)
\]

and

\[
J \frac{D}{dt} \lambda_\alpha = -\langle \dot{\gamma}, J\lambda_\alpha \rangle \frac{3}{2\pi f_\alpha} \lambda_\alpha + O(f_\alpha^3).
\]

Plugging this in to the formula for \( 2r_\alpha(t)r_\alpha'(t) \), and noting that

\[
\max\{|\langle \lambda_\alpha, \dot{\gamma} \rangle|, |\langle J\lambda_\alpha, \dot{\gamma} \rangle|\} < r_\alpha,
\]

we get:

\[
2r_\alpha(t)r_\alpha'(t) = \frac{3}{\pi f_\alpha} \langle \lambda_\alpha, \dot{\gamma} \rangle \langle \dot{\gamma}, J\lambda_\alpha \rangle^2 - \frac{3}{\pi f_\alpha} \langle \lambda_\alpha, \dot{\gamma} \rangle \langle \dot{\gamma}, J\lambda_\alpha \rangle^2 + O(r_\alpha f_\alpha^3)
\]

\[
= O(r_\alpha f_\alpha^3).
\]

\[\diamondsuit\]

**Lemma 3.11.** Fix \( c > 1 \). For every \( (\sigma, \chi) \in \mathcal{B} \) and any \( \gamma \), if \( \gamma(t) \in \Omega(\sigma, \chi, c) \), then

\[
r_\alpha^2(t) = (f_\alpha'(t))^2 + \frac{2\pi}{3} f_\alpha(t) f_\alpha''(t) + O(f_\alpha^4(t)),
\]

for every \( \alpha \in \sigma \).
Proof. Since $\lambda_\alpha = \text{grad} \ell_\alpha^{1/2}$, it follows that $f'_\alpha = \langle \lambda_\alpha, \dot{\gamma} \rangle$; differentiating this expression, we obtain using part (1) of Proposition 3.7:

$$f''_\alpha = \frac{d}{dt} \langle \lambda_\alpha, \dot{\gamma} \rangle = \langle \nabla_\gamma \lambda_\alpha, \dot{\gamma} \rangle = \frac{3}{2\pi f_\alpha(t)} \langle \dot{\gamma}, J\lambda_\alpha \rangle^2 + O(f_\alpha^3).$$

Now multiplying this last expression by $f_\alpha$ and adding it to $f'^2_\alpha$, the result then follows from the definition of $r^2_\alpha$.

Let

$$k^2(t) = \sup_{v \in T^1_\gamma(t) T} -(R(v, \dot{\gamma}(t)) \dot{\gamma}(t), v).$$

We next bound $k^2$ in terms of $r_\alpha$ and $f_\alpha$; the preceding estimates on $r_\alpha$ and $f_\alpha$ will then allow us to bound solutions to the differential inequality $u' \leq k^2 - u^2$.

**Lemma 3.12.** Fix $c > 1$. For any $(\sigma, \chi) \in B$ and any unit speed geodesic $\gamma$, if $(\sigma, \chi)$ is a combined length basis in $U \subset \Omega(\sigma, \chi, c)$, and $\gamma(t) \in U$, then

$$k^2(t) = \sum_{\alpha \in \sigma} O \left( \frac{r^2_\alpha(t)}{f^2_\alpha(t)} \right).$$

Proof. Since $(\sigma, \chi)$ is a combined length basis, we can write $v \in T^1 \Omega(\sigma, \chi, c)$ and $\dot{\gamma}$ as

$$v = \sum_{\alpha \in \sigma} (a_\alpha \lambda_\alpha + b_\alpha J\lambda_\alpha) + \sum_{\beta \in \chi} c_\beta \text{grad}(\ell_\beta)$$

and

$$\dot{\gamma} = \sum_{\alpha \in \sigma} (d_\alpha' \lambda_\alpha + e_\alpha' J\lambda_\alpha) + \sum_{\beta \in \chi} c_\beta' \text{grad}(\ell_\beta).$$

Now $v$ and $\dot{\gamma}$ are unit vectors, the above estimates on the metric say that all coefficients $a_\alpha, b_\alpha, c_\beta, a'_\alpha, b'_\alpha, c'_\beta$ are $O(1)$. Moreover by these same estimates and the definition of $r_\alpha$, we have

$$r^2_\alpha = \frac{1}{4\pi^2} (a'^2_\alpha + b'^2_\alpha) + O(f_\alpha^3).$$

It now follows from Proposition 3.8 that

$$-\langle R(v, \dot{\gamma}) \dot{\gamma}, v \rangle = -\sum_{\alpha \in \sigma} (a^2_\alpha b'^2_\alpha + a'^2_\alpha b_\alpha^2) \langle R(\lambda_\alpha, J\lambda_\alpha) J\lambda_\alpha, \lambda_\alpha \rangle + O(1)$$

$$= \sum_{\alpha \in \sigma} O \left( \frac{r^2_\alpha}{f^2_\alpha} \right) + O(1).$$

\qed
3.4. **Estimates on** $r_\alpha/f_\alpha$. In the next two subsections, we establish how bounds on $k^2$ can be used to bound solutions to the Riccati inequality $u' \leq k^2 - u^2$. The main result in this subsection will give us control over $k^2$.

**Proposition 3.13.** Fix $c > 1$. There is a constant $A = A(c) > 0$ such that for any $(\sigma, \chi) \in B$, for any unit speed WP segment $\gamma: [-\delta, \delta] \to \Omega(\sigma, \chi, c)$ and any $\alpha \in \sigma$, we have

$$\frac{r_\alpha(t)}{f_\alpha(t)} \leq A \max\left(1, \frac{r_\alpha(t_0)}{r_\alpha(t_0)|t-t_0|+f_\alpha(t_0)}\right) \quad \text{for } 0 \leq t \leq \delta,$$

where $t_0$ is the unique time in $[0, \delta]$ such that $f_\alpha(t) \geq f_\alpha(t_0)$ for $0 \leq t \leq \delta$.

**Proof of Proposition 3.13.** The time $t_0$ is uniquely defined since $f_\alpha(t)$ is a convex function of $t$. It will suffice to prove the proposition under the additional assumption that $f_\alpha(t)$ is increasing for $t \geq 0$. If $t_0 = 0$, we apply this restricted form of the proposition directly to the geodesic $\gamma$; if $t_0 = \delta$, we apply it to the geodesic $t \mapsto \gamma(\delta - t)$; and if $0 < t_0 < \delta$, we consider both of the geodesics $t \mapsto \gamma(t - t_0)$ and $t \mapsto \gamma(t_0 - t)$.

We choose $C \geq 1$ large enough so that:

- (C1) the $O(f_\alpha^4)$ term in the equation $r_\alpha^2 = (f_\alpha')^2 + \frac{2\pi}{3} f_\alpha f_\alpha'' + O(f_\alpha^4)$ given by Lemma 3.11 is at most $Cf_\alpha^4$;
- (C2) $\frac{r_\alpha}{f_\alpha} \leq \frac{1}{2}$;
- (C3) $|r_\alpha'| \leq C f_\alpha^3$ (which is possible by Lemma 3.10).

Conditions (C1) and (C2) give a lower bound on $f_\alpha''$ when $r_\alpha/f_\alpha \geq C$ and $|f_\alpha'|$ is small.

**Lemma 3.14.** If $\frac{r_\alpha}{f_\alpha} \geq C$ and $|f_\alpha'| \leq \frac{r_\alpha}{2}$, then $f_\alpha'' \geq \frac{3}{4\pi} \frac{r_\alpha^2}{f_\alpha}$.

**Proof.** By (C1) and (C2),

$$r_\alpha^2 = (f_\alpha')^2 + \frac{2\pi}{3} f_\alpha f_\alpha'' + O(f_\alpha^4) \leq \frac{2\pi}{3} 2 f_\alpha + \frac{r_\alpha^2}{4C} \leq \frac{2\pi}{3} + \frac{r_\alpha^2}{2} + \frac{2\pi}{3} f_\alpha f_\alpha''.$$

$\diamond$

We continue with the proof of Proposition 3.13. Recall we are assuming $t_0 = 0$. We have that $f(t)$ increasing for $t \geq 0$. We shall show that

$$\frac{r_\alpha(t)}{f_\alpha(t)} \leq \max\left(4C, \frac{32\pi r_\alpha(0)}{f_\alpha(0)+tr_\alpha(0)}\right) \quad \text{for } 0 \leq t \leq \delta.$$

If $\frac{r_\alpha(t)}{f_\alpha(t)} \leq 4C$ for $0 \leq t \leq \delta$ we are done. Otherwise, let

$$b = \sup\{t \in [0, \delta] : \frac{r_\alpha(t)}{f_\alpha(t)} \geq 4C\}.$$
Since $\frac{r_\alpha(t)}{f_\alpha(t)} \leq 4C$ for $b \leq t \leq \delta$, it will suffice to show that

$$(8) \quad \frac{r_\alpha(t)}{f_\alpha(t)} \leq \frac{32\pi r_\alpha(0)}{f_\alpha(0) + tr_\alpha(0)} \quad \text{for } 0 \leq t \leq b.$$ 

The function $r_\alpha$ is approximately constant and $r_\alpha/f_\alpha$ is large on the interval $[0, b]$.

**Lemma 3.15.** For $0 \leq t \leq b$ we have:

(i) $\frac{r_\alpha(0)}{2} \leq r_\alpha(t) \leq 2r_\alpha(0)$;

(ii) $\frac{r_\alpha(t)}{f_\alpha(t)} \geq C$.

**Proof.** By $(C_3)$, $|r_\alpha'| \leq Cf_\alpha^3$ on the interval $[0, b]$. Since $b \leq 1$ and $f_\alpha$ is increasing on $[0, b]$, we have $|r_\alpha(b) - r_\alpha(t)| \leq Cf_\alpha^3(b)$ for $0 \leq t \leq b$. The definition of $b$ and $(C_2)$ ensure that $f_\alpha(b) \leq \frac{r_\alpha(b)}{2C} \leq \frac{1}{4}$ by $(C_2)$. Hence

$$\frac{|r_\alpha(b) - r_\alpha(t)|}{r_\alpha(b)} \leq \frac{Cf_\alpha^3(b)}{2Cf_\alpha(b)} \leq \frac{1}{2}f_\alpha^2(b) \leq \frac{1}{32}.$$ 

Thus $\frac{31}{32} \leq \frac{r_\alpha(t)}{r_\alpha(b)} \leq \frac{33}{32}$ for $0 \leq t \leq b$, and (i) follows easily. Claim (ii) follows from (i) since $r_\alpha(b)/f_\alpha(b) \geq 2C$ and $f_\alpha$ is increasing on $[0, b]$. \diamond

Using this lemma we see that inequality (8) will follow if we prove

$$(9) \quad 16\pi f_\alpha(t) \geq f_\alpha(0) + tr_\alpha(0) \quad \text{for } 0 \leq t \leq b.$$ 

Lemma 3.15 ensures that $r_\alpha(0) > 0$, so we can set $a = \frac{f_\alpha(0)}{r_\alpha(0)}$. Now for $0 \leq t \leq \min(a, b)$, we have

$$f_\alpha(0) + tr_\alpha(0) \leq f_\alpha(0) + ar_\alpha(0) = 2f_\alpha(0) \leq 2f_\alpha(t),$$

since $f_\alpha$ is increasing on $[0, \delta]$. This gives (9) for $0 \leq t \leq \min(a, b)$.

We are done if $a \geq b$. It remains to show that if $a \leq b$, then inequality (9) also holds for $a \leq t \leq b$. Since $f_\alpha$ is convex and (9) already holds for $t = a$, it will suffice to show that if $a \leq b$ that

$$(10) \quad 16\pi f_\alpha'(a) \geq r_\alpha(0)$$

We may assume that $4f_\alpha'(a) \leq r_\alpha(0)$, since otherwise there is nothing to prove. Then $f_\alpha'(t) \leq f_\alpha'(a) \leq r_\alpha(0)/a$ for $0 \leq t \leq a$, because $f_\alpha$ is convex and increasing on $[0, a]$. Since $a \leq b$, we can now apply Lemma 3.15 to see that on $[0, a]$ we have

$$f_\alpha'(t) \leq r_\alpha(0)/4 \leq r_\alpha(t)/2 \quad \text{and} \quad \frac{r_\alpha(t)}{f_\alpha(t)} \geq C.$$
Thus both hypotheses of Lemma 3.14 are satisfied on \([0, a]\). Lemmas 3.14 and 3.15 give us

\[
\frac{f''_{\alpha}(t)}{4\pi f_{\alpha}(t)} \geq \frac{3r_{\alpha}(0)}{16\pi f_{\alpha}(t)} > \frac{1}{8\pi f_{\alpha}(t)}
\]

for \(0 \leq t \leq a\). Since \(f'_{\alpha} \leq r_{\alpha}(0)/4\) on \([0, a]\), we have

\[
f_{\alpha}(a) \leq f_{\alpha}(0) + ar_{\alpha}(0)/4 = f_{\alpha}(0) + f_{\alpha}(0)/4 < 2f_{\alpha}(0),
\]

and hence

\[
f''_{\alpha}(t) \geq \frac{1}{16\pi f_{\alpha}(0)},
\]

for \(0 \leq t \leq a\). Finally, since \(f_{\alpha}\) is increasing on \([0, a]\), we have \(f'_{\alpha}(0) \geq 0\) and

\[
f'_{\alpha}(a) \geq \frac{a r_{\alpha}(0)}{16\pi} = \frac{r_{\alpha}(0)}{16\pi},
\]

which is the desired inequality (10).

Combining Lemma 3.12 and Proposition 3.13 we obtain the immediate corollary:

**Corollary 3.16.** Fix \(c > 1\). There is a constant \(B = B(c) > 0\) such that for any \((\sigma, \chi) \in B\), if \((\sigma, \chi)\) is a combined length basis in an open set \(U \subset \Omega(\sigma, \chi, c)\) and \(\gamma: [-\delta, \delta] \to U\) is a unit-speed WP geodesic segment, then

\[
\bar{K}(t) \leq B \max_{\alpha \in \sigma} \left(1, \frac{r_{\alpha}(t_{\alpha})}{r_{\alpha}(t_{\alpha})|t - t_{\alpha}| + f_{\alpha}(t_{\alpha})}\right)
\]

for \(0 \leq t \leq \delta\), where \(t_{\alpha}\) is the unique time in \([-\delta, \delta]\) such that \(f_{\alpha}(t) \geq f_{\alpha}(t_{\alpha})\) for \(-\delta \leq t \leq \delta\).

### 3.5. Controlled functions and bounds on Riccati solutions.

**Lemma 3.17.** Let \(\kappa: (-\delta, \delta) \to \mathbb{R}_{> 0}\) be a continuous function such that the right derivative

\[
D_R \kappa(t) = \lim_{h \to 0^+} \frac{\kappa(t + h) - \kappa(t)}{h}
\]

is defined for all \(t \in (\delta, \delta)\), for some \(\delta > 0\). Suppose there is a constant \(Q > 1\) such that

\[
D_R \kappa(t) \geq -(Q - 1)\kappa^2(t)
\]

for all \(t \in (-\delta, \delta)\).

Let \(u(t)\) be a solution of the differential inequality

\[
u' \leq \kappa^2 - u^2
\]

that is defined for all \(t \in (\delta, \delta)\). Then

\[
u(t) \leq 2Q \max \left\{\frac{1}{\delta}, \kappa(t)\right\}
\]

for all \(t \in (0, \delta)\).
Proof. Replacing \( \kappa \) by \( \hat{\kappa} = \max\{\delta^{-1}, \kappa\} \), we obtain a new function satisfying
\[
D_R \hat{\kappa}(t) \geq -(Q - 1)\hat{\kappa}^2(t)
\]
for all \( t \in (-\delta, \delta) \). Moreover, any solution to (11) is also a solution to
\[
u' \leq \hat{\kappa}^2 - \kappa^2.
\]
We may therefore assume that \( \kappa(t) > \delta^{-1} \), for all \( t \in (-\delta, \delta) \). Moreover, increasing the value of \( Q \) slightly, we may assume that
\[
D_R \kappa(t) > -(Q - 1)\kappa^2(t).
\]
Suppose that \( u \) is solution of (11) with \( u(t_0) > 2Q\kappa(t_0) \) for some \( t_0 \in (0, \delta) \); we prove that \( u(t) \) is not defined for all \( t < t_0 \). It will suffice to show that if \( v_0 \) is a solution of differential equation
\[
(12) \quad v' = \kappa^2 - v^2
\]
with \( v_0(t_0) > 2Q\kappa(t_0) \) for some \( t_0 \in (0, t_0) \), then \( v_0(t) \) blows up in time \( < \delta \) as \( t \) decreases from \( t_0 \). We begin by finding a solution \( v_1(t) \) that is defined for all \( t \in (-\delta, \delta) \) and satisfies \( 0 \leq v_1 \leq Q\kappa. \)

Consider a solution \( v \) of (12). If \( v(t) = Q\kappa(t) \) for some \( t \), then
\[
v'(t) = (1 - Q^2)\kappa^2(t) \leq (Q - Q^2)\kappa^2(t) = -Q(Q - 1)\kappa^2(t) < QD_R\kappa(t).
\]
It follows that if \( v \) is a solution of (12) defined on an interval containing a time \( t_0 \) such that \( v(t_0) \leq Q\kappa(t_0) \), then \( v(t) \leq Q\kappa(t) \) for all \( t \geq t_0 \) in the interval. Moreover if \( v(t) = 0 \) for some \( t \), then \( v'(t) = \kappa^2(t) \geq 0 \). Thus if \( 0 \leq v(t_0) \leq Q\kappa(t_0) \) for some \( t_0 \), then \( v(t) \) is defined for \( t_0 \leq t < \delta \) and satisfies \( 0 \leq v \leq Q\kappa \) on the interval \([t_0, \delta)\). Hence the solution \( v_1 \) with initial condition \( v_1(\delta) = 0 \) satisfies \( 0 \leq v_1 \leq Q\kappa \) on \((-\delta, \delta)\).

We now show that \( v_0(t) \) and \( v_1(t) \) diverge rapidly as \( t \) decreases from \( t_0 \). Since \( v_0(t_0) > v_1(t_0) \) and the graphs of two solutions of (12) cannot intersect, we have
\[
(13) \quad v_0 > v_1 \geq 0
\]
on the interval where \( v_0 \) is defined.

Let \( \Delta(s) = v_0(t_0 - s) - v_1(t_0 - s) \). Observe that \( \Delta(0) > 2Q\kappa(t_0) - Q\kappa(t_0) = Q\kappa(t_0) > \frac{1}{\delta} \). Then \( \Delta(s) > 0 \) and it follows from (12) and (13) that
\[
\Delta'(s) = v_0^2(t_0 - s) - v_1^2(t_0 - s) \geq [v_0(t_0 - s) - v_1(t_0 - s)]^2 = \Delta(s)^2.
\]
Hence for \( s \geq 0 \) we have
\[
\Delta(s) \geq \frac{1}{(1/\Delta(0) - s)},
\]
which blows up as \( s \to 1/\Delta(0) < \delta \). Since \( v_1 \) stays bounded, it follows that \( v_0(t) \to \infty \) as \( t \searrow t_0 - 1/\Delta(0) \in (-\delta, t_0) \).

Let us say that a function \( \kappa \) is \( Q \)-controlled if it satisfies the hypotheses in the previous lemma. If \( \kappa \) is \( Q \)-controlled, then it is \( Q' \)-controlled, for all \( Q' > Q \). If \( \kappa \) is \( Q \)-controlled, then so is \( t \mapsto \kappa(t - t_0) \) for any \( t_0 \), and for any
$A > 0$, the function $A\kappa$ is $A(Q - 1) + 1$-controlled. The maximum of two $Q$-controlled functions is $Q$-controlled. Moreover $\kappa$ is $1$-controlled if $\kappa(t) = 1$ and $2$-controlled if $\kappa(t) = \frac{1}{|t| + a}$ where $a > 0$.

