Stable commutator length in graphs of groups

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Stable commutator length (scl)

Definition

Space $X$, null-homologous loop $\gamma$. Then $\gamma$ bounds a surface.

Admissible surfaces $(S, f)$

$$\text{scl}_X(\gamma) := \inf_{(S, f)} \frac{-\chi^- (S)}{2 \cdot n}$$

Multiplicative!

$$\chi^- (S) = \chi(S - D^2 \cdot S^2 \cdot S).$$

$$\text{scl}_G(g) = \text{scl}_X(\gamma)$$
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Multiplicative!

$$\chi^-(S) = \chi(S - D^2 s', S^2 s').$$

$$\text{scl}_G(g) = \text{scl}_X(\gamma)$$

**Example**

$$\text{scl}_G([x, y]) \leq 1/2.$$  

Space of surfaces has little structure.

Challenge: prove lower bounds of scl or compute it.
Properties

- **algebraic**: only depends on $G = \pi_1(X)$, and has a group-theoretic definition using commutators.

- **non-increasing**: $\forall \varphi : G \rightarrow H$ homomorphism, $\text{scl}_G(g) \geq \text{scl}_H(\varphi(g))$.

- **characteristic**: $\forall \varphi \in \text{Aut}(G)$, $\text{scl}_G(\varphi(g)) = \text{scl}_G(g)$.

- **dual** to quasimorphisms, related to $H^2_b(G)$.

- $\text{scl}_G \equiv 0$ if $G$ is amenable or a higher rank irreducible lattice.

- **non-trivial** on word-hyperbolic groups, mapping class groups, etc.
Gromov–Thurston norm

Gromov