**Proposition 3.18.** There exist constants $C \geq 2$, $\delta \in (0, 1)$ such that for any positive $\delta' < \delta$ and any geodesic segment $\gamma: (-\delta', \delta') \to T$, there exists a $C$-controlled function $\kappa: (-\delta', \delta') \to \mathbb{R}_{>0}$ such that for every $t \in (-\delta', \delta')$:

1. $k^2(t) \leq \kappa^2(t)$, where
   
   $$\kappa^2(t) = \sup_{v \in T^1_{\gamma(t)}T} -\langle R(v, \dot{\gamma}(t))\dot{\gamma}(t), v \rangle;$$

2. $\int_{-\delta'}^{\delta'} \kappa(s) \, ds \leq C|\ln(\rho_{\delta'}(\dot{\gamma}(0)))|,$

and $\kappa(\delta') \leq C(\rho_{\delta'}(\dot{\gamma}(0)))^{-1}$, where $\rho_{\delta'}(\dot{\gamma}(0))$ is the distance from the geodesic segment $\gamma[-\delta', \delta']$ to $\partial T$.

**Proof.** Let $c$, $\eta$ and $\delta$ be the constants and let $U$ be the collection of open sets in $T$ given by Proposition 3.2. We write

$$T = U(\eta) \sqcup \Theta;$$

the set $\Theta = T \setminus U(\eta)$ lies in the thick part of Teichmüller space in which the WP sectional curvatures are negative and bounded below by a constant $-b^2$. By shrinking the value of $\delta$ if necessary, we may assume that for every $X \in \Theta$, and $Y \in T$, if $d(X, Y) < \delta$, then:

$$\sup_{v, w \in T^1_{\gamma(t)}T} -\langle R(v, w)w, v \rangle < b^2.$$

Let $B = B(c) > 0$ be the constant given by Corollary 3.16.

Fix $\delta' < \delta$. It follows from Proposition 3.2 that if $\gamma: (-\delta', \delta') \to T$ is a unit-speed WP geodesic, then either $\gamma(0) \in \Theta$, or $\gamma(-\delta', \delta') \subset U$,

for some $U \in U$.

If $\gamma(0) \in \Theta$, then we define $\kappa$ to be the constant function $b$. Then by construction we have $k^2 \leq \kappa^2$. Since the WP distance from any point in $T$ to $\partial T$ is bounded above by a uniform constant, it also follows that in this case:

$$\int_{-\delta'}^{\delta'} \kappa(s) \, ds = 2b\delta' = O(|\ln(\rho_{\delta'}(\dot{\gamma}(0)))|),$$

and $\kappa(\delta') = O(\rho_{\delta'}(\dot{\gamma}(0)))^{-1}$. 


Suppose on the other hand that \( \gamma(\delta', \delta') \subset U \), for some \( U \in \mathcal{U} \). Let \((\sigma, \chi)\) be the combined length basis in \( U \) given by Proposition 3.2 satisfying:

\[
1/c < \ell_{\chi}(X) \leq \tilde{\ell}_{\chi}(X) < c,
\]

for every \( X \in U \). For \( \alpha \in \sigma \), define \( \kappa_\alpha : (-\delta', \delta') \rightarrow \mathbb{R}_{>0} \) by:

\[
\kappa_\alpha(t) = \frac{r_\alpha(t_\alpha)}{r_\alpha(t_\alpha) |t - t_\alpha| + f_\alpha(t_\alpha)},
\]

where \( t_\alpha \) is the unique time in \([-\delta', \delta']\) such that \( f_\alpha(t) \geq f_\alpha(t_\alpha) \) for \( t \in [-\delta', \delta'] \). Observe that \( \kappa_\alpha \) is a 2-controlled function and attains its maximum value of \( r_\alpha(t_\alpha) f_\alpha(t_\alpha) \) at \( t = t_\alpha \).

Applying Corollary 3.16, we obtain that for all \( t \in (-\delta', \delta') \):

\[
k(t) \leq B \max_{\alpha \in \sigma} \{1, \kappa_\alpha\}.
\]

We define \( \kappa : (-\delta', \delta') \rightarrow \mathbb{R}_{>0} \) by:

\[
\kappa = B \max_{\alpha \in \sigma} \{1, \kappa_\alpha\}.
\]

Since \( \kappa_\alpha \) is 2-controlled, for each \( \alpha \), it follows that \( \kappa \) is \( B+1 \)-controlled. By its construction \( \kappa \) satisfies the inequality \( \kappa^2 < \kappa^2 \) on \( (-\delta', \delta') \).

It remains to estimate the integral of \( \kappa \) over the interval \( (-\delta', \delta') \). Simple integration shows that

\[
\int_{-\delta'}^{\delta'} \kappa_\alpha(s) \, ds = O(\max\{\delta', |\ln(f_\alpha(t_\alpha))|\}),
\]

since \( r_\alpha(t_\alpha) = O(1) \).

Note that \( f_\alpha(t_\alpha) \) is the minimum value of the function \( f_\alpha^{1/2} \) along the geodesic segment \( \gamma[-\delta, \delta] \). Lemma 3.9 implies that there exists a constant \( r > 0 \) such that \( f_\alpha(t_\alpha) \geq r \rho_t(\dot{\gamma}(0)) \). This implies that

\[
\int_0^t \kappa(s) \, ds \leq B \max_{\alpha \in \sigma} \int_0^t \kappa_\alpha(s) \, ds = O(\ln(\rho_t(\dot{\gamma}(0)))).
\]

Similarly,

\[
\kappa(0) \leq B \max_{\alpha \in \sigma} \frac{r_\alpha(t_\alpha)}{f_\alpha(t_\alpha)} = O(\ln(\rho_t(\dot{\gamma}(0)))).
\]

\( \diamond \)

Combining Proposition 3.18 and Lemma 3.17, we obtain:

**Proposition 3.19.** There exists \( C > 1 \) such that for any infinite geodesic ray \( \gamma : (-\infty, 1) \rightarrow T \), if \( u(t) \) is a solution to the differential inequality

\[
u' \leq k^2 - u^2
\]

(14)
that is defined for all $t < 1$, then for every $t \in (0, 1)$:
\[ \int_0^t u(s) \, ds \leq C |\ln(\rho_t(\dot{\gamma}(0)))|, \]
and
\[ u(0) \leq C(\rho_t(\dot{\gamma}(0)))^{-1}. \]

3.6. **Proof of Theorem 3.1.** The theorem follows immediately from Proposition 2.1, which bounds the derivative of the time $t$ map in terms of solutions $u$ to the differential inequality $u' \leq k^2 - u^2$, and the estimates in Proposition 3.19 for $u$ and its integral.

4. **Bounding the second derivative of the geodesic flow**

The results in this section will be used to derive bounds on the second derivative of the WP geodesic flow, on the order of an inverse power of the distance to the singular locus. Since these results hold in a more general context, we return to the setting of Riemannian geometry.

4.1. **More on the Sasaki metric and statement of the general result.**

Let $M$ be a Riemannian manifold, and let $\pi: TM \to M$ be the canonical projection. The Sasaki metric on $T^1M$ induces a Sasaki metric on $T T^1M$, which for brevity we will also call the Sasaki metric (although strictly speaking it is some sort of Sasaki Sasaki metric). In general we will denote the Sasaki distance on $T^1M$ by $d_{Sas}$ and on $T T^1M$ by $d_{Sas}$.

Recall that for $v \in T^1M$, each vector $\xi \in T_v T^1M$ can be naturally identified with a pair $(u, w) \in (T^1_{\pi(v)}M)^2$ with $w \perp v$. The distance $d_{Sas}$ on $T T^1M$ induced by this metric can be estimated as follows. Let $\xi_0 = (u_0, w_0) \in (T^1_{\pi(v_0)}M)^2$ and $\xi_1 = (u_1, w_1) \in (T^1_{\pi(v_1)}M)^2$ be tangent vectors in $T T^1M$ based at $v_0$ and $v_1$ respectively. Let $\sigma$ be a Sasaki geodesic in $T^1M$ from $v_0$ to $v_1$. Let $\sigma: T^1_{\pi(v_0)}M \to T^1_{\pi(v_1)}M$ be parallel translation along the curve of basepoints $\pi \circ \sigma$ in $M$. The following lemma follows from the discussion in Section 2.

**Lemma 4.1.** For each $v_0$ there exists an $\epsilon > 0$ such that if $d_{Sas}(v_0, v_1) < \epsilon$, then
\[ d_{Sas}(\xi_0, \xi_1) \leq d_{Sas}(v_0, v_1) + \|u_1 - P_\sigma(u_0)\| + \|w_1 - P_\sigma(w_0)\| \leq 2 d_{Sas}(\xi_0, \xi_1). \]

If $R$ is the curvature tensor of a metric on $M$ and $x \in N$, we define
\[ \|R_x\| = \sup_{v_1, v_2, v_3 \in T^1_2 N} \|R_x(v_1, v_2)v_3\|. \]
and
\[ \|\nabla R_x\| = \sup_{v_1, v_2, v_3, v_4 \in T^1_4 N} \|\nabla_{v_1} R_x(v_2, v_3)v_4\|. \]

The main result in this section is:
Proposition 4.2. Let $M$ be an $m$-dimensional Riemannian manifold, possibly incomplete, and let $t_0 \leq 1$ be a positive number. Let $\gamma : [-t_0, t_0] \to M$ be a unit-speed geodesic segment whose image lies in the interior of $M$.

Suppose that there exist constants $C_1, C_2, C_3 > 1$ and $\epsilon_0 > 0$ such that for all $t \in (-t_0, t_0)$:

1. if $v \in T^1M$ satisfies $d_{\text{Sas}}(v, \dot{\gamma}(0)) < \epsilon_0$, then
   \[
   \max\{\|D_v\varphi_t\|, \|D_{\varphi_t(v)}\varphi_{-t}\|\} \leq C_1,
   \]

2. if $p \in M$ satisfies $d(p, \gamma(t)) < \epsilon_0$, then
   \[
   \|R_p\| \leq C_2 \quad \text{and} \quad \|\nabla R_p\| \leq C_3.
   \]

Then there exists $\epsilon_1 > 0$ such that for every $t \in (-t_0, t_0)$, for every pair $v_0, v_1 \in T^1M$, with $d_{\text{Sas}}(v_1, \dot{\gamma}(0)) < \epsilon_1$, and for all $\xi_i \in T_{v_i}T^1M$, $(i = 0, 1)$, we have:

\[
\text{d}_{\text{Sas}}(D\varphi_t(\xi_0), D\varphi_t(\xi_1)) \leq (8mC_1^3C_2^2C_3)\text{d}_{\text{Sas}}(\xi_0, \xi_1).
\]

4.2. Variations of solutions to linear ODEs. To prove Proposition 4.2, we first treat the linearized version of the problem. We begin with a basic fact about solutions to linear ODEs. Recall that if $A(t) : [0, T] \to L(\mathbb{R}^m)$ is any continuous family of linear transformations of $\mathbb{R}^m$, then the ODE

\[
x'(t) = A(t)x(t); \quad x : [0, T] \to \mathbb{R}^m
\]

has a unique fundamental solution $F(t) : [0, T] \to L(\mathbb{R})$, which satisfies $F(0) = I$ and

\[
F'(t) = A(t)F(t);
\]

every solution to (15) with initial condition $x(0) = x_0$ then takes the form $x(t) = F(t)x_0$.

Lemma 4.3. Let $F_i : [0, T] \to L(\mathbb{R}^m)$ be the fundamental solution to the differential equation $x'(t) = A_i(t)x(t)$, for $i = 0, 1$. Then

\[
\|F_0(T) - F_1(T)\| \leq T\|F_0\|_0\|F_0^{-1}\|_0\|A_0 - A_1\|_0\|F_1\|_0,
\]

where the $\| \cdot \|_0$ is the supremum of the operator norm over the interval $[0, T]$.

Proof. For $t \in [0, T]$, let $W(t) = F_0(t) - F_1(t)$. Since $F_i'(t) = A_i(t)F_i(t)$ and $F_i(0) = I$ for $i = 0, 1$, we obtain by subtraction that $W$ is a solution to the affine initial value system:

\[
W' = A_0F_0 - A_1F_1 = A_0W + B \quad W(0) = 0,
\]

where $B(t) = (A_0(t) - A_1(t))F_1(t)$. This system can be solved explicitly; we obtain that

\[
W(t) = F_0(t)\left(\int_0^t F_0(s)^{-1}B(s)\,ds\right),
\]

as can be verified directly by differentiation ($W' = F_0F_0^{-1}W + B = A_0W + B$). The desired bound on $\|W(T)\|$ now follows immediately. \(\diamondsuit\)
Consider next the second-order linear ODE
\begin{equation}
    x''(t) = -R(t)x(t)
\end{equation}
where $R: [0, T] \rightarrow L(\mathbb{R}^m)$; in our application $R(t)$ will be a matrix representing the sectional curvature operator along a geodesic $\gamma$ and (16) will be the Jacobi equation in a suitably chosen coordinate system along $\gamma$.

Then (16) can be transformed into a first order system in the standard way by introducing the variable $z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \in \mathbb{R}^{2m}$ and the additional constraint $x'(t) = y(t)$. Then $z$ satisfies the first order ODE:
\begin{equation}
    z'(t) = A(t)z(t),
\end{equation}
where
\[ A(t) = \begin{pmatrix} 0 & I \\ -R(t) & 0 \end{pmatrix}. \]

The fundamental solution $F(t)$ to this equation has the property that if $x(t)$ is a solution to (16) with initial values $x(0) = x_0$, $x'(0) = y_0$, then
\[ \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} = F(t) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \]

We obtain the corollary:

**Corollary 4.4.** Let $F_i: [0, T] \rightarrow L(\mathbb{R}^{2m})$ be the fundamental solution to the differential equation $x''(t) = -R_i(t)x(t)$, for $i = 0, 1$. Then
\[ \|F_0(T) - F_1(T)\| \leq T\|F_0\|\|F_0^{-1}\|\|R_0 - R_1\|\|F_1\|. \]

4.3. **Proof of Proposition 4.2.** We now return to the setting of differential geometry and finish the proof of Proposition 4.2. Let $\gamma: [-t_0, t_0] \rightarrow M$ be given. We start with a lemma.

**Lemma 4.5.** With the assumptions of Proposition 4.2 suppose $p_0, p_1 \in M$ satisfy $d(p_i, \gamma(t)) < \epsilon_0$, then for all $v_i, w_i \in T^1_{p_i}M$, the curvature tensor $R$ satisfies:
\[ \|R(v_0, w_0)w_0\| \leq C_2, \]
and
\[ d_{Sas}(R(v_0, w_0)w_0, R(v_1, w_1)w_1) \leq C_3(d_{Sas}(v_0, v_1) + d_{Sas}(w_0, w_1)). \]

**Proof.** This follows in a straightforward way from the Mean Value Theorem and the hypotheses that $\|R_p\| \leq C_2$ and $\|\nabla R_p\| \leq C_3$, for all $p \in M$ with $d(p, \gamma(t)) < \epsilon_0$. \(\diamondsuit\)

Let $v_0, v_1 \in T^1M$ be unit tangent vectors in a neighborhood of $\dot{\gamma}(0)$, and let $\sigma: (-2, 2) \rightarrow T^1M$ be a Sasaki geodesic with $\sigma(0) = v_0$ and $\sigma(1) = v_1$. Each $\sigma(s)$ determines a unit speed geodesic $\gamma_s: (-t_0, t_0) \rightarrow M$ with $\dot{\gamma}_s(0) = \sigma(s)$. In this way $\sigma$ determines a variation of geodesics $\alpha: (-2, 2) \times (-t_0, t_0) \rightarrow M$ with the property that $\alpha(s, t) = \gamma_s(t)$. 

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We may assume that the norms of the derivatives of \( \alpha \) are uniformly bounded from above by a constant, say 1. For \( s \in (-2, 2) \), let \( L_s(t) = \partial \alpha / \partial s(s, t) \) be the induced Jacobi field along along \( \gamma_s \). Choose \( \epsilon_1 \) such that if \( d_{Sas}(v_1, \dot{\gamma}(0)) < \epsilon_1 \) for \( i = 0, 1 \), then \( \dot{d}_{Sas}(\dot{\gamma}_s(t), \dot{\gamma}_s(t)) < \epsilon_0 \) for all \( (s, t) \in (-2, 2) \times (-t_0, t_0) \), where \( \epsilon_0 \) is given by the hypotheses of the proposition. If \( d_{Sas}(v_1, \dot{\gamma}(0)) < \epsilon_1 \), then for any \( (s, t) \in (-2, 2) \times (-t_0, t_0) \) we have

\[
d_{Sas}(\dot{\gamma}_s(t), \dot{\gamma}_0(t)) \leq \int_0^s \| (L_u(t), L'_u(t)) \|_{Sas} du.
\]

Since \( \sigma \) is a Sasaki geodesic the above inequality is an equality in the case of \( t = 0 \); that is

\[
\int_0^s \| L_u(0), L'_u(0) \|_{Sas} du = d_{Sas}(\dot{\gamma}_s(0), \dot{\gamma}_0(0)).
\]

By the bounds (1) on the first derivative of the geodesic flow (which bounds the growth of Jacobi fields), we also have that for any \( u, t \),

\[
\| (L_u(t), L'_u(t)) \|_{Sas} \leq C_1 \| (L_u(0), L'_u(0)) \|_{Sas},
\]

and so

\[
\int_0^s \| (L_u(t), L'_u(t)) \|_{Sas} du \leq C_1 \int_0^s \| L_u(0), L'_u(0) \|_{Sas} du.
\]

Putting these inequalities together, we obtain:

\[
d_{Sas}(\dot{\gamma}_s(t), \dot{\gamma}_0(t)) \leq C_1 d_{Sas}(\dot{\gamma}_s(0), \dot{\gamma}_0(0)).
\]

Our goal is to bound the Lipschitz norm of the derivative of the time-\( t \) map of the geodesic flow \( \varphi_t \) at \( \dot{\gamma}_0(0) \). The conclusion of Proposition 4.2 will follow if we show that for any \( (s, t) \in (-2, 2) \times (-t_0, t_0) \), and any \( \xi_0 \in T_{\gamma_0(0)}T^1 M \) and \( \xi_s \in T_{\dot{\gamma}_s(0)}T^1 M \), we have:

\[
d_{Sas}(D\varphi_t(\xi_0), D\varphi_t(\xi_s)) \leq (4mC_1^2C_2^2C_3) d_{Sas}(\xi_0, \xi_s).
\]

Recall that under the standard identification of \( \xi_s \in T_{\gamma_0(0)}T M \) with a pair \((u_s, w_s) \in (T_{\gamma_0(0)}M)^2 \), the vector \( D\varphi_t(\xi_0) \) is identified with \((J_s(t), J'_s(t))\), where \( J_s \) is the solution to the (second-order) Jacobi equation

\[
J'' + R(J, \dot{\gamma_s})\dot{\gamma}_s = 0
\]

with initial condition \((J_s(0), J'_s(0)) = (u_s, w_s)\).

To analyze the variation of solutions to this ODE, we fix convenient coordinates for the tangent bundle to the geodesic \( \gamma_s \) in order to express (20) as a matrix equation of the form (16). To this end, let \( \{e_j(0, 0) : j = 1, \ldots, m\} \) be an orthonormal frame at \( \gamma_0(0) = \alpha(0, 0) \) spanning the tangent space \( T_{\gamma_0(0)}M \). We first parallel translate this frame along \( \alpha(s, 0) \) to obtain an orthonormal frame \( \{e_j(s, 0)\} \) at \( \gamma_s(0) \), for \( s \in (-2, 2) \). We next parallel translate the frame \( \{e_j(s, 0)\} \) along \( \gamma_s(t) \), for \( t \in (-t_0, t_0) \) to obtain a frame \( \{e_j(s, t)\} \) at each point \( \alpha(s, t) \).
Lemma 4.6. For \( j \in \{1, \ldots, m\} \), we have:
\[
d_{Sas}(e_j(0, t), e_j(s, t)) \leq d(\gamma_0(0), \gamma_s(0)) + 2tC_1C_2 d_{Sas}(\dot{\gamma}_0(0), \dot{\gamma}_s(0)),
\]
for all \((s, t) \in (-2, 2) \times (-t_0, t_0)\).

Proof. Fix \( j \). Our construction of \( e_j \) (using parallel translation) gives that for all \( s, t \):
\[
\frac{D}{Ds} e_j(s, 0) = 0, \quad \text{and} \quad \frac{D}{Dt} e_j(s, t) = 0;
\]
we would like to estimate \( \frac{D}{Dt} \frac{D}{Ds} e_j(s, t) \) for general \( s, t \). To do this, we first estimate \( \frac{D}{Dt} \frac{D}{Ds} e_j(s, t) \).

It follows directly from the definition of the Riemannian curvature tensor and the joint integrability of the pair \( \{L_s, \dot{\gamma}_s\} \) that
\[
R(L_s(t), \dot{\gamma}_s(t)) e_j(s, t) = \frac{D}{Dt} \frac{D}{Ds} e_j(s, t) - \frac{D}{Dt} \frac{D}{Ds} e_j(s, t),
\]
where we have used the second part of (21) in the last step. Applying the bound
\[
\|R(L_s(t), \dot{\gamma}_s(t)) e_j\| \leq C_2 \|L_s(t)\|,
\]
we obtain that
\[
\|\frac{D}{Dt} \frac{D}{Ds} e_j(s, t)\| \leq C_2 \|L_s(t)\|.
\]
Integrating this expression with respect to \( t \), we then have the bound:
\[
\|\frac{D}{Ds} e_j(s, t)\| \leq \|\frac{D}{Ds} e_j(s, 0)\| + C_2 \int_0^t \|L_s(u)\| \, du = C_2 \int_0^t \|L_s(u)\| \, du.
\]
Integrating again, this time with respect to \( s \), and using Lemma 4.1 and (18), we obtain:
\[
d_{Sas}(e_j(0, t), e_j(s, t)) \leq d(\gamma_0(0), \gamma_s(0)) + \int_0^s \|\frac{D}{Ds} e_j(u, t)\| \, du
\]
\[
\leq d(\gamma_0(0), \gamma_s(0)) + \int_0^s \int_0^t \|L_u'(u)\| \, du \, dw
\]
\[
+ C_2 \int_0^s \int_0^t \|L_u(u)\| \, du \, dw
\]
\[
\leq d(\gamma_0(0), \gamma_s(0)) + 2C_2 \int_0^s \int_0^t \|(L_u(u), L_u'(u))\|_{Sas} \, du \, dw
\]
\[
\leq d(\gamma_0(0), \gamma_s(0)) + 2tC_1C_2 \int_0^s \|(L_u(0), L_u'(0))\|_{Sas} \, dw
\]
\[
= d(\gamma_0(0), \gamma_s(0)) + 2tC_1C_2 d_{Sas}(\dot{\gamma}_0(0), \dot{\gamma}_s(0)),
\]
which is the desired bound. \( \diamond \)
For \((s,t) \in (-2,2) \times (-t_0,t_0)\), the frame \(\{e_j(s,t)\}\) gives an isometric isomorphism between \(\mathbb{R}^m\) and \(T_{\alpha(s,t)}M\):
\[
(x_1, \ldots, x_m) \mapsto \sum_{j=1}^{m} x_j e_j(s,t).
\]
This in turn induces for each \((s,t)\) an isometric isomorphism
\[
I_{s,t} : \mathbb{R}^{2m} \to T_{\gamma_s}(t)TM \cong (T_{\alpha(s,t)}M)^2.
\]
Lemma 4.6 has the following immediate corollary.

**Corollary 4.7.** For each \((s,t) \in (2,2) \times (-t_0,t_0)\) and (Euclidean) unit vector \(z \in \mathbb{R}^{2m}\), we have
\[
d_{Sas}(I_{s,t}(z), I_{0,t}(z)) \leq d(\gamma_0(0), \gamma_s(0)) + 2tC_1C_2 d_{Sas}(\dot{\gamma}_0(0), \dot{\gamma}_s(0)).
\]
Expressing the Jacobi equation (20) along \(\gamma_s\) in the coordinates on \(T_{\gamma_s}Teich(s)\) given by \(I_{s,t}\), we obtain the ODE:
\[
x''(t) = -R_s(t)x(t),
\]
where \(((R_s(t))_{i,j} = \langle R(e_i(s,t), \dot{\gamma}_s(t)), \dot{\gamma}_s(t), e_j(s,t) \rangle\).

It is here one can see the convenience in choosing the frame \(\{e_j\}\) to be parallel along the geodesic \(\gamma_s\); otherwise, the Jacobi equation would include terms involving \(x'\), which would complicate the linear analysis.

Denote by \(F_s : (-t_0,t_0) \to L(\mathbb{R}^{2m})\) the fundamental solution to (22). Corollary 4.4 implies that for any \((s,t) \in (-2,2) \times (-t_0,t_0)\), we have
\[
\|F_0(t) - F_s(t)\| \leq t\|F_0\|\|F_s^{-1}\|\|R_0 - R_s\|\|F_s\|.
\]
Now the main hypotheses of the proposition, when combined with 4.6 and (18), and the triangle inequality can be seen to give the upper bound:
\[
\|R_0(t) - R_s(t)\| \leq (mC_1C_2^2C_3) d_{Sas}(\dot{\gamma}_0(0), \dot{\gamma}_s(0)),
\]
where we omit the details and so
\[
\|F_0(t) - F_s(t)\| \leq (tmC_1C_2^2C_3)\|F_0\|\|F_s\|\|F_s^{-1}\|\|R_0 - R_s\|\|F_s\|.
\]
for all \((s,t) \in (-2,2) \times (-t_0,t_0)\). The bounds on the first derivative of \(\varphi_t\) imply that for all \((s,t) \in (-2,2) \times (-t_0,t_0)\), we have \(\max\{|\|F_s(t)\|, |\|F_s^{-1}(t)\|\} \leq C_1\), which implies that
\[
\|F_0(t) - F_s(t)\| \leq (tmC_1^2C_2^2C_3) d_{Sas}(\dot{\gamma}_0(0), \dot{\gamma}_s(0)).
\]
Finally, suppose that \(\xi_0 = I_{0,0}(z_0)\) and \(\xi_s = I_{s,0}(z_s)\) are arbitrary unit tangent vectors to \(T^1M\) based at \(\dot{\gamma}_0(0)\) and \(\dot{\gamma}_s(0)\), respectively (where \(z_0, z_s\) are Euclidean unit vectors in \(\mathbb{R}^{2m}\)). Since \(\frac{D}{Ds} I_{s,0} = 0\), Lemma 4.1 implies that the Sasaki distance \(d_{Sas}(\xi_0, \xi_s)\) between \(\xi_0\) and \(\xi_s\) is uniformly comparable to \(|z_0 - z_1| + d_{Sas}(\dot{\gamma}_0(0), \dot{\gamma}_s(0))\); in particular:
\[
|z_0 - z_1| + d_{Sas}(\dot{\gamma}_0(0), \dot{\gamma}_s(0)) \leq 2d_{Sas}(\xi_0, \xi_s).
\]
We may then conclude using Corollary 4.7 and our previous estimates that:

\[
\begin{align*}
\mathbf{d}_{\text{Sas}}(D\varphi_t(\xi_0), D\varphi_t(\xi_s)) &= \mathbf{d}_{\text{Sas}}(I_{0,t}(F_0(t)z_0), I_{s,t}(F_s(t)z_s)) \\
&\leq \mathbf{d}_{\text{Sas}}(I_{0,t}(F_0(t)z_0), I_{0,t}(F_s(t)z_s)) \\
&\quad + \mathbf{d}_{\text{Sas}}(I_{0,t}(F_s(t)z_s), I_{s,t}(F_s(t)z_s)) \\
&\leq \|F_0(t)z_0 - F_s(t)z_s\| \\
&\quad + \|F_s(t)z_s\| (d(\gamma_0(0), \gamma_s(0)) + 2tC_1C_2 d_{\text{Sas}}(\gamma_0(0), \gamma_s(0))) \\
&\leq \|(F_0(t) - F_s(t))z_0\| + \|F_s(t)(z_0 - z_s)\| \\
&\quad + C_1 (d(\gamma_0(0), \gamma_s(0)) + 2tC_1C_2 d_{\text{Sas}}(\gamma_0(0), \gamma_s(0))) \\
&\leq (tmC_4^4C_2^3) d_{\text{Sas}}(\gamma_0(0), \gamma_s(0)) + C_1\|z_0 - z_s\| \\
&\quad + C_1 (d(\gamma_0(0), \gamma_s(0)) + 2tC_1C_2 d_{\text{Sas}}(\gamma_0(0), \gamma_s(0))).
\end{align*}
\]

Using the fact that \(t_0 < 1\) we obtain

\[
\begin{align*}
\mathbf{d}_{\text{Sas}}(D\varphi_t(\xi_0), D\varphi_t(\xi_s)) &\leq (mC_4^4C_2^3) d_{\text{Sas}}(\gamma_0(0), \gamma_s(0)) + C_1\|z_0 - z_s\| \\
&\quad + 3C_4^4C_2^2 d_{\text{Sas}}(\gamma_0(0), \gamma_s(0)) \\
&\leq (4mC_4^4C_2^3) (\|z_0 - z_s\| + d_{\text{Sas}}(\gamma_0(0), \gamma_s(0))) \\
&\leq (8mC_4^4C_2^3) \mathbf{d}_{\text{Sas}}(\xi_0, \xi_s),
\end{align*}
\]

where we used (23) in the last step. This proves the desired inequality (19) and completes the proof of Proposition 4.2.

5. Higher order control of the WP metric

In this section, we show how to control higher order derivatives of the WP metric. If \(R\) is the curvature tensor of a Riemannian metric, we define \(\|\nabla R_x\|\) as in the previous section and let

\[
\|\nabla^2 R_x\| = \sup_{v_1, \ldots, v_5 \in T_x M} \|\nabla^2_{v_1, v_2} R_x(v_3, v_4)v_5\|,
\]

where \(\nabla^2 R\) is the second covariant derivative of the curvature tensor:

\[
\nabla^2_{X,Y} R = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) R.
\]

The main result in this section is

**Proposition 5.1.** There exist \(C, \beta_1 > 0\) such that for any \(X_0 \in T\), the WP curvature tensor \(R_{WP}\) satisfies:

\[
\text{max}\{\|\nabla R_{WP}\|_{X_0}, \|\nabla^2 R_{WP}\|_{X_0}\|\} \leq C\rho_0^{-\beta_1},
\]

where \(\rho_0 = \rho_0(X_0)\) is the distance from \(X_0\) to the singular locus \(\partial T\).

We remark that similar bounds on higher derivatives of the WP curvature tensor can also be obtained using the methods in this section. The preliminary step in proving this proposition is to establish comparisons between the WP and Teichmüller metrics.
5.1. Comparison of Weil-Petersson and Teichmüller metrics. The goal of this section is to prove the following proposition, which compares Teichmüller and Weil-Petersson lengths of vectors. Much of the material for this section is taken from [36] and [23]. There are similar computations to be found in [21].

For a given Riemann surface $X$, recall that $\ell(X)$ denotes the length of the shortest simple closed curve in the hyperbolic metric.

**Proposition 5.2.** There exists $C > 0$ such that for any $X \in \text{Teich}(S)$ and any tangent vector $[\mu] \in T_X \text{Teich}(S)$, we have

$$\|\mu\|_{WP} \geq C \ell(X)^{1/2} \|\mu\|_T.$$ 

The proposition follows from the following lemma.

**Lemma 5.3.** For any holomorphic quadratic differential $\phi \in Q(X)$,

$$\|\phi\|_{WP} \leq C \ell(X)^{-1/2} \|\phi\|_T.$$ 

We now give the proof of the proposition assuming Lemma 5.3.

**Proof.** We have

$$\|\mu\|_T = \sup_{\|\phi\|_T = 1} |\phi(\mu)| \leq \sup_{\|\phi\|_T = 1} \|\phi\|_{WP} \|\mu\|_{WP} \leq \sup_{\|\phi\|_T = 1} C \ell(X)^{-1/2} \|\phi\|_T \|\mu\|_{WP} = C \ell(X)^{-1/2} \|\mu\|_{WP}.$$

\[\Box\]

We now begin the discussion leading to the proof of Lemma 5.3. We again use notation $x \asymp y$ to indicate that the two quantities $x, y$ are comparable up to multiplicative constants that depend only on $g, n$.

5.1.1. Blowing up nodes and $(s, t)$ coordinates. Now suppose $[f: S \to X_\sigma]$ is a marked noded Riemann surface in $T_\sigma$, where the simplex $\sigma$ has $p \leq 3g - 3 + n$ nodes.

For each small $\eta$ consider the neighborhood $\Omega(\sigma, \eta)$ of $[f: S \to X_\sigma]$ in $T$. Now let $\Gamma_\sigma$ be the abelian group generated by the Dehn twists about the curves in $\sigma$, and let $V_\sigma = \Omega(\sigma, \eta)/\Gamma_\sigma$. Let $P_\sigma: V_\sigma \to \Omega(\sigma, \eta)$ denote the projection map. We make our computations using coordinates in $V_\sigma$ which we now describe. In what follows the marking is irrelevant.

Separating the nodes of $X_\sigma$ into pairs of punctures, denoted $P_i, Q_i$, produces the Riemann surface $\tilde{X}_\sigma$. Choose conformal neighborhoods $V_i = \{z_i : 0 < |z_i| < 1\}$ and $W_i = \{w_i : 0 < |w_i| < 1\}$ of $P_i$ and $Q_i$. These may be taken to be mutually disjoint. Choose a nonempty open set $O$ on $\tilde{X}_\sigma$ disjoint from $U_i(V_i \cup W_i)$. (If $\tilde{X}_\sigma$ is disconnected $O$ consists of an open set in each component). There exist Beltrami differentials $\nu_1, \ldots, \nu_{3g-3+n-p}$ supported in $O$ whose equivalence classes form a basis for $T_{\tilde{X}_\sigma} T_\sigma$. This implies that for
any $\hat{X}_\sigma$ sufficiently close to $\hat{X}_\sigma$, there is a tuple $s(\hat{X}_\sigma) = (s_1, \ldots, s_{3g-3+n-p})$ of complex numbers close to 0 and a quasiconformal map $h : \hat{X}_\sigma \to \hat{X}_\sigma$ such that the dilatation $\mu(h)$ of $h$ satisfies

$$\mu(h) = \sum_{i=1}^{3g-3+n-p} s_i \nu_i.$$ 

This gives a parametrization $\{\hat{X}_\sigma(s)\}$ of surfaces in a neighborhood of $\hat{X}_\sigma$ that lie in $\partial T$ by a neighborhood $D^{3g-3+n-p}$ of 0 in $\mathbb{C}^{3g-3+n-p}$.

Since the map $h$ (on each component) is conformal in $U_i \cup V_i$ the coordinates $z_i, w_i$ are local holomorphic coordinates in neighborhoods $V_i, W_i$ of the punctures on each $\hat{X}_\sigma(s)$.

We now parametrize $V_\sigma$ by pairs $(s, t)$ where $s \in D^{3g-3+n-p}$ and $t = (t_1, \ldots, t_p)$ are small complex numbers. For each surface $\hat{X}_\sigma$ and for each $1 \leq i \leq p$ remove the disc of radius $|t_i|^{1/2}$ from each of $V_i$ and $W_i$ and then glue $z_i$ to $t_i/w_i$, identifying the circle $|z_i| = |t_i|^{1/2}$ with the circle $|w_i| = |t_i|^{1/2}$. We denote the resulting surface by $X(s, t)$; this gives a parametrization

$$\{X(s, t) : s \in D^{3g-3+n-p}, t \in D^p\}$$

of $V_\sigma$. We note that in this notation we have $X(s, 0) = \hat{X}_\sigma(s)$; that is, if all $t_i = 0$, then there are no discs to remove.

For each $t_i$ we can form the annulus

$$A_{t_i} = \{z_i : |t_i| \leq |z_i| \leq 1\} = \{w_i : |t_i| \leq |w_i| \leq 1\}$$

as a subset of $X(s, t)$.

5.1.2. Metric in coordinates. At points $X(s, t)$ with $t_i \neq 0$ for all $i$, we may regard

$$\{\partial/\partial t_1, \ldots, \partial/\partial t_p, \partial/\partial s_1, \ldots, \partial/\partial s_{3g-3+n-p}\}$$

as a basis for the tangent space to $V_\sigma$ and therefore as (equivalence classes of) Beltrami differentials on these surfaces. There exists a dual basis

$$\phi_1, \ldots, \phi_p, \phi_{p+1}, \ldots, \phi_{3g-3+n}$$

of quadratic differentials for the cotangent space at $X(s, t)$ whose expressions in terms of the coordinates $z_i$ and $w_i$ on the surface $X(s, t)$ take the following form:

$$\phi_j = \frac{\partial^2}{\partial z^2_j} (c_{i,j}(s, t) + f_{i,j}(z_i, s, t) + g_{i,j}(w_i, s, t)), $$

where

(a) $f_{i,j}(z_i, s, t)$ is holomorphic in $|z_i| < 1$ and $t$ and real analytic in $s$;
(b) $g_{i,j}(w_i, s, t)$ is holomorphic in $|w_i| < 1$ and $t$ and real analytic in $s$;
(c) $f_{i,j}(0, s, t) = g_{i,j}(0, s, t) = 0$;
(d) if $j > p$ or $j \leq p$ and $i \neq j$, then $c_{i,j}(s, t) = 0$; and
(e) if $j \leq p$, then $c_{j,j}(s, t)$ is bounded above and below away from 0.
For each \( j \leq p \) the term \( c_{j,j}(s,t) \) is called the \textit{residue} of \( \phi_j \). It is the coefficient of the \( 1/z^2 \) term in the Laurent expansion of \( \phi_j \) in \( A_{t_j} \). As we will see below, it is the dominant term in the calculation of integrals in \( A_{t_j} \). Now we prove Lemma 5.3.

\textbf{Proof.} We first note that the hyperbolic metric \( \rho(z_i)|dz_i| \) on \( X(s,t) \) restricted to the annulus \( A_{t_i} \) satisfies

\begin{equation}
\rho(z_i) \asymp \frac{1}{|z_i|(|\log |z_i||}. \tag{24}
\end{equation}

The hyperbolic length of the short curve \( \alpha_i \) is given by

\begin{equation}
\ell_{\alpha_i}(X) \asymp \int_{|z_i|=|t_i|^{1/2}} \frac{|dz_i|}{-|z_i||\log |z_i||} \asymp -\frac{1}{|\log |t_i||}. \tag{25}
\end{equation}

Now suppose \( \phi \in QD(X(s,t)) \) satisfies

\[ \|\phi\|_T = \int_{X(s,t)} |\phi| = 1. \]

Write

\[ \phi = \sum_{j=1}^{p} a_j \phi_j + \sum_{j=p+1}^{3g-3+n} b_j \phi_j, \]

and set

\[ \psi = \sum_{j=p+1}^{3g-3+n} b_j \phi_j. \]

We note that

\[ \frac{dz^2_i}{z^2_i} = \frac{dw^2_i}{w^2_i} \]

under the change of variables \( w_i = t_i/z_i \). This means that we can use \( z_i \) coordinates on \( A_{t_i} \) when calculating the integral of \( f_{i,j} \) and the \( w_i \) coordinates for calculating the integral of \( g_{i,j} \). Now (a) and (c) say that \( f_{i,j}/z^2_i \) has at most a simple pole at \( z_i = 0 \), so

\[ \int_{A_{t_i}} \left| \frac{f_{i,j}}{z^2_i} \right| |dz^2_i| = O(1), \]

and similarly for \( g_{i,j} \). Combining this with (d), for \( j \neq i \) we obtain

\begin{equation}
\int_{A_{t_i}} |\phi_j| = O(1). \tag{26}
\end{equation}

On the other hand it follows from (e) that for \( i \leq p \), we have

\begin{equation}
\int_{A_{t_i}} |\phi_i| \asymp -\log |t_i|. \tag{27}
\end{equation}
Each $\phi_j$ has $L^1$ norm bounded below away from 0 in the complement of the union of the $A_{t_i}$. This implies that $|b_j| = O(1)$, and
\begin{equation}
1 = \|\phi\|_T \asymp \max_i \{\|\psi\|_T, \|a_i\phi_i\|_T\}.
\end{equation}

The residue of $\phi_j$ is 0 for $j \geq p + 1$. Then the expansion (24) for the hyperbolic metric in $A_{t_i}$ shows that
\[ \int_{A_{t_i}} |\psi|^2 \asymp \int_{|t_i|}^1 (\log |z_i|)^2 |dz_i|^2 = O(1). \]

On the other hand, we also have
\[ \int_{A_{t_i}} |\psi| = O(1), \]
which shows that
\[ \|\psi\|_{WP} \asymp \|\psi\|_T. \]
Thus if the maximum in (28) is given by $\|\psi\|_T$ we are done.

Therefore we are reduced to the case that the maximum is given by $\|a_i\phi_i\|_T$, for some $i \leq p$. Then from (27) we see that
\[ |a_i| \asymp \frac{1}{-\log |t_i|}. \]
Thus
\[ \|\phi_i\|_{WP} \asymp \int_{A_{t_i}} |\phi_i|^2/\rho^2 \asymp \int_{A_{t_i}} |dz_i|^2 (\log |z_i|)^2 / |z_i|^2 \]
\[ = \int_{|t_i|}^1 dr (\log r)^2 / r \asymp (-\log |t_i|)^3, \]
which by (25) and the above estimate for $|a_i|$ gives the desired estimate:
\[ \|a_i\phi_i\|_{WP} \leq C(-\log |t_i|)^{1/2} \leq \ell(X)^{-1/2} \|\phi\|_T. \]
This completes the proof of Lemma 5.3.

5.2. Estimates on the WP metric in special coordinates. Following [21], we introduce coordinates on $\text{Teich}(S)$ in which we can bound the derivatives of the WP metric. In the section we denote by $N$ the complex dimension of $\text{Teich}(S)$. Let $\Delta^N$ denote the Euclidean unit polydisk in $\mathbb{C}^N$. We will denote by $z = (z_1, \ldots, z_N)$ an element of $\Delta^N$, where $z_k$ is a complex coordinate, and by $x_k = \text{Re}(z_k)$, $y_k = \text{Im}(z_k)$ the real coordinates. Let $e_i$ be the vector field $\partial/\partial x_i$, for $1 \leq i \leq N$, and $\partial/\partial y_{N-i}$, for $N + 1 < i \leq 2N$.

The following lemma is proved in [21] and follows from Nehari’s bound and the fact that the Teichmüller and Kobayashi metrics agree on the image of the Bers embedding.
Lemma 5.4. [21, cf. Theorem 2.2 and Proof of Theorem 8.2] There exists $C \geq 1$ such that for any $X_0 \in \text{Teich}(S)$, there is a holomorphic embedding $\psi = \psi_{X_0} : \Delta^N \to \text{Teich}(S)$, sending $0 \in \Delta^N$ to $X_0$ and such that for every $v \in T \Delta^N$, we have:

$$\frac{1}{C} \|v\| \leq \|D\psi(v)\|_T \leq C \|v\|$$

where $\|\cdot\|$ is the Euclidean norm on $\Delta^N$ and $\|\cdot\|_T$ is the Teichmüller Finsler norm on $\text{Teich}(S)$.

Fix a point $X_0 \in \text{Teich}(S)$, and let $\psi = \psi_{X_0}$ be the holomorphic embedding given by this lemma. Since the metric $g_{WP}$ on $\text{Teich}(S)$ is Kähler with respect to the 2-form $\omega_{WP}$, and $\psi$ is holomorphic, it follows that the pullback metric $\psi^*g_{WP}$ on $\Delta^N$ is Kähler with respect to the pullback form $\psi^*\omega_{WP}$ and the standard almost complex structure on $\Delta^N$. For $z \in \Delta^N$, denote by $G(z)$ the matrix with entries $G_{ij}(z) = (\psi^*g_{WP})_z(e_i, e_j)$.

The main content of this section is the proof of the following proposition.

Proposition 5.5. There exists $C > 0$ such that, for every $X_0 \in \text{Teich}(S)$, and every $z \in \Delta^N$ with $\|z\| \leq 1/2$ we have:

1. $\|G^{-1}(z)\| \leq C \ell(X_0)^{-1}$, and
2. for any $i, j \in \{1, \ldots, 2N\}$ and any $k \geq 0$, we have

$$\sup_{(\xi_1, \ldots, \xi_k) \in \{x_1, \ldots, x_N, y_1, \ldots, y_N\}^k} \left| \frac{\partial^k G_{i,j}}{\partial \xi_1 \cdots \partial \xi_k}(z) \right| \leq C k!.$$ 

We will use Proposition 5.5 to bound the covariant derivatives of the WP curvature in terms of the distance to the singular strata. An immediate corollary that we record now is:

Corollary 5.6. For each $X_0 \in \text{Teich}(S)$, the image of the polydisk $\{ |z| \leq 1/2 \} \subset \Delta^N$ under the embedding $\psi = \psi_{X_0}$ given by Lemma 5.4 contains the WP metric ball around $X_0$ of radius $(2C)^{-1}\ell(X_0)^{1/2}$, where $C$ is given by Proposition 5.5.

Proof. We prove Proposition 5.5. Part (1) is just Proposition 5.2 and Lemma 5.4. The proof of part (2) uses in a crucial way results of McMullen in [21].

Using the embedding $\psi$, we define an embedding $\Psi : \Delta^N \times \Delta^N \to QF(S)$ by

$$\Psi(z, w) = (\psi(z), \overline{\psi(w)}).$$

Since $\psi$ is holomorphic and $X \mapsto \overline{X}$ is antiholomorphic, the map $\Psi$ is holomorphic. Note that the image of the antidiagonal $A = \{(z, \overline{z}) : z \in \Delta^N\}$ under $\Psi$ lies in the Fuchsian locus $F(S) \subset QF(S)$. Denote by $\alpha : \Delta^N \to \Delta^N \times \Delta^N$ the antidiagonal embedding $\alpha(z) = (z, \overline{z})$, and by $\tilde{\alpha} : \text{Teich}(S) \to QF(S)$ the antidiagonal embedding $\tilde{\alpha}(X) = (X, \overline{X})$. Then we have the following commutative diagram:
Note that the maps $\alpha$ and $\hat{\alpha}$ are not holomorphic, although their derivatives are bounded in the Euclidean and Teichmüller metrics, respectively.

Since $\text{Teich}(S)$ and $QF(S)$ are complex manifolds, so are their cotangent bundles $T^*\text{Teich}(S)$ and $T^*QF(S)$, and $T^*QF(S) = T^*\text{Teich}(S) \oplus T^*\text{Teich}(\overline{S})$. Fixing $Z \in \text{Teich}(\overline{S})$ we define a map $\tau: QF(S) \rightarrow T^*\text{Teich}(S)$ by:

$$\tau(X, Y) = \sigma_{QF}(X, Y) - \sigma_{QF}(X, Z).$$

Since $T^*\text{Teich}(S)$ embeds as the first factor in $T^*QF(S)$, we may regard $\tau$ as a 1-form on $QF(S)$. McMullen proves that $\tau$ has the following properties:

1. $\tau$ is holomorphic.
2. the 1-form $\theta = -\hat{\alpha}^*\tau$ on $\text{Teich}(S)$ is a primitive for the WP Kähler form:
   $$d(i\theta) = \omega_{WP}.$$
3. For any $(X, Y) \in \text{Teich}(S)$, the covector $\tau(X, Y)$ is bounded in the Teichmüller Finsler norm on $\text{Teich}(S)$:
   $$\|\tau(X, Y)\|_T = \sup\{\|\tau(X, Y)(\mu)\| : \mu \in TX\text{Teich}(S), \|\mu\|_T = 1\} \leq C,$$
   where $C$ is independent of $(X, Y)$.

Pulling the holomorphic 1-form $\tau$ back to $\Delta^N \times \Delta^N$, we thus obtain a holomorphic 1-form $\kappa = \Psi^*\tau$.

**Lemma 5.7.** $\kappa$ is bounded in the Euclidean metric on $\Delta^N \times \Delta^N$.

**Proof.** $\tau$ is bounded in the Teichmüller metric, and the Euclidean metric is comparable to the $\Psi$-pullback of the Teichmüller metric, by Lemma 5.4. $\diamond$

**Lemma 5.8.** The holomorphic 2-form $\Omega = d(i\kappa)$ on $\Delta^N \times \Delta^N$ satisfies $\alpha^*\Omega = \psi^*\omega_{WP}$, which is the Kähler 2-form for the pullback metric $\psi^*g_{WP}$.

**Proof.** This follows from the commutativity of the diagram above. $\diamond$

We now finish the proof of Proposition 5.5. In complex coordinates $(z_1, \ldots, z_n, w_1, \ldots, w_N)$ on $\Delta^N \times \Delta^N$ one can write

$$\kappa = \sum_{i=1}^N a_i dz_i,$$
where $a_i: \Delta^N \times \Delta^N \to \mathbb{C}$ are bounded holomorphic functions. Now
\[
\Omega = d(i \kappa) = \sum_{j,k=1}^{N} i\frac{\partial a_j}{\partial z_k} \, dz_k \wedge d z_j + i\frac{\partial a_i}{\partial w_k} \, dw_k \wedge dz_j.
\]
Finally
\[
\alpha^*\Omega = \sum_{j,k=1}^{N} i\frac{\partial a_j}{\partial z_k} \, dz_k \wedge d z_j + i\frac{\partial a_j}{\partial z_k} \, d \bar{z}_k \wedge d z_j.
\]
The Euclidean coefficients of the Kähler metric $\psi^*g_{WP}$ are hence linear combinations, with bounded coefficients, of $\frac{\partial a_j}{\partial z_k}$ and $\frac{\partial a_j}{\partial z_k}$, which in turn are pullbacks of the complex partial derivatives $\frac{\partial a_j}{\partial z_k}$ and $\frac{\partial a_j}{\partial w_k}$. Since the $a_j$ are bounded holomorphic functions, Cauchy’s Theorem implies that the derivatives $\frac{\partial a_j}{\partial z_k}$ and $\frac{\partial a_j}{\partial z_k}$ are bounded for $\|(z,w)\| < 1/2$; it follows that the (real) partial derivatives of $a_i$ are bounded for $\|z\| < 1/2$. The same applies to all higher order partial derivatives (where the kth order derivatives get a bound of order $k!$). This proves (2).

5.3. Proof of Proposition 5.1. Fix $X_0 \in \text{Teich}(S)$ and local coordinates $\psi = \psi_{X_0}$ as in the previous section. For $z \in \Delta^N$, let $G(z) = G_{X_0}(z) = (G_{ij}(z))$ be the matrix for the pullback metric $\psi^*g_{WP}$, and let $G^{ij}(z) = (G(z)^{-1})_{ij}$.

The curvature tensor for $G$ can be calculated in these Euclidean coordinates using the Christoffel symbols:
\[
\Gamma^m_{ij} = \frac{1}{2} C^k_{ij} \left( \frac{\partial}{\partial x^i} G_{kj} + \frac{\partial}{\partial x^j} G_{ik} - \frac{\partial}{\partial x^k} G_{ij} \right)
\]
and the Riemannian curvature tensor coefficients:
\[
R^\ell_{ijk} = \frac{\partial}{\partial x^j} \Gamma^\ell_{ik} - \frac{\partial}{\partial x^k} \Gamma^\ell_{ij} + \Gamma^\ell_{js} \Gamma^s_{ik} - \Gamma^\ell_{ks} \Gamma^s_{ij},
\]
where $G^{ij}$ At each $z \in \Delta^N$ the curvature tensor $R_z$ for the WP pullback metric $G$ is defined by setting $R_z(e_i, e_j)e_k = \sum \ell R^\ell_{ijk}e_\ell$. It is a function from $\Delta^N \times \mathbb{C}^{3N}$ to $\mathbb{C}^N$. Finally, the covariant tensor defined by
\[
R_{i,j,k,\ell} = \sum_{\nu} G_{\nu \ell} R^\nu_{ijk},
\]
satisfies $\langle R_z(e_i, e_j)e_k, e_\ell \rangle = R^\ell_{ijk}$. The formula for the covariant derivative of a covariant tensor $T_{b_1 \ldots b_s}$ is given by:
\[
(\nabla^c T)_{b_1 \ldots b_s} = \frac{\partial}{\partial x^c} T_{b_1 \ldots b_s} - \Gamma^d_{bc} T_{d \ldots b_s} - \cdots - \Gamma^d_{bce} T_{b_1 \ldots b_{s-1}d}.
\]
Since $\|D\psi\|$ and $\|D\psi^{-1}\|$ are bounded, the quantities $\|\nabla R_{WP}\psi(z)\|$ and $\|\nabla^2 R_{WP}\psi(z)\|$ can therefore be bounded by a (universal) polynomial function of the quantities $|G^{ij}(z)|$, $|G_{ij}(z)|$ and

$$\left| \frac{\partial^k G_{i,j}}{\partial \xi_1 \cdots \partial \xi_k}(z) \right|,$$

for $k = 1, \ldots, 4$. But Proposition 5.5 implies that the entries $G^{ij}(z)$ are $O(\ell(X_0)^{-1})$ and the entries $G_{ij}(z)$ and their first $k$ derivatives are $O(1)$; the conclusion of Proposition 5.1 then follows.

6. General criteria for ergodicity of the geodesic flow

Let $M$ be an open, contractible Riemannian manifold, negatively curved, possibly incomplete. Let $\Gamma$ be a group that acts freely and properly discontinuously on $M$ by isometries, and denote by $N$ the quotient manifold $N = M/\Gamma$. We denote by $d$ both the path metric on $M$ and the quotient metric on $N$, which is just the path metric for the induced Riemannian metric on $N$. The quotient map $p: M \to N$ is a covering map and a local isometry.

Recall that the completion $\overline{X}$ of a metric space $(X, d)$ is the set of all Cauchy sequences $\langle x_n \rangle$ in $X$ modulo the equivalence relation:

$$\langle x_n \rangle \sim \langle y_n \rangle \iff \lim_{n \to \infty} d(x_n, y_n) = 0,$$

with the induced metric

$$d(\langle x_n \rangle, \langle y_n \rangle) = \lim_{n \to \infty} d(x_n, y_n).$$

Let $\overline{M}$ be the metric completion of $M$ and let $\overline{N}$ be the completion of $N$. Let $\partial N = \overline{N} \setminus N$. We will use $d$ to denote the metric on all of these spaces.

Consider the following additional assumptions on $M$ and $N$:

I. $M$ is a geodesically convex: for every $p, p' \in M$ there is a unique geodesic segment in $M$ connecting $p$ to $p'$.

II. $\overline{N}$ is compact.

III. $\partial N$ is volumetrically cusplike: there exist constants $C > 1$ and $\nu > 0$ such that:

$$\text{Vol}\{ p \in N : d(p, \partial N) < \rho \} \leq C \rho^{2+\nu},$$

for every $\rho > 0$.

Note that if II.–III. hold, then $N$ has finite volume. In this case, we denote by $m$ the Riemannian volume on $N$, normalized so that $m(N) = 1$.

A metric space $X$ is $\text{CAT}(0)$ if it is a geodesic space and and every geodesic triangle in $X$ satisfies the $\text{CAT}(0)$ inequality with the comparison Euclidean triangle (see [7, p.159]).

Lemma 6.1. If I. holds, then $M$ and $\overline{M}$ are both $\text{CAT}(0)$ spaces.
Proof. The fact that \( M \) is CAT(0) follows from [7, Theorem II.1A.6] and Alexandrov’s Patchwork [7, Proposition II.4.9]. The metric completion of a CAT(0) space is CAT(0), by [7, Corollary II.3.11]. ⋄

**Proposition 6.2** (The flow is a.e. defined for all time). If I.–III. hold, then for almost every \( v \in T^1M \), there exists an infinite geodesic (necessarily unique) tangent to \( v \).

Before proving this we state and prove another lemma that will be useful later as well. Let

\[ U_\rho = \{ v \in T^1N : d(\pi(v), \partial N) < \rho \}, \]

and let \( S^+(\rho) \) be the set of all tangent vectors that flow into \( U_\rho \) in some forward time \( 0 \leq t \leq 1 \).

**Lemma 6.3.** If I.–III. hold, then

\[ m(S^+(\rho)) = O(\rho^{1+\eta}). \]

Proof. We look at the “shell” \( S_k^+(\rho) \) of vectors \( v \) that flow into \( U_\rho \) in time between \( k\rho \) and \( (k + 1)\rho \). Any two vectors in this shell arrive in \( U_{2\rho} \) after time \( (k + 1)\rho \). Volume-preservation of the flow implies that the the volume of \( S_k^+(\rho) \) is at most the volume of \( U_{2\rho} \), which is \( O(\rho^{2+\eta}) \), by assumption III.

The set \( S^+(\rho) \) is contained in a union of the shells \( S_0^+(\rho), \ldots, S_m^+(\rho) \), where \( m = O(\rho^{-1}) \). It follows that the volume of \( S^+(\rho) \) is \( O(\rho^{1-\rho^{2+\eta}}) = O(\rho^{1+\eta}). \)

⋄

Proof of Proposition 6.2. The set of vectors such that the flow is not defined for some \( 0 \leq t \leq 1 \) is contained in \( S^+(\rho) \) for all \( \rho > 0 \). By Lemma 6.3 this set has measure 0. The set \( E_n \) of vectors such that the flow is not defined for some time \( 0 \leq t \leq n \) is the union of \( E_n \) and the set of vectors that flow into \( E_{n-1} \) within time 1. Inductively \( E_n \) has measure 0, and since the flow is measure preserving, so does \( E_n \). It follows that the set of vectors for which the flow is defined for all time has full measure. ⋄

The next proposition will be used later to prove ergodicity of the geodesic flow on \( T^1N \), under further assumptions on the metric.

Suppose that \( v \in TM \) determines an infinite geodesic ray \( \gamma_v : [0, \infty) \to M \) tangent to \( v \) at 0. Since \( M \) is a CAT(0) space, the functions \( b^s_{v,t} : M \to \mathbb{R} \) defined by

\[ b^s_{v,t}(y) = d(y, \gamma_v(t)) - t \]

converge uniformly on compact sets as \( t \to \infty \) to a function \( b^s_v : M \to \mathbb{R} \), called a (stable) Busemann function [7, Lemma II.8.18]. For a fixed \( v \), the Busemann function \( b^s_v \) is clearly Lipschitz continuous, with Lipschitz norm 1. If we assume that I. holds, then we can say more.
Proposition 6.4. Assume that I. holds. For any $v$ that determines an infinite geodesic ray $\gamma_v$, the function $b^s_v$ is convex and $C^1$, and $\|\text{grad } b^s_v\| \equiv 1$.

For every $y \in T^1 M$, the unit vector

$$w^s_v(y) := -\text{grad } b^s_v(y)$$

defines an infinite geodesic ray $\gamma_{w^s_v(y)} : [0, \infty) \to \overline{M}$ tangent to $w_v(y)$ at 0 with the property that

$$d(\gamma_v(t), \gamma_{w^s_v(y)}(t)) \leq d(\gamma_v(0), y),$$

for all $t \geq 0$.

Proof. Since $\gamma_v$ is an infinite ray, and $M$ is a geodesically convex Riemannian manifold, the functions $b^s_{v,t}$ are convex, $C^1$ and have the property that $\|\text{grad } b^s_{v,t}(y)\| = 1$, for every $y \in M$. Since $M$ is nonpositively curved, and $b^s_{v,t}$ converges uniformly on compact sets in $M$ to $b^s_v$, the desired properties of $C^1$ smoothness of $b^s_v$, convexity and $\|\text{grad } b^s_v\| \equiv 1$ follow from [4, Lemma 3.4, and the following Remark]. The final conclusion follows from [7, Proposition II.8.2]. \diamond

Suppose that $v \in T^1 M$ determines a geodesic ray whose projection to $N$ is forward recurrent. Proposition 6.4 implies that for each $t \in \mathbb{R}$, the set $\mathcal{H}^s(v) := (b^s_v)^{-1}(t)$ is a connected, codimension-1, $C^1$ submanifold of $M$, called a stable horosphere. For a such a $v$, we define:

$$\mathcal{W}^s(v) = \{ w^s_v(y) : y \in \mathcal{H}^s_v(0) \}.$$

The set of basepoints $\pi(\mathcal{W}^s(v))$ in $M$ is the horosphere $\mathcal{H}^s_v(0)$, and $\mathcal{W}^s(v)$ is a continuous, codimension-1 submanifold of $T^1 M$. Similarly, if $\gamma_v$ projects to a backward recurrent geodesic ray in $N$, we define the unstable Busemann function

$$b^u_v(y) = \lim_{t \to \infty} d(y, \gamma_v(-t)) - t$$

and unstable manifold

$$\mathcal{W}^u(v) = \{ w^u_v(y) : y \in \mathcal{H}^u_v(0) \},$$

where $w^u_v(y) = -\text{grad } b^u_v(y)$, and $\mathcal{H}^u_v(t) := (b^u_v)^{-1}(t)$ is the unstable horosphere at level $t$ determined by $v$. Our next proposition justifies the terminology “stable and unstable manifolds” for $\mathcal{W}^s(v)$ and $\mathcal{W}^u(v)$. The results stated up to this point all hold true when $M$ is nonpositively curved, but the following proposition uses the negative curvature assumption on $M$ in an essential way.

We say that a geodesic ray $\gamma : [0, \infty) \to N$ is (forward) recurrent if the tangent vector $\dot{\gamma}(0)$ is an accumulation point for the tangent vectors \{ $\dot{\gamma}(t) : t > 0$ \}. We similarly define backward recurrence for a geodesic ray $\gamma : (-\infty, 0] \to N$. An infinite geodesic is recurrent if it is both forward and backward recurrent. Under assumptions I.-III., Proposition 6.2 and Poincaré recurrence imply that almost every $v \in T^1 N$ determines an infinite recurrent geodesic $\gamma_v : \mathbb{R} \to N$ with $\dot{\gamma}_v(0) = v$. 

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Proposition 6.5 (Contraction of horospheres). Assume I.–III. Let $v \in T_x M$ be tangent to an infinite geodesic ray $\gamma_v$ whose projection to $N$ is forward recurrent. Let $y \in M$ be any other point, and let $w = w^v_s(y) \in T_y M$. Then $w$ is tangent to an infinite geodesic ray $\gamma_w: [0, \infty) \to M$, and
\[
\lim_{t \to -\infty} d(\gamma_v(t), \gamma_w(t + b^v_s(y))) = 0;
\]
moreover,
\[
\lim_{t \to -\infty} d_{\text{Sas}}(\varphi_t(v), \varphi_{t+b^v_s(y)}(w)) = 0.
\]
In particular, if $\gamma_v$ projects to a forward recurrent geodesic ray in $N$, then for every $t > 0$, $\varphi_t(W^s(v)) = W^s(\varphi_t(v))$, and for every $w \in W^s(v)$, we have
\[
\lim_{t \to -\infty} d_{\text{Sas}}(\varphi_t(v), \varphi_t(w)) = 0.
\]
Similarly, if $v$ is tangent to a backward ray $\gamma_w: (-\infty, 0] \to M$ whose projection is recurrent, then $w = w^v_u(w)$ is tangent to a backward ray $\gamma_w: (-\infty, 0] \to M$, and
\[
\lim_{t \to -\infty} d_{\text{Sas}}(\varphi_t(v), \varphi_{t+b^v_u(y)}(w)) = 0.
\]
In particular, for every $w \in W^u(v)$, we have
\[
\lim_{t \to -\infty} d(\varphi_t(v), \varphi_t(w)) = 0.
\]

Before beginning the proof we remark that in [9] a property called non-refraction was proved for the WP metric. Using that result a short proof of the above proposition was given in the WP case in [6]. Since we are in a general Riemannian setting in this section, we prefer not to make this additional assumption of nonrefraction.

Proof. Let $\gamma_v: [0, \infty) \to M$ be an infinite geodesic ray whose projection to $N$ is recurrent, and let $x = \gamma_v(0)$ be the footpoint of $v$. Suppose that $x' \in M$ is another point, and let $v' = w^v_s(x')$. Since $\overline{M}$ is CAT(0), the distance $d(\gamma_v(t), \gamma_{v'}(t))$ is a convex function of $t$; since it is bounded, it must be nonincreasing, and hence bounded above for all $t$ by $d(x, x')$. We claim that if $d(x, x') < d(x, \partial M)$, then the image of $\gamma_{v'}$ must lie entirely in $M$.

Since the projection of $\gamma_v$ to $N$ is recurrent, there exist sequences $g_n \in \Gamma$ and $t_n \to \infty$ such that
\[
d(x, g_n \gamma_{v'}(t_n)) < d(x, \partial M) - d(x, x')
\leq d(x, \partial M) - d(\gamma_v(t_n), \gamma_{v'}(t_n)),
= d(x, \partial M) - d(g_n \gamma_v(t_n), g_n \gamma_{v'}(t_n)),
\]
which implies that $d(x, g_n \gamma_{v'}(t_n)) < d(x, \partial M)$. Hence $g_n \gamma_{v'}(t_n) \in M$, and so $\gamma_{v'}(t_n) \in M$; geodesic convexity of $M$ implies that $\gamma_{v'}[0, t_n] \subset M$, for all $n$, which proves the claim.

Now a standard ruled surface argument using geodesic convexity and the negative curvature of $M$ (see e.g. [6, Theorem 4.1], where it is proved in the WP context) shows that for every $\gamma_v$ that projects to a recurrent geodesic ray in $N$, and any $y \in M$ with the property that $\gamma_{w^v_s(y)}[0, \infty) \subset M$, the
distance \(d(\gamma_{w^t_v}(y) (t), \gamma_v[0, \infty))\) is strictly decreasing in \(t\) and tends to 0 as \(t \to \infty\). (Alternately, one can show this using Jacobi fields). What is more, this convergence takes place in the tangent bundle:

\[
\lim_{t \to \infty} d_{Sas}(\gamma_{w^t_v}(y)(t), \gamma_v[0, \infty)) = 0.
\]

Now suppose that \(y \in M\) is an arbitrary point. Connect \(y\) to \(x = \gamma_v(0)\) by a geodesic arc \(\sigma\) in \(M\). Fix \(\epsilon_0 > 0\) such that \(d(x, \partial M) < \epsilon_0\). We claim that if \(x'\) is any point on \(\sigma\) that satisfies

\[
\lim_{t \to \infty} d(\gamma_{w^t_v}(x')(t), \gamma_v[0, \infty)) = 0,
\]

then for any point \(y'\) on \(\sigma\) such that \(d(x', y') < \epsilon_0/3\):

\[
\lim_{t \to \infty} d(\gamma_{w^t_v}(y')(t), \gamma_v[0, \infty)) = 0.
\]

From the claim it follows that \(\lim_{t \to \infty} d_{Sas}(\gamma_{w^t_v}(y)(t), \gamma_v[0, \infty)) = 0\).

To prove the claim, suppose that \(x'\) and \(y'\) are given. Since the distance \(d((\gamma_{w^t_v}(x')(t), \gamma_{w^t_v}(y')(t))\) is bounded for all \(t > 0\) and convex, it is nonincreasing, and hence bounded above by \(\epsilon_0/3\), for all \(t > 0\). If \(T > 0\) is sufficiently large, then the distance from \(\gamma_{w(t)}(x')(t)\) to \(\gamma_v\) is less than \(\epsilon_0/3\) for all \(t > T\). Since \(\gamma_v\) projects to a recurrent ray in \(N\), there exist \(g_n \in \Gamma\) and \(t_n \to \infty\) such that \(d(\gamma_v(t), g x) < \epsilon_0/3\). It follows that \(\gamma_{w^t_v}(y')(t_n) \in M\) for all \(t_n \to T\), which implies that \(\gamma_{w(t)}[0, \infty) \subset M\). The conclusion follows.

A simple application of the triangle inequality shows that the property \(\lim_{t \to \infty} d(\gamma_{w^t_v}(y)(t), \gamma_v[0, \infty)) = 0\) implies that

\[
\lim_{t \to \infty} d(\gamma_v(t), \gamma_{w^t_v}(y)(t + b^t_v(y))) = 0.
\]

Since \(\lim_{t \to \infty} d_{Sas}(\gamma_{w^t_v}(y)(t), \gamma_v[0, \infty)) = 0\) for every \(y \in M\), we conclude that

\[
\lim_{t \to \infty} d_{Sas}(\varphi(t), \varphi_{t+b^t_v(y)}(w^t_v(y)) = 0.
\]

\(\diamondsuit\)

Our final set of assumptions will be used to establish ergodicity of the geodesic flow. We assume there exist constants \(C > 1\) and \(\beta > 0\) such that:

IV. \(N\) has controlled curvature: for all \(x \in N\), the curvature tensor \(R\) satisfies:

\[
\max\{\|R_x\|, \|\nabla R_x\|, \|\nabla^2 R_x\|\} \leq C d(x, \partial N)^{-\beta}.
\]

V. \(N\) has controlled injectivity radius: for every \(x \in N\),

\[
\text{inj}(x) \geq C^{-1} d(x, \partial N)^{\beta}.
\]

VI. The derivative of the geodesic flow is controlled: for every infinite geodesic \(\gamma\) in \(N\) and every \(t \in [0, 1]\):

\[
\|D_{\dot{\gamma}(0)} \varphi_t\| \leq C d(\gamma([-t, t], \partial N)^{-\beta};
\]
Theorem 6.6. Under assumptions I.-VI., the geodesic flow $\varphi_t$ on $T^1 N$ is nonuniformly hyperbolic and ergodic. The entropy $h(\varphi_1)$ of $\varphi_1$ is positive and finite, equal to the sum of the positive Lyapunov exponents of $\varphi_t$, counted with multiplicity.

Remark: It seems that Assumption II. (compactness of $N$) can be relaxed to the assumption that $N$ has finite diameter, but we have not verified all of the details.

The proof of Theorem 6.6 proceeds in several steps. The first is to establish nonuniform hyperbolicity. To this end, we need the following lemma.

Lemma 6.7. Let $\varphi_1$ be the time-1 map of the geodesic flow, which is defined $m$-almost everywhere in $T^1 N$. Then

$$\int_{T^1 N} \log^+ ||D\varphi_1|| \, dm < \infty.$$ 

Proof. We consider the set $S^+(1/n)$ of vectors which flow into the set of points of distance between $1/(n+1)$ and $1/n$ from $S$ for some $0 \leq t \leq 1$. Lemma 6.3 implies that $m(S^+(1/n)) = O((1/n)^{1+\eta})$.

On $S^+(1/n)$ we have

$$\log^+ ||D\varphi_1|| = O(\log n).$$

Therefore

$$\int_{S^+(1/n)} \log^+ ||D\varphi_1|| \, dm = O(\log n/n^{1+\eta}).$$

Summing over $n$ we obtain the conclusion. ☐

Since $\log ||D\varphi_1||$ is integrable, Oseledeč’s theorem implies that for $m$-almost every $v \in T^1 N$, there exist real numbers $\lambda_1(v) \leq \lambda_2(v) \leq \cdots \leq \lambda_{2n-1}(v)$ and a splitting $T_v T^1 N = \bigoplus_{i=1}^{2n-1} E_i(v)$ such that for every nonzero vector $\xi \in E_i(v)$:

$$\lim_{t \to \pm \infty} -\frac{1}{t} \log ||D \varphi_t(\xi)|| = \lambda_i(v).$$

The numbers $\lambda_i(v)$ are called the Lyapunov exponents of $\varphi_t$ at $v$, and $E_i(v)$ the Lyapunov subspaces. The stable and unstable Lyapunov subspaces at $v$ are $E^s(v) := \bigoplus_{\lambda_i(v) < 0} E_i(v)$ and $E^u(v) := \bigoplus_{\lambda_i(v) > 0} E_i(v)$, respectively. The functions $v \mapsto \lambda_i(v)$ and $v \mapsto E_i(v)$ are measurable. Since the generating vector field $\dot{\varphi}$ is preserved by $D\varphi_1$, it follows that $\lambda_n(v) = 0$. Moreover, since the orthocomplement $\dot{\varphi}^\perp$ is $D\varphi_1$-invariant, and the restriction of $D\varphi_1$ preserves a natural symplectic form, the Lyapunov exponents of $\varphi_t$ are paired: $\lambda_i(v) = -\lambda_{2n-i}(v)$, for $i = 1, \ldots, n-1$.

Recall that the geodesic flow $\varphi_t$ is nonuniformly hyperbolic if $\lambda_{n-1}(v) < 0$ for almost every $v$; equivalently, $T_v T^1 N = E^s(v) \oplus \mathbb{R}\dot{\varphi}(v) \oplus E^u(v)$, for almost every $v$. We have:
Proposition 6.8 (Nonuniform hyperbolicity). Under assumptions I.-VI., the geodesic flow is nonuniformly hyperbolic. That is, there exists a full measure, $\varphi_t$-invariant subset $\Lambda_0 \subset T^1 N$ and a measurable, $D\varphi_t$-invariant splitting of the tangent bundle:

$$T\Lambda_0(T^1 N) = E^s \oplus E^0 \oplus E^u$$

such that, for every $v \in \Lambda_0$:

1. $E^0(v)$ is tangent to the orbits of the flow: $E^0(v) = \mathbb{R} \varphi(v)$;
2. for every $\xi^u \in E^u(v)$, $\xi^s \in E^u(v)$:

$$\lim_{t \to \pm \infty} \frac{1}{t} \log \|D\varphi_t (\xi^u)\| > 0,$$

and

$$\lim_{t \to \pm \infty} \frac{1}{t} \log \|D\varphi_t (\xi^s)\| < 0,$$

and the limits are finite.

Moreover, there exist continuous, disjoint cone fields $C^u$ and $C^s$ over $T^1 N$, spanning the space of perpendicular Jacobi fields in $TT^1 N$, such that for every $v \in \Lambda_0$ and all $t > 0$:

$$E^u(v) \subset C^u(v), \quad E^s(v) \subset C^s(v),$$

$$D\varphi_t (C^u(v)) \subset C^u(\varphi_t(v)), \quad D\varphi_{-t} (C^s(v)) \subset C^s(\varphi_{-t}(v)).$$

For each $v \in \Lambda_0$, $E^u(v)$ is spanned by the unstable perpendicular Jacobi fields at $v$, and $E^s(v)$ is spanned by the stable perpendicular Jacobi fields at $v$.

Proof. This proof is standard for compact negatively curved manifolds (see, e.g., [16, Section 17.6]). A few modifications adapt the proof to this setting.

Define positive continuous functions $\overline{k}, \underline{k} : N \to \mathbb{R}_{>0}$ by

$$\overline{k}(x) = \sup_{v, w \in T^1_{\pi N}} \sqrt{-\langle R(v, w)w, v \rangle}, \quad \underline{k}(x) = \inf_{v, w \in T^1_{\pi N}} \sqrt{-\langle R(v, w)w, v \rangle},$$

and $\delta : T^1 N \to \mathbb{R}_{>0}$ by

$$\delta(v) = \frac{k^2(\pi(v))}{2(1 + \overline{k}^2(\pi(v)))}.$$ 

For $v \in T^1 N$, we identify $T_v T^1 N \cong (T_\pi(v) N)^2$ in the standard way, and let $\dot{\varphi}^\perp(v) = \{(w_0, w_1) : \langle w_0, v \rangle = \langle w_1, v \rangle = 0\}$ be the orthocomplement of $\dot{\varphi}$; it is a $2n - 2$-dimensional subspace of $T_v T^1 N$ and is naturally identified with the space of perpendicular Jacobi fields to $\gamma_v$. We define the cone fields $\mathcal{C}^u, \mathcal{C}^s$ by

$$\mathcal{C}^u(v) = \{(w_0, w_1) \in \dot{\varphi}^\perp(v) : \langle w_0, w_1 \rangle \geq \delta^2(v) \| (w_0, w_1) \|_{\text{Spa}}^2\}$$

and

$$\mathcal{C}^s(v) = \{(w_0, w_1) \in \dot{\varphi}^\perp(v) : \langle w_0, w_1 \rangle \leq -\delta^2(v) \| (w_0, w_1) \|_{\text{Spa}}^2\}.$$

By our choice of $\delta$, these cone fields satisfy the strict invariance conditions

$$D\varphi_t (\mathcal{C}^u \setminus \{0\}) \subset \text{int} (\mathcal{C}^u), \quad \text{and} \quad D\varphi_{-t} (\mathcal{C}^s \setminus \{0\}) \subset \text{int} (\mathcal{C}^s),$$

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for all $t > 0$, wherever these objects are defined; the proof follows closely
the proof of [16, Lemma 17.6.1].

These conefields are symplectic in the sense of [41], as we now explain.
The restriction of the symplectic form on $TTN$ defined by
$$
\omega((v_0, v_1), (w_0, w_1)) = \langle v_0, w_1 \rangle - \langle v_1, w_0 \rangle.
$$
to the subbundle $\dot{\varphi}$ of perpendicular Jacobi fields in $TT^1N$ is also symplectic,
and the derivative of the time-$t$ map $D\varphi_t$ is a linear symplectomorphism
when restricted to $\dot{\varphi}$.

Let $\alpha: T^1N \to \mathbb{R}_{>0}$ be the positive solution to $\alpha(1 + \alpha^2)^{-1} = \delta$. For
$v \in T^1N$, let
$$
L_1(v) = \{(w_0, \alpha(v)w_0) : w_0 \in T_{\pi(v)}N, \langle v, w_0 \rangle = 0\},
$$
and let
$$
L_2(v) = \{(\alpha(v)w_1, w_1) : w_1 \in T_{\pi(v)}N, \langle v, w_1 \rangle = 0\}.
$$
Observe that $L_1$ and $L_2$ are transverse Lagrangian subbundles spanning $\dot{\varphi}$.
One can uniquely express an arbitrary element $(w_0, w_1) \in \dot{\varphi}$ as a sum of
elements in $L_1$ and $L_2$, as follows:
$$
(w_0, w_1) = \left(\frac{w_0 - \alpha w_1}{1 - \alpha^2}, \frac{w_0 - \alpha w_1}{1 - \alpha^2}\right) + \left(\frac{w_1 - \alpha w_0}{1 - \alpha^2}, \frac{w_1 - \alpha w_0}{1 - \alpha^2}\right).
$$
Using this decomposition, we define a symplectic quadratic form $K_{L_1, L_2}$ on $\dot{\varphi}$ by:
$$
K_{L_1, L_2}(w_0, w_1) = \omega\left(\left(\frac{w_0 - \alpha w_1}{1 - \alpha^2}, \frac{w_0 - \alpha w_1}{1 - \alpha^2}\right), \left(\frac{w_1 - \alpha w_0}{1 - \alpha^2}, \frac{w_1 - \alpha w_0}{1 - \alpha^2}\right)\right).
$$
A straightforward calculation shows that
$$
K_{L_1, L_2}(w_0, w_1) = (-\alpha\|w_0\|^2 - \alpha\|w_1\|^2 + (1 + \alpha^2)\langle w_0, w_1 \rangle).
$$
Hence $K_{L_1, L_2}(w_0, w_1) \geq 0$ if and only if
$$
\langle w_0, w_1 \rangle \geq \frac{\alpha}{1 + \alpha^2}(\|w_0\|^2 + \|w_1\|^2) = \delta\|w_0, w_1\|_{sas};
$$
in other words:
$$
C^u = \{\xi \in \dot{\varphi} : K_{L_1, L_2}(\xi) \geq 0\}.
$$
We thus have established that $D\varphi_t$ is a symplectomorphism of $\dot{\varphi}$, the loga-
rithm of $\|D\varphi_t\|$ is integrable, and the conefield $C^u$ defined by the symplectic
quadratic form $K_{L_1, L_2}$ is everywhere strictly invariant under $D\varphi_t$, for any
t $> 0$. It follows immediately from [15, Corollary 2.2] that $D\varphi_t$ has exactly
$n - 1 = \text{dim}(N) - 1$ positive Lyapunov exponents, $m$-almost everywhere.
Similarly, $D\varphi_t$ has exactly $n - 1$ negative Lyapunov exponents almost ev-
erywhere. The subbundle $E^u$ is defined to be the direct sum of the positive
Lyapunov subbundles, and $E^s$ is the sum of the negative Lyapunov subbun-
dles. Invariance of the subbundles and recurrence give that $E^u \subset C^u$ and
$E^s \subset C^s$ almost everywhere. ∅
Proposition 6.9 (Existence and absolute continuity of local stable manifolds). Assume I.-VI. Let \( n = \dim(N) \), and let \( \Lambda_0 \subset T^1 N \) be given by Proposition 6.8. There exists a full volume, \( \varphi_t \)-invariant subset \( \Lambda_1 \subset \Lambda_0 \), a measurable function \( \tau : \Lambda_1 \to \mathbb{R}_{>0} \) and measurable families of \( C^\infty \), \( (n-1) \)-dimensional embedded disks \( W^s_{loc} = \{ W^s_{loc}(v) : v \in \Lambda_1 \} \) and \( W^u_{loc} = \{ W^u_{loc}(v) : v \in \Lambda_1 \} \) with the following properties. For each \( v \in \Lambda_1 \):

1. \( W^s_{loc}(v) \) is tangent to \( E^s(v) \) at \( v \), and \( W^u_{loc}(v) \) is tangent to \( E^u(v) \) at \( v \);
2. for all \( t > 0 \),
   \[ \varphi_t(W^s_{loc}(v)) \subset W^s_{loc}(\varphi_t(v)), \quad \text{and} \quad \varphi_{-t}(W^u_{loc}(v)) \subset W^u_{loc}(\varphi_{-t}(v)) \]
3. \( w \in W^s_{loc}(v) \) if and only if \( d(v, w) < r(v) \) and
   \[ \lim_{t \to \infty} d_{Sas}(\varphi_t(v), \varphi_t(w)) = 0; \]
4. \( w \in W^u_{loc}(v) \) if and only if \( d(v, w) < r(v) \) and
   \[ \lim_{t \to -\infty} d_{Sas}(\varphi_t(v), \varphi_t(w)) = 0. \]

Moreover, for \( * \in \{s, u\} \), the family \( W^*_{loc} \) is absolutely continuous. In particular:

5. if \( Z \subset T^1 N \) has volume \( m(Z) = 0 \), then for \( m \)-almost every \( v \in \Lambda_1 \),
   the set \( Z \cap W^*_{loc}(v) \) is a zero set in \( W^*_{loc}(v) \) (with respect to the induced \( (n-1) \)-dimensional Riemannian volume); and
6. if \( D \subset T^1 N \) is any \( C^1 \)-embedded, \( n \)-dimensional open disk, and \( B \subset D \) has induced Riemannian volume zero in \( D \), then \( m(\text{Sat}^*_{loc}(B)) = 0 \), where
   \[ \text{Sat}^*_{loc}(B) := \bigcup_{\{v \in \Lambda_1 : W^*_{loc}(v) \cap B \neq \emptyset\}} W^*_{loc}(v). \]

The conclusions of Proposition 6.9 follow from the main results in [17]. To apply these results, it is necessary to verify a list of hypotheses, some of a technical nature, concerning the \( C^3 \) properties of the Sasaki metric and the geodesic flow. We defer the verification of these properties to the next subsection and now show how Proposition 6.9 can be used to prove ergodicity of \( \varphi_t \). Properties (5) and (6) in Proposition 6.9 are the heart of the matter in proving ergodicity and account for the technical hurdles we’ve been jumping. Property (5) is a form of “leafwise absolute continuity” and (6) is a form of “transverse absolute continuity.” Both are implied by the condition of absolute continuity in [17].

Let \( \Omega_1 \) be the full measure set of \( v \in T^1 M \) such that \( \gamma_v \) projects to a (forward and backward) recurrent geodesic in \( TN \). For each \( v \in \Omega_1 \), there exists a stable manifold \( W^s(v) \) and an unstable manifold \( W^u(v) \). For \( v \in \Omega_1 \) and \( \delta > 0 \), denote by \( W^s(v, \delta) \) the connected component of \( W^s(v) \cap B_{T^1 M}(v, \delta) \) containing \( v \), where \( B_{T^1 M}(v, \delta) \) is the Sasaki ball of radius \( \delta \) in \( T^1 M \) centered at \( v \). For \( v' = Dp(v) \in Dp(\Omega_1) \), and \( \delta < \text{inj}(\pi(v')) \) we
denote by $W^s(v', \delta)$ the projection $Dp(W^s(v, \delta))$; it is an $(n-1)$-dimensional embedded disk.

Notice that, for every $v \in \Omega_1$, if $v' = Dp(v)$ belongs to the full measure set $\Lambda_1$ of Proposition 6.9, then the local stable manifold $W^s_{loc}(v')$ through $v'$ must coincide with $W^s(v', r(v'))$, where $r: \Lambda_1 \to \mathbb{R}_{>0}$ is the function given by Proposition 6.9. We will use this connection to prove the following.

**Proposition 6.10** (Smoothness and absolute continuity of horospherical laminations). Assume $I$.-VI. There is a full volume subset $\Omega_2 \subset \Omega_1$ such that for $* \in \{s, u\}$ and for $v \in \Omega_2$, the Busemann function $b^*_v: M \to \mathbb{R}$ is $C^\infty$. The leaves of the lamination $W^* = \{W^*(v): v \in \Omega_2\}$ are $C^\infty$ submanifolds of $T^1M$ diffeomorphic to $\mathbb{R}^{n-1}$.

Let $\Lambda_2 = Dp(\Omega_2)$. The family of manifolds

$$\{W^*(v, \delta) : v \in \Lambda_2, \delta < \text{inj}(\pi(v))\}$$

has the following absolute continuity properties.

1. if $Z \subset T^1N$ has volume $m(Z) = 0$, then for $m$-almost every $v \in \Lambda_2$, and every $\delta < \text{inj}(\pi(v))$, the set $Z \cap W^*(v, \delta)$ is a zero set in $W^*(v, \delta)$ (with respect to the induced $(n-1)$-dimensional Riemannian volume); and

2. if $D \subset T^1N$ is any smoothly embedded, $n$-dimensional open disk, and $B \subset D$ has induced Riemannian volume zero in $D$, then for any $\delta < \frac{1}{2} \inf_{v \in D} \text{inj}(\pi(v))$, we have $m(\text{Sat}^*(B, \delta)) = 0$, where

$$\text{Sat}^*(B, \delta) := \bigcup_{\{v \in \Lambda_2 : W(v, \delta) \cap B \neq \emptyset\}} W^*(v, \delta).$$

**Proof.** We first show that $W^s(v)$ is a $C^\infty$ submanifold of $T^1M$, for almost every $v \in T^1M$. For any $\epsilon > 0$ there exists a compact set $\Delta_\epsilon \subset \Lambda_1$ of measure $m(\Delta_\epsilon) > 1 - \epsilon$ such that the restriction of the function $r$ from Proposition 6.9 to $\Delta_\epsilon$ is continuous and bounded from below by a constant $r_\epsilon > 0$. Fix $\epsilon > 0$, and let $\Delta^s_\epsilon \subset \Delta_\epsilon$ be the set of vectors $v' \in \Delta_\epsilon$ such that $\varphi_{k_n}(v') \in \Delta_\epsilon$ for a sequence of integers $k_n \to \infty$. Poincaré recurrence implies that $m(\Delta_\epsilon \setminus \Delta^s_\epsilon) = 0$.

Fix $v' \in \Delta^s_\epsilon \cap Dp(\Omega_1)$. Let $v \in Dp^{-1}(v')$ be an arbitrary lift of $v'$, and let $w \in W^s(v)$. We show that $W^s(v)$ is $C^\infty$ in a neighborhood of $w$; as $w$ is arbitrary, this implies that $W^s(v)$ is $C^\infty$. Since $v' = Dp(v) \in \Delta^s_\epsilon$, there exists a sequence $k_n \to \infty$ such that $\varphi_{k_n}(v') \in \Delta_\epsilon$. At the same time, Proposition 6.5 implies that

$$\lim_{t \to \infty} d_{Sas}(\varphi_t(v), \varphi_t(w)) = 0,$$

and so for $n$ sufficiently large, $d_{Sas}(\varphi_{k_n}(v), \varphi_{k_n}(w)) < r_\epsilon/2$, where $r_\epsilon > 0$ is the lower bound on the restriction of $r$ to $\Delta_\epsilon$. But this implies that $Dp(\varphi_{k_n}(w)) \in W^s_{loc}(\varphi_{k_n}(v'))$. Since $\varphi_{k_n}$ is a diffeomorphism, we conclude that there is a neighborhood of $w$ in $W^s(v)$ that is diffeomorphic to the $C^\infty$ submanifold $W^s_{loc}(\varphi_{k_n}(v'))$. Since $w$ was arbitrary, this implies that $W^s(v)$
is a $C^\infty$ submanifold of $T^1M$. The intersection $\Lambda_2^s := \bigcap_{\epsilon > 0} \Delta^s \cap Dp(\Omega_1)$ is a full volume subset of $T^1N$, and we have shown that for every $v \in \Omega_2^s := Dp^{-1}(\Lambda_2^s)$, the submanifold $W^s(v)$ is $C^\infty$.

For each $v \in \Omega_2^s$, consider the map $\psi$ from $\mathcal{H}_v^s \times \mathbb{R}$ to $M$ that sends $(y, t)$ to $\pi((w^v_\epsilon)(y))$, where $w^v_\epsilon(y) = -\text{grad} b^v_\epsilon(y)$. Since $W^s(v)$ is $C^\infty$, the function $w^v_\epsilon(y)$ is $C^\infty$ along $\mathcal{H}_v^s$; it follows that $\psi$ is a diffeomorphism. In the coordinates on $M$ given by $\psi$, the Busemann function $b^v_s$ assigns the value $-t$ to the point $(x, t)$. It follows that $b^v_s$ is $C^\infty$, for every $v \in \Omega_2^s$. Similarly, there is a set $\Omega_2^u$ of full measure such that $b^v_u$ is $C^\infty$ for every $v \in \Omega_2^u$. Setting $\Omega_2 = \Omega_2^s \cap \Omega_2^u$, we obtain the full measure set where the conclusions of the proposition will hold.

We establish the absolute continuity properties of $W^s$; analogous arguments show the properties for $W^u$. The preceding arguments show that for every $v \in \Lambda_2$ there exists an integer $k \geq 0$ such that

$$\varphi_k(W^s(v, \delta)) \subset W^s_{\text{loc}}(\varphi(v)), \text{ for every } \delta < \text{inj}(\pi(v)) \tag{29}$$

For a fixed $k > 0$, denote by $X_k$ the set of $v \in \Lambda_2$ for which (29) holds. Then $\Lambda_2 = \bigcup_{k \geq 0} X_k$.

Suppose that $m(Z) = 0$, for some $Z \subset T^1N$. Then the set $Z = \bigcup_{k \geq 0} \varphi_k(Z)$ also has measure 0. It follows from Proposition 6.9 that for almost every $w \in \Lambda_1$, the induced Riemannian measure of $Z$ in $W^s_{\text{loc}}(w)$ is zero. But this implies in particular that for every $k \geq 0$ and for almost every $v \in X_k$, the induced Riemannian measure of $\varphi_k(Z) \subset Z$ in $\varphi_k(W^s(v, \delta)) \subset W^s_{\text{loc}}(\varphi_k(v))$ is zero; hence the induced volume of $Z$ in $W^s(v, \delta)$ is 0, for all $\delta < \text{inj}(\pi(v))$. This establishes (1).

Suppose that $D$ is a $C^1$-embedded, $n$-dimensional disk in $T^1N$. Fix $\delta < \frac{1}{2} \inf_{v \in D} \text{inj}(\pi(v))$. Suppose that $B \subset D$ has induced Riemannian volume 0. Let

$$B_k = B \cap \bigcup_{w \in X_k} W^s(w, \delta)$$

and note that

$$\text{Sat}^s(B, \delta) = \bigcup_{k \geq 0} \text{Sat}^s(B_k, \delta);$$

hence it suffices to show that $m(\text{Sat}^s(B_k, \delta)) = 0$, for all $k \geq 0$.

Fix $k \geq 0$. For each $w \in B_k$, there an $n$-dimensional open ball $D_w \subset D$ centered at $w$ in the induced Riemannian metric in $D$, such that $\bigcup_{j=0}^k \varphi_j(D_w) \subset T^1N$. Since $\varphi_k$ is a diffeomorphism, the set $\varphi_k(B_k \cap D_w)$ has induced Riemannian volume zero in the $n$-dimensional disk $\varphi_k(D_w)$. It follows from Proposition 6.9 that $m(\text{Sat}^s_{\text{loc}}(\varphi_k(B_k \cap D_w))) = 0$, and so

$$m(\varphi_k(\text{Sat}^s_{\text{loc}}(\varphi_k(B_k \cap D_w)))) = 0.$$
and so $m(\text{Sat}^s(B_k \cap D_w, \delta)) = 0$. Now fix a countable cover $\{D_{w_i} : w_i \in B_k\}$ of $B_k$ in $D$ by such balls (this is possible by the Besicovitch covering theorem, since $D$ is an embedded $C^1$ submanifold). Then

$$\text{Sat}^s(B_k, \delta) \subset \bigcup_i \text{Sat}^s(B_k \cap D_{w_i}, \delta),$$

and so $m(\text{Sat}^s(B_k, \delta)) = 0$. Conclusion (2) follows. \qed

**Proof of ergodicity.** Assume I.-VI. The proof that $\varphi_t$ is ergodic is an adaptation of the standard “Hopf Argument,” along the lines of the proof of local ergodicity in [15]. To prove ergodicity, it suffices to show that for every continuous function $f: T^1 N \to \mathbb{R}$ with compact support:

$$(30) \quad \lim_{T \to \infty} \frac{1}{T} \int_0^T f(\varphi_t(v)) \, dt = \int_{T^1 N} f \, dm, \quad \text{for } m - a.e. \, v \in T^1 N$$

Indeed, if (30) holds for a dense set of functions $f$ in $L^2$, then by continuity of the projection $f \mapsto B(f) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f \circ \varphi_t \, dt$, (30) will hold for every $f$ in $L^2$.

Fix then a continuous function $f$ with compact support and define measurable functions $f^s$ and $f^u$ by:

$$f^s(v) = \limsup_{T \to \infty} \frac{1}{T} \int_0^T f(\varphi_t(v)) \, dt, \quad \text{and} \quad f^u(v) = \limsup_{T \to \infty} \frac{1}{T} \int_{-T}^0 f(\varphi_t(v)) \, dt.$$  

The Birkhoff Ergodic Theorem implies that there is a set $G \subset T^1 N$ of full measure such that for every $v \in G$, we have $f^s(v) = f^u(v) = B(f)(v)$. Since $f$ is continuous with compact support, and the leaves of $\mathcal{W}^s$ are contracted by $\varphi_t$, it follows that $f^s$ is constant along leaves of $\mathcal{W}^s$. Similarly, $f^u$ is constant along leaves of $\mathcal{W}^u$. Finally, all three functions $f^s, f^u, B(f)$ are invariant under the flow $\varphi_t$.

Now fix an arbitrary element $v \in T^1 N$. We will show that there is a neighborhood $U_v$ of $v$ on which $B(f)$ is almost everywhere constant. Since $T^1 N$ is connected, this will imply that $B(f)$ is constant on $T^1 N$. Since $\int_{T^1 N} B(f) \, dm = \int_{T^1 N} f \, dm$, it will then follow that (30) holds, and so $\varphi$ is ergodic.

Let $\delta = \delta(v) = \frac{1}{4} \min\{\text{inj}(\pi(v)), d(v, \partial N)\}$, and let $V$ be the $\delta$-neighborhood of $v$ in $T^1 N$. For $w \in \Lambda_2 \cap V$, consider the set

$$N_\delta(w) = \text{Sat}^u(\varphi_{(-,\delta)}(\mathcal{W}^s(w, \delta)), \delta).$$

We claim:

(a) for almost every $w \in \Lambda_2 \cap V$, $B(f)$ is almost everywhere constant on $N_\delta(w)$;

(b) there is a neighborhood $U_v \subset V$ of $v$ such that for almost every $w \in U_v$, the set $N_\delta(w) \cap U_v$ has full measure in $U_v$.  

Together, these statements imply that there is a neighborhood $U_v$ of $v$ on which $B(f)$ is a.e. constant, completing the proof of ergodicity.

We first establish part (a) of this claim. Let $G$ be the full measure subset of $v \in \Lambda_2$ where the limit (30) exists and $f^u(v) = f^s(v) = B(f)(v)$. Absolute continuity property (1) of $W^s$ in Proposition 6.10 implies that for almost every $w \in V \cap \Lambda_2$, the intersection $G \cap W^s(w, \delta)$ has full volume in $W^s(w, \delta)$ (that is, its complement has induced volume 0). Fix such a $w$. On $W^s(w, \delta)$, $f^s$ takes a constant value $f^s \equiv a$. On the full volume subset $G \cap W^s(w, \delta)$, $f^u$ coincides with $f^s$ and therefore also takes the constant value $a$. Since $f^u$ is $\varphi_t$-invariant, and $\varphi_t$ is a $C^\infty$ flow, $f^u$ takes the constant value $a$ on the full measure subset $G' := \varphi_{(-\delta,\delta)}(G \cap W^s(w, \delta))$ of the $n$-dimensional $C^\infty$ submanifold $D = \varphi_{(-\delta,\delta)}(W^s(w, \delta))$.

But $f^u$ is constant along $W^u$ manifolds and so takes the constant value $a$ on $Sat^u(G', \delta)$. Since $W^u$ satisfies the absolute continuity property (2) in Proposition 6.10, and $G'$ has full measure in $D$, it follows that $Sat^u(G', \delta)$ has full measure in $Sat^u(D, \delta) = N_\delta(w)$. Hence $f^u$ is constant on a full measure subset of $N_\delta(w)$. Since $f^u = B(f)$, a.e., it follows that $B(f)$ is almost everywhere constant on $N_\delta(w)$, proving part (a).

We next establish part (b) of the claim. Absolute continuity property (1) of $W^s$ implies that for almost every $w' \in \Lambda_2 \cap V$, the disk $W^s(w', \delta)$ is almost everywhere tangent to $E^s$, which is contained in the continuous conefield $C^\ast$. Hence for almost every $w'$, the tangent bundle $T(W^s(w', \delta))$ is everywhere contained in $C^\ast$. Invariance of $W^s$ under $\varphi_t$ implies that for any $w \in \Lambda_2 \cap V$, the tangent bundle to the disk $D(w) = \varphi_{(-\delta,\delta)}(W^s(w, \delta))$ is everywhere contained in $C^\ast \oplus E^0$. The conefields $C^u$ and $C^s$ and the line field $E^0 = \mathbb{R} \varphi$ are continuous (indeed, they are smooth), and so locally the angle between any two of them is bounded below. It follows that there exists a neighborhood $U_v \subset V$ of $v$ such that for any $w, w' \in \Lambda_2 \cap U_v$:

$$W^u(w', \delta) \cap D(w) \neq \emptyset;$$

in other words, for every $w \in \Lambda_2 \cap U_v$:

$$N_\delta(w) = Sat^u(D(w), \delta) \supset \Lambda_2 \cap U_v.$$

This completes the proof of part (b) of the claim, and the proof ergodicity. \hfill \diamond

6.1. Proof of Proposition 6.9: verifying the conditions of Katok-Strelcyn-Ledrappier. Let $V \subset T^1N$ be the set of $v \in T^1N$ such that $\varphi_t(v) \in T^1N$, for all $t \in (-1, 1)$. Fix $t_0 \in (0, 1)$ and consider the restriction of the time-$t_0$ map $\varphi_{t_0}$ to $V$. In this section we verify that the main hypotheses in [17] hold for the map $\varphi_{t_0}: V \to T^1N$. The main results in [17] then imply the conclusions of Proposition 6.9. To paraphrase [17], the conditions we will verify ensure that the set of singularities of the map $\varphi_{t_0}$ is “thin” and that the first and second derivatives of $\varphi_{t_0}$ grow moderately near this set. Throughout this subsection, we assume that assumptions I.-VI. hold.
In the setup of [17], the background hypotheses are: $X$ is a compact metric space, and $V$ is an open and dense subset of $X$ carrying a Riemann structure with controlled singularities near $X \setminus V$. In our application, $V$ is the set described above, endowed with the Sasaki Riemann structure, and $X = T^1N$ is the completion of $T^1N$ in the Sasaki distance metric $d_{Sas}$. We first verify that $X$ is compact, which establishes condition (A) of [17].

**Lemma 6.11.** $(\overline{T^1N}, d_{Sas})$ is compact.

*Proof.* Let $\langle v_{n,m} \rangle_m$ be a sequence of elements of $\overline{T^1N}$, where for each $m \geq 1$, $\langle v_{n,m} \rangle$ is a Sasaki Cauchy sequence in $T^1N$. Since $d_{Sas}(v, w) \geq d(\pi(v), \pi(w))$, it follows that for each $m$, the sequence $\langle \pi(v_{n,m}) \rangle$ is Cauchy in $N$; since $N$ is compact, by passing to a subsequence in the $m$’s, we may assume that $\langle \pi(v_{n,m}) \rangle_m$ converges to a Cauchy sequence $\langle x_n \rangle$ in $N$. What this means is that for every $\epsilon > 0$ there exists an $m_0 > 0$ such that for $m \geq m_0$, we have

$$\lim_{n \to \infty} d(\pi(v_{n,m}), x_n) < \epsilon.$$ 

Now for each $n$, consider the collection $\{\hat{v}_{n,m} \mid m \geq 1\} \subset T_{x_n}^1N$ obtained by parallel translating each $v_{n,m}$ along a geodesic from $T_{\pi(v_{n,m})}N$ to $T_{x_n}N$. Using compactness of $T_{x_n}N$ and a diagonal argument, we obtain a subsequence $m_k$ such that for each $n$, $\hat{v}_{n,m_k}$ converges as $k \to \infty$ to an element $\hat{v}_n \in T_{x_n}^1N$, uniformly in $n$; that is, for every $\epsilon > 0$, there exists $k_0 > 0$ such that for all $k > k_0$ we have

$$\lim_{n \to \infty} \|\hat{v}_{n,m_k} - \hat{v}_n\| < \epsilon.$$

Since the Sasaki distance $d_{Sas}(v_{n,m_k}, \hat{v}_n)$ is bounded by $d(\pi(v_{n,m_k}), x_n) + \|\hat{v}_{n,m_k} - \hat{v}_n\|$, it follows that for every $\epsilon > 0$ there exists a $k_1 > 0$ such that for all $k \geq k_1$,

$$\lim_{n \to \infty} d_{Sas}(v_{n,m_k}, \hat{v}_n) \leq \lim_{n \to \infty} d(\pi(v_{n,m}), x_n) + \|\hat{v}_{n,m_k} - \hat{v}_n\| < \epsilon.$$ 

Hence $\langle v_{n,m_k} \rangle_k$ converges as $k \to \infty$ to the Sasaki Cauchy sequence $\langle \hat{v}_n \rangle \in \overline{T^1N}$. 

Clearly $V$ is an open and dense subset of $\overline{T^1N}$. Let $S = \overline{T^1N} \setminus V$. The Sasaki distance from $v$ to the singular set $S$ is bounded above by the distance from $\pi(v)$ to $\partial N$.

6.1.1. More (yet) on the Sasaki metric. Condition (B) in [17], which concerns the Riemann structure on $V$, has three parts that require verification. In this subsection, we establish bounds on the derivatives of the Sasaki exponential map $\exp : TV \to V$, which we will then use to verify these conditions as well as later conditions on $\varphi_{t_0}$. To control the Sasaki exponential map, we will need to control the first three derivatives of the Sasaki metric; these can be related to the higher order derivatives of the metric on $N$ via the following lemma.
Lemma 6.12. There exists a cubic polynomial \( C : \mathbb{R}^3 \to \mathbb{R} \) such that for any Riemannian manifold \( N \) and any \( v \in T^1_xN \), the Sasaki curvature tensor \( R_{Sas} \) satisfies

\[
\| (R_{Sas})_v \| + \| \nabla (R_{Sas})_v \| \leq C(\| R_x \|, \| \nabla R_x \|, \| \nabla^2 R_x \|),
\]

where \( R \) is the Riemannian curvature tensor on \( N \).

Proof. The sectional curvatures of the Sasaki metric on the unit tangent bundle can be computed as follows [18]. We use the usual identification \( T(x,u)TN \cong (T_xN)^2 \). Let \( \Pi \) be a plane in \( T(x,u)TN \), and choose an orthonormal basis \( \{(v_1, w_1), (v_2, w_2)\} \) for \( \Pi \) satisfying \( \| v_i \|^2 + \| w_i \|^2 = 1 \) for \( i = 1, 2 \) and \( \langle v_1, v_2 \rangle = \langle w_1, w_2 \rangle = 0 \). Then the Sasaki sectional curvature of \( \Pi \) is given by

\[
K_{Sas}(\Pi) = \langle R_x(v_1, v_2)v_2, v_1 \rangle + \langle R_x(v_1, v_2)w_2, w_1 \rangle + \| w_1 \|^2 \| w_2 \|^2 \left( -\frac{3}{4} \| R_x(v_1, v_2)w_1 \|^2 + \frac{1}{4} \| R_x(v_1, v_2)v_1 \|^2 + \frac{1}{4} \| R_x(u, w_1)v_2 \|^2 \right) + \frac{1}{2} \langle R_x(u, w_1)v_2, R_x(u, w_2)v_1 \rangle - \langle R_x(u, w_1)v_1, R_x(u, w_2)v_2 \rangle + \langle (\nabla v_1)R_x(u, w_2)v_2, v_1 \rangle + \langle (\nabla v_2)R_x(u, w_1)v_2, v_1 \rangle.
\]

The conclusion follows from the Chain Rule and the well-known identities relating the sectional curvatures with the norm of the curvature tensor. \( \diamond \)

The next lemma will be used to bound the derivative of the Sasaki exponential map.

Lemma 6.13. Let \( Y \) be a Riemannian manifold, and let \( J \) be a Jacobi field along a geodesic \( \gamma : [-\delta_0, \delta_0] \to Y \) satisfying \( J(0) = 0 \) and \( \| J'(0) \| = 1 \). Suppose that

\[
\sup_{|t| < \delta_0} \| R_{\gamma(t)} \| \leq R_0
\]

for some \( R_0 > 1 \). Let \( \epsilon \in (0, 1) \) be given, and let \( t_0 = \min\{\delta_0, \epsilon/(3R_0)\} \). Then for all \( |t| \leq t_0 \) we have

\[
(1 - \epsilon)|t| \leq \| J(t) \| \leq (1 + \epsilon)|t|
\]

and

\[
|J'(t)| \leq 1 + \epsilon
\]

Proof. Let \( a(t) = \| J(t) \| \), and let \( b(t) = \| J'(t) \| \). Then the Cauchy-Schwarz inequality implies

\[
\| (a^2)' \| = \| 2aa' \| = \| 2\langle J, J' \rangle \| \leq 2ab,
\]

and since \( |t| < \delta_0 \):

\[
\| (b^2)' \| = \| 2bb' \| = \| 2\langle J', J'' \rangle \| = \| 2\langle J', R(\gamma, J)\gamma \rangle \| \leq 2R_0ab.
\]

We conclude that wherever \( |a| \) and \( |b| \) are not zero, we have

\[
|a'| \leq b \quad \text{and} \quad |b'| \leq R_0a.
\]
We are assuming that $a(0) = 0$ and $b(0) = 1$. Without loss of generality, assume that $|a(t)| > 0$ for $t > 0$ (otherwise, we may replace $t = 0$ with a positive value of $t$ in the following argument). From this we obtain the integral inequality, for $t \geq 0$:

$$\int_0^t |b'(s)| \, ds \leq 1 + R_0 \int_0^t a(s) \, ds.$$

Suppose that, for some $t_1 \in (0, t_0)$ we have $|a'(t)| < 1 + \epsilon$ for all $t \in [0, t_1)$ and $|a'(t_1)| = 1 + \epsilon$. Then $a(t) < (1 + \epsilon)t$, for all $t \in [0, t_1)$; combined with (31), this gives that

$$|a'(t_1)| \leq 1 + R_0 \int_0^{t_1} (1 + \epsilon)s \, ds < 1 + \frac{R_0(1 + \epsilon)}{2} t_1^2 < 1 + \epsilon,$$

since $\epsilon \in (0, 1)$ implies that

$$t_1^2 < t_0^2 \leq \frac{\epsilon^2}{2 R_0^2} < \frac{2 \epsilon}{R_0(1 + \epsilon)}.$$

This contradicts our assumption that $|a'(t_1)| = 1 + \epsilon$. We conclude that $|a'(t)| < 1 + \epsilon$ for all $t \in (0, t_0)$; similarly, $|a'(t)| < 1 + \epsilon$, for all $t \in (-t_0, 0)$. From this we conclude that $a(t) \leq (1 + \epsilon) |t|$ for all $|t| \leq t_0$.

We now prove the lower bound. Since $b(0) = 1$ and $|b'(t)| \leq R_0 a(t)$, for $|t| \leq t_0$ we have

$$b(t) \geq 1 - \frac{(1 + \epsilon) R_0 t_0^2}{2}.$$

On the other hand, we know that

$$(a^2)'' = 2b^2 - 2(R(J, \dot{\gamma}\dot{\gamma}, J) \geq 2b^2 - 2 R_0 a^2$$

$$> 2 \left[ \left( 1 - \frac{(1 + \epsilon) R_0 t_0^2}{2} \right)^2 - (1 + \epsilon)^2 R_0 t_0^2 \right] > 2[1 - 2(1 + \epsilon)^2 R_0 t_0^2],$$

(using the lower bound for $b(t)$ and upper bound of $(1 + \epsilon)|t|$ for $a(t)$). Now, since

$$t_0^2 \leq t_0^2 \leq \frac{\epsilon^2}{9 R_0^2} < \frac{\epsilon^2}{2(1 + \epsilon)^2 R_0},$$

we find that

$$(a^2)''(t) > 2[1 - 2(1 + \epsilon)^2 R_0 t_0^2] > 2(1 - \epsilon^2).$$

But then $2a(t)a'(t) = (a^2)'(t) > 2(1 - \epsilon^2)|t|$, and again using the upper bound on $a$, we get

$$a'(t) > \frac{(1 - \epsilon^2)|t|}{a(t)} > \frac{(1 - \epsilon^2)|t|}{(1 + \epsilon)|t|} = 1 - \epsilon;$$

hence $a(t) > (1 - \epsilon)|t|$.

Finally, since $b(0) = 1$ and $|b'| \leq R_0 a \leq R_0 (1 + \epsilon)$, it follows that $|b(t)| \leq 1 + |t| R_0 (1 + \epsilon)$, and so for $|t| < |t_0|$, we have $|b(t)| \leq 1 + \epsilon (1 + \epsilon)/3 < 1 + \epsilon$. The final conclusion follows. ⊗
We apply this lemma to the Sasaki exponential map \( \exp : TV \to V \) to obtain:

**Proposition 6.14.** There exist constants \( \delta_1 > 0 \) and \( k_1 > 1 \) such that for every \( v_0 \in V \), if \( d_{Sas}(v_0, S) < \delta_1 \), then for all \( v \in V \) with \( d_{Sas}(v, v_0) < d_{Sas}(v_0, S)^{k_1} \):

\[
1 - d_{Sas}(v_0, S) \leq \| D_v \exp_{v_0}^{-1} \|^{-1} \leq \| D_\xi \exp_{v_0} \| \leq 1 + d_{Sas}(v_0, S),
\]

where \( \xi = \exp_{v_0}^{-1}(v) \).

**Proof.** Let \( V \) be the 1-parameter family of image geodesics in \( R \) tangent to \( \delta \) we have responding Jacobi field \( J \). Let \( \xi = \exp_{v_0}^{-1}(v) \) be the unit vector in the direction of \( \xi \). Suppose \( \xi' \in T_{v_0}V \) is an orthogonal unit vector. Let \( a(s, t) = (\xi + s\xi')t \) be the 1-parameter family of rays through the origin in \( T_{v_0}V \). Let

\[
a(s, t) = \exp_{v_0} \circ a(s, t)
\]

be the 1-parameter family of image geodesics in \( V \). We consider the corresponding Jacobi field \( J(t) \) along \( a(0, t) \) defined by \( J(t) = \partial a(s, t) / \partial s \) at \( s = 0 \). Clearly \( J(0) = 0 \) and \( J'(0) = \xi' \). Setting \( t_1 = ||\xi|| \), by the chain rule we have

\[
\| J(t_1) \| = ||t_1 D_\xi \exp_{v_0}(\xi')||.
\]

Thus we have to bound \( \| J(t_1) \| \) above and below.

By Lemma 6.12, the sectional curvatures of the Sasaki metric on the unit tangent bundle are bounded polynomially in terms of the absolute value of the curvature and the derivative of the curvature of the original metric. Assumption IV. gives a bound for these latter quantities, and therefore a polynomial bound on the curvatures in the Sasaki metric, in the reciprocal of the distance to the singular set \( S \). It follows that there exist \( k_0 > 1 \) and \( \delta_0 > 0 \) such that for all \( v_0 \in V \) with \( d_{Sas}(v, S) < \delta_0 \), the Sasaki curvature tensor \( R_{Sas} \) satisfies

\[
\|(R_{Sas})_v\| < d_{Sas}(v, S)^{-k_0}.
\]

Let \( k_1 = k_0 + 2 \). Then there exists \( \delta_1 \in (0, 1/3) \) such that if \( d_{Sas}(v_0, S) < \delta_1 \) and

\[
d_{Sas}(v_0, v) \leq d_{Sas}(v_0, S)^{k_1},
\]

then the maximum norm \( R_0 \) of the Sasaki curvature tensor along the geodesic joining \( v_0 \) to \( v \) also satisfies \( R_0 < d_{Sas}(v_0, S)^{-k_0} \). Lemma 6.13 implies that

\[
1 - \epsilon \leq \left| \frac{J(t_1)}{t_1} \right| < 1 + \epsilon,
\]

provided \( \epsilon > 3R_0|t_1| = 3R_0d_{Sas}(v_0, v) \). Hence if \( d_{Sas}(v_0, S) < \delta_1 \) and \( d_{Sas}(v, v_0) \leq d_{Sas}(v_0, S)^{k_1} \), then (32) holds for \( \epsilon = d_{Sas}(v_0, S) \), since

\[
3R_0d_{Sas}(v_0, v) < 3d_{Sas}(v_0, S)^{k_0} \cdot d_{Sas}(v_0, S)^{k_1} = 3d_{Sas}(v_0, S)^2 < d_{Sas}(v_0, S) = \epsilon.
\]
The next proposition gives bounds on the second derivative of $\exp$, which we will later use to verify condition (1.3) of [17].

**Proposition 6.15.** There exist constants $\delta_2 > 0$ and $k_2 > 1$ such that for every $v_0 \in V$, if $d_{Sas}(v_0, S) < \delta_2$, then for all $v \in V$ with $d_{Sas}(v, v_0) < d_{Sas}(v_0, S)^{k_2}$:

$$\max\{\|D^2_\xi \exp_{v_0}\|, \|D^2_v \exp^{-1}_{v_0}\|\} < d_{Sas}(v_0, S)^{-k_2}$$

where $\xi = \exp^{-1}_{v_0}(v)$.

**Proof.** Fix $v_0 \in V$ and $v \in V$ in a neighborhood of $v_0$. Let $\xi = \exp^{-1}_{v_0}(v)$. Let $\hat{\xi} = \frac{\xi}{\|\xi\|}$ be the unit vector in the direction of $\xi$, and suppose $\xi' \in T^1_{v_0} V$ is an orthogonal unit vector. As in the proof of Proposition 6.14, we consider the variation of geodesics

$$\alpha(s, t) = \exp_{v_0} \circ a(s, t),$$

where $a(s, t) = (\hat{\xi} + s\xi')t$. Define $Z, J$ and $Q$ by

$$Z(s, t) = \frac{d}{dt}\alpha(s, t); \quad J(s, t) = \frac{d}{ds}\alpha(s, t); \quad Q(s, t) = \frac{d^2}{ds^2}\alpha(s, t) = \frac{d}{ds}J(s, t).$$

The chain rule implies that

$$Q(0, t) = D^2_{a(0, t)} \exp_{v_0}(t\xi', t\xi'),$$

and so

$$\|D^2_\xi \exp_{v_0}(\xi', \xi')\| = \frac{1}{\|\xi\|^2}\|Q(0, \|\xi\|)\|,$$

since $\xi = a(0, \|\xi\|)$.

Observe that for a fixed $s$, $J(s, \cdot)$ is a Jacobi field down the geodesic $\alpha(s, \cdot)$ and so satisfies the Jacobi equation

$$\frac{d}{dt^2}J = -R_{Sas}(Z, J)Z.$$

From this, the symmetry of the Levi-Civita connection, and the definition of $Q$ it follows that

$$\frac{d}{dt^2}Q = \frac{d}{ds} \frac{d}{dt^2}J = -\frac{d}{ds}R_{Sas}(Z, J)Z$$

$$= -\left(\left(\frac{d}{ds}R_{Sas}\right)(Z, J)Z + R_{Sas}(J', J)Z + R_{Sas}(Z, Q)Z + R_{Sas}(Z, J)J'\right),$$

where $'$ denotes the derivative with respect to $t$. Then $\|Q''(0, t)\| \leq C_1(t) + \|Q(0, t)\|C_2(t)$, where

$$C_1(t) = \| (\nabla R_{Sas})_{\exp_{v_0}(t\hat{\xi})} \| \|J(0, t)\| + 2\| (R_{Sas})_{\exp_{v_0}(t\hat{\xi})} \| \|J(0, t)\| \|J'(0, t)\|,$$

and

$$C_2(t) = \| (R_{Sas})_{\exp_{v_0}(t\hat{\xi})} \|.$$
Assumption IV. and Lemma 6.12 imply that there exists $k_0 > 1$ such that
\[ \max\{\| (R_{Sas})_{v_0} \|, \| (\nabla R_{Sas})_{v_0} \| \} < d_{Sas}(v_0, S)^{-k_0}. \]

Fix $\delta_2 \in (0, 1/10)$ such that if $d_{Sas}(v_0, S) < \delta_2$, then
\[ \sup_{|t| \leq d_{Sas}(v_0, S)^{k_0+1}} \max\{\| (R_{Sas})_{\exp_v(t)} \|, \| (\nabla R_{Sas})_{\exp_v(t)} \| \} < d_{Sas}(v_0, S)^{-k_0-1}. \]

Assume that $d_{Sas}(v_0, S) < \delta_2$. Lemma 6.13 implies that for $|t| < d_{Sas}(v_0, S)^{k_0+2}$, both $\| J(0, t) \|$ and $\| J'(0, t) \|$ are bounded by 2, and so
\[ C_1(t) \leq 10d_{Sas}(v_0, S)^{-k_0-1} < d_{Sas}(v_0, S)^{-k_0-2}, \]
and
\[ C_2(t) \leq d_{Sas}(v_0, S)^{-k_0-1} < d_{Sas}(v_0, S)^{-k_0-2}. \]

Let $q(t) = \| Q(0, t) \|$ and let $r(t) = \| Q'(0, t) \|$. As in the proof of Lemma 6.13, we have that
\[ |qq'| = |\langle Q, Q' \rangle| \leq qr, \quad \text{and} \quad |rr'| = |\langle Q', Q'' \rangle| \leq r(C_1 + qC_2). \]

Note that $q(0) = r(0) = 0$. An analysis similar to that in the proof of Lemma 6.13 (whose details we omit) shows that for $|t| < d_{Sas}(v_0, S)^{k_0+2}$, we have
\[ q(t) \leq t^2 d_{Sas}(v_0, S)^{-k_0-2}. \]

Hence, if $\| \xi \| \leq d_{Sas}(v_0, S)^{k_0+2}$, then
\[ \| D^2_{\xi} \exp_{v_0}(\xi', \xi') \| = \frac{q(\| \xi \|)}{\| \xi \|^2} \leq d_{Sas}(v_0, S)^{-k_0-2}. \]

Hence the upper bound for $\| D^2_{\xi} \exp_{v_0} \|$ in the conclusion of the proposition holds, with the exponent $k_2 = k_0 + 2$. An upper bound for $\| D^2_{\xi} \exp_{v_0}^{-1} \|$ on the order of $d_{Sas}(v_0, S)^{-k_2}$ follows from this upper bound on $\| D^2_{\xi} \exp_{v_0}^{-1} \|$, the upper bound on $\| D_v \exp_{v_0}^{-1} \|$ given by Proposition 6.14, and the fact that
\[ D^2 \exp^{-1} = - (D \exp^{-1}) (D^2 \exp) (D \exp^{-1}). \]

\[ \diamond \]

6.1.2. Verifying condition (B) in [17]. For $v \in V$, let $\text{inj}(v)$ denote the radius of injectivity of the Sasaki exponential map $\exp_v : T_v V \to V$. Since $d_{Sas}(v, w) \geq d(\pi(v), \pi(w))$, the controlled injectivity assumption V. implies that
\[ \text{inj}(v) \geq \text{inj}(\pi(v)) \geq Cd(\pi(v), \partial N)^d \geq Cd_{Sas}(v, S)^d. \]

This implies condition (Ba) of [17]. Conditions (Bb) and (Bc) in [17] follow in a straightforward way from Proposition 6.14.
Verifying conditions (1.1) – (1.4) of [17]. Conditions (1.1) – (1.4) of [17] concern the volume of the singular set $S$ and the behavior of $\phi_t$ near $S$. Condition (1.1) of [17], which concerns the volume of a neighborhood of $S$, follows directly from Lemma 6.3. Condition (1.2) of [17], concerning the integrability of $\log dS$, follows immediately from Lemma 6.7. Condition (1.3) of [17] requires a bound on $\Delta \phi$, which follows from the next proposition.

Proposition 6.16. There exist constants $\delta_3 > 0$ and $k_3 > 1$ such that for every $v_0 \in V$, if $d_{Sas}(v_0, S) < \delta_3$, then for all $v \in V$ with $d_{Sas}(v, v_0) < d_{Sas}(v_0, S)^{k_3}$:

$$\|D^2_\xi \Phi(v_0)\| < d_{Sas}(v_0, S)^{-k_3},$$

where $\xi = \exp^{-1}(v)$.

Proof. Choose constants $k_2 > 1$ and $\delta_2 > 0$ satisfying the conclusions of Proposition 6.15 and such that for every $|t| \leq t_0$:

$$\sup_{d_{Sas}(v, \phi_t(v)) \leq d_{Sas}(\phi_t(v), S)^{k_2}} \max\{\|R_{Sas}(v)\|, \|\nabla R_{Sas}(v)\|\} < d_{Sas}(v_0, S)^{-k_2}. $$

By assumption VI., there exist $\delta_3 < \min\{\delta_2, 1/(8n)\}$ and $k'_3 > k_3$ such that for $d_{Sas}(v_0, S) < \delta_3$, and all $|t| \leq t_0$:

$$\max\{\|D_{\phi_t}(v_0)\|, \|D_{\phi_t}(v_0)\|\} \leq d_{Sas}(v_0, S)^{-k'_3}. $$

Proposition 4.2 implies that if $\xi, \xi' \in TV$ satisfy $d_{Sas}(\pi_V(\xi), \pi_V(\xi')) < d_{Sas}(\pi_V, S)^{k_2}$ then

$$d_{Sas}(D_{\phi_t}(\xi), D_{\phi_t}(\xi')) \leq (8n) d_{Sas}(v_0, S)^{-4k'_3-3k_2} d_{Sas}(\xi, \xi').$$

To bound the norm $\|D^2_\xi \Phi(v_0)\|$ it suffices to bound the Lipschitz constant of the map $D_\xi \Phi(v_0)$ in a small neighborhood of $\xi$. This in turn is bounded by the product of the Lipschitz constants of the three factors $D_{\exp^{-1}_{\phi_t(v_0)}}$, $D_{\phi_t}$ and $D_{\exp(v_0)}$ in the composition defining $D_\xi \Phi(v_0)$.

The Lipschitz constants for $D_{\exp(v_0)}$ and $D_{\exp^{-1}_{\phi_t(v_0)}}$ are bounded for nearby points by the norms $\|D^2 \exp(v_0)\|$ and $\|D^2 \exp^{-1}_{\phi_t(v_0)}\|$, which by Proposition 6.15 are both bounded on the order of $d_{Sas}(v_0, S)^{-k_2}$. We have just shown that the Lipschitz constant for $D_{\phi_t}$ is bounded on the order of $d_{Sas}(v_0, S)^{-7k_2}$. Hence the Lipschitz constant of $D_\xi \Phi(v_0)$ is bounded on the
order of $d_{\text{Sas}}(v_0, S)^{-k_4}$, for $k_4 = 2k_2 + 7k_2'$. The details of this argument are left to the reader. ⋄

This completes the verification of the hypotheses in [17] implying the conclusions of Proposition 6.9.

6.2. Additional conditions in [17] implying finite, positive entropy. The final conclusion of Theorem 6.6 that remains to be proved concerns the entropy of $\phi$. The positivity of the entropy follows from [17] and the hypotheses we have just verified. Finitude of the entropy requires that an additional hypothesis – Condition (C) – hold. As stated in [17], condition (C) is the requirement that the capacity of the space $X = T^1N$ be finite. In fact, a slightly weaker condition is required, which is given by the following proposition. Recall that $U_\rho$, for $\rho > 0$, denotes the set of $v \in T^1N$ such that $d(\pi(v), \partial N) < \rho$.

**Proposition 6.17.** There exists $q > 1$ such that if $\rho_0 > 0$ is sufficiently small, then for any $\rho < \rho_0$ there is a cover of $T^1N \setminus U_{\rho_0}$ by open balls of radius $\rho$, whose cardinality does not exceed $\rho^{-q}$.

**Proof.** Proposition 6.14 implies that there exist $\delta > 0$ and $k > 1$ such that for $\rho_0 < \delta$ and all $v \in T^1N \setminus U_{\rho_0}$, the derivative of the Sasaki exponential map $D_{\xi} \exp_v$ and its inverse have norm bounded by 2, for all $\|\xi\| < \rho_0^k$. Hence on a ball of radius $\rho_0^k$ in $T^1N \setminus U_{\rho_0}$, the Sasaki metric is uniformly comparable to Euclidean; in particular, the volume of a ball of radius $\rho \leq \rho_0^k$ is bounded below by $c^{-1} \rho^n$, where $c > 1$ is a universal constant, and the local geometry of covers by balls is the same as Euclidean.

For $\rho < \rho_0$, the Besicovitch covering lemma then implies that $T^1N \setminus U_{\rho_0}$ can be covered by balls $\{B_\ell(\rho)\}_{\ell=1}^{p(\rho)}$ of radius $\rho^{k+1}$ with universally bounded overlap $C$. Hence

$$p(\rho) \left(c^{-1} \rho^{n(k+1)}\right) \leq \sum_{\ell=1}^{p(\rho)} m(B_\ell(\rho)) \leq C m \left( \bigcup_{\ell=1}^{p(\rho)} B_\ell \right) \leq C m(T^1N) = C$$

and so

$$p(\rho) \leq C c \rho^{-n(k+1)},$$

for all $\rho < \rho_0$. This implies the conclusion of the proposition, with $q = n(k + 1) + 1$.

⋄

7. Ergodicity and finite entropy of the WP geodesic flow

Fix a Riemann surface $S$, and let $\mathcal{T} = \text{Teich}(S)$, $\text{MCG} = \text{MCG}(S)$ and $\mathcal{M} = \mathcal{M}(S)$. We describe here first how the results of Section 6 can be applied to obtain ergodicity and finite entropy of the geodesic flow on the quotient $\mathcal{M}^1 = T^1\mathcal{T}/\text{MCG}$. Note that the results in Section 6 cannot be
applied directly with \( M = T \) and \( \Gamma = \text{MCG} \), since MCG does not act freely on \( T \). Our strategy is to prove ergodicity first for a finite branched cover \( T^1/\text{MCG}[3] \). Here \( \text{MCG}[k] \) is the level \( k \) congruence subgroup:

\[
\text{MCG}[k] = \{ \phi \in \text{MCG} : \phi_* = 0 \text{ acting on } H_1(S, \mathbb{Z}/k\mathbb{Z}) \},
\]

which is clearly a finite index subgroup of MCG. If \( k \geq 3 \), \( \text{MCG}[k] \) is torsion-free and so acts freely and properly discontinuously by isometries on \( T \). The quotient \( T^1/\text{MCG}[k] \) has finite volume. We obtain ergodicity for the flow on \( T^1/\text{MCG}[k] \) for any \( k \geq 3 \) using the setup of the previous section. This result has independent interest.

7.1. **Ergodicity of the flow on** \( T^1/(T/\text{MCG}[k]) \). Fix \( k \geq 3 \). To establish ergodicity and finite metric entropy of the WP geodesic flow on \( T^1(T/\text{MCG}[k]) \), we show that the assumptions I.-VI. of the previous section are satisfied for \( M = T \), \( \Gamma = \text{MCG}[k] \) and the WP metric.

The fact that in the Weil-Petersson metric \( T \) is geodesically convex was proved by Wolpert [38]. Since the completion \( \overline{M} \) of \( M \) is compact, and \( T/\text{MCG}[k] \) is a finite branched cover of \( M \), it follows that the completion of \( T/\text{MCG}[k] \) is compact as well. Hence assumptions I. and II. hold true.

The curvature bound in assumption IV. is due to Wolpert and was stated as Proposition 3.8. The bounds on \( \|\nabla R_{WP}\| \) and \( \|\nabla^2 R_{WP}\| \) in assumption IV. is the content of Theorem 5.1. Assumption VI. was proved in Theorem 3.1. It remains to prove Assumptions III. and V.

**Verifying assumption III.**: \( \partial (T/\text{MCG}[k]) \) is volumetrically cusplike.

Given \( \rho \), let

\[
E_\rho = \{ X : \rho(X) \leq \rho \}.
\]

**Lemma 7.1.** We have \( \text{Vol}(E_\rho) = O(\rho^4) \)

**Proof.** Cover \( E_\rho \) with a finite number of open sets in which we can use the plumbing coordinates \((s,t)\) introduced in Section 5.1.1. Fix one such neighborhood \( U \) with \( p \geq 1 \) short curves. The matrix \((g_{i,j})\) of the metric is the inverse of the matrix \((\langle \phi_i, \phi_j \rangle)\).

By the estimates in Section 5.1 the determinant of the matrix \((\langle \phi_i, \phi_j \rangle)\) is

\[
\asymp \prod_{i=1}^{p} -|t_i|^2(\log |t_i|)^3.
\]

The volume element is then the determinant of the inverse and the volume is then found by integration:

\[
\text{Vol}(U) = \prod_{i=1}^{p} \int_{0}^{|t_i|} \frac{|dt_i|}{-|t_i|^2(\log |t_i|)^3} = \prod_{i=1}^{p} \frac{1}{(-\log |t_i|)^2}.
\]

Thus \( \text{Vol}(E_\rho) = O(\rho^{4p}) \).

**Verifying assumption IV.**: \( T/\text{MCG}[k] \) has controlled injectivity radius.
For any $X \in \mathcal{T}$, again let $\rho_0(X)$ denote the distance of $X$ to $\partial \mathcal{T}$. Let $\tau_\alpha$ denote the Dehn twist about the curve $\alpha$.

Given a simplex $\sigma = \{\alpha_1, \ldots, \alpha_p\} \in \mathcal{C}(S)$, let $\Gamma(\sigma) = \langle \tau_1, \ldots, \tau_p \rangle$ be the abelian group generated by the Dehn twists about the $\alpha_i$.

**Lemma 7.2.** For each $\epsilon > 0$ there exists $c > 0$ and $j \leq (3g - 3 + n)!$ such that if $\phi \in \text{MCG}[k]$ and $d(X, \phi(X)) < c$, then there exists $\sigma \in \mathcal{C}(S)$ such that $X \in \Omega(\sigma, \epsilon)$ and $\phi^{(j)} \in \Gamma(\sigma)$.

**Proof.** Let $\epsilon > 0$ be given. Since $\text{MCG}[k]$ acts properly discontinuously without fixed points, the first conclusion must be true: there exists $c_0 > 0$ such that if $\phi \in \text{MCG}[k]$ and $d(X, \phi(X)) < c_0$, then $X \in \Omega(\sigma, \epsilon)$, for some $\sigma \in \mathcal{C}(S)$. Now suppose the second statement is not true; i.e., suppose that $\phi^{(j)} \not\in \Gamma(\sigma)$ for any $j$ smaller than the order of the permutation group of $\sigma$.

Then there is a sequence $X_m \in \mathcal{T}$, $\phi_m$ such that $\phi_m^{(j)} \in \text{MCG}[k] \setminus \Gamma(\sigma)$ for all $j$, $X_m \in \Omega(\sigma, \epsilon)$ and

$$d_{WP}(X_m, \phi_m(X_m)) \to 0.$$  

Passing to a subsequence and applying an element $\psi_m \in \Gamma(\sigma)$ if necessary, we can assume $X_m$ converges to a noded surface $X_\sigma$. There is a constant $c_1 = c_1(\delta)$ such that if there is a curve $\beta$ and a pair of surfaces $Z_1$ and $Z_2$ such that $\ell_\beta(Z_1) \leq \delta/2$ and $\ell_\beta(Z_2) \geq \delta$, then

$$d_{WP}(Z_1, Z_2) \geq c_1.$$  

This implies that $\phi_m(\sigma) = \sigma$ for $m$ sufficiently large and so for some $j$, $\phi_m^{(j)}$ preserves the individual curves of $\sigma$.

We are assuming that $\phi_m^{(j)} \not\in \Gamma(\sigma)$. By the classification of elements of MCG, restricted to each piece of $X_\sigma$, $\phi_m^{(j)}$ is either pseudo-Anosov or finite order. Since $\phi_m^{(j)} \in \text{MCG}[k]$ it cannot be finite order on each piece and so must be pseudo-Anosov on some piece. There is a uniform lower bound [?] for $d_{WP}(X_\sigma, \phi_m^{(j)}(X_\sigma))$ and thus a lower bound for $d_{WP}(X_m, \phi(X_m))$ for $m$ sufficiently large, a contradiction. \hfill \diamond

**Lemma 7.3.** There is a constant $c > 0$ such that for any $X \in \mathcal{T}/\text{MCG}[k]$:

$$\text{inj}(X) \geq c\rho(X)^3.$$  

**Proof.** By Proposition 15 of [36] there is a positive constant $c > 0$ such that for $X \in \Omega(\sigma, \epsilon)$, $d_{WP}(X, \Gamma(\sigma)(X)) \geq c\rho(X)^3$. This bounds the injectivity radius from below. \hfill \diamond

Applying Theorem 6.6, we have now proved

**Theorem 7.4.** The Weil-Petersson flow is ergodic and has finite entropy on $T^1\mathcal{T}/\text{MCG}[k]$.  

7.2. Ergodicity of the flow on $\mathcal{M}^1(S)$: Proof of Theorem 1. The manifold $T/MCG[k]$ is a finite branched cover over $\mathcal{M}$. Let $h : X \to X$ a conformal automorphism of finite order. Then the action $h : T \to T$ has a fixed point set $F(h)$. It is known [30] that if $S$ is compact, unless $h$ is the hyperelliptic involution in genus 2, then $F(h)$ has complex dimension at most $3g−5$. In fact $F(h)$ is the Teichmüller space of the quotient orbifold $X/h$. In genus 2 the hyperelliptic involution fixes every point. In the noncompact case where $S$ has punctures, the complex dimension of $F(h)$ is at most $3g−4$.

Let $F$ denote the union of the fixed point sets of the action of all finite order elements of $MCG(S)$, excluding the genus 2 hyperelliptic case. This is a countable union of lower dimensional Teichmüller spaces.

**Lemma 7.5.** $F$ is a closed subset of $T$, of codimension at least 2.

**Proof.** We have already seen that each fixed point set has real codimension at least 2 so we need only check that the union is locally finite. Fix a compact set $K \subset T$. By the proper discontinuity of the action of $MCG$ on $T$, there cannot be an infinite set of finite order elements each with a fixed point in $K$. Thus $K$ is intersected by only finitely many of the fixed point sets $F(h)$, and so the union of these sets is closed. ⋄

We now finish the proof of ergodicity. Since the fixed point set has codimension at least 2, the flow is defined almost everywhere on the quotient $\mathcal{M}^1$. If one has a positive measure invariant set in $E \subset \mathcal{M}^1$ then the lift of $E$ is a positive measure invariant measure set in $T^1T/MCG[k]$, which by ergodicity must have 0 or full measure. The same is then true for $E$. Hence the geodesic flow on $\mathcal{M}^1$ is ergodic.

Since the geodesic flow on $T^1T/MCG[k]$ covers the geodesic flow on a full measure subset of $\mathcal{M}^1$, it follows that the entropy of the flow on $\mathcal{M}^1$ is also finite.

**References**


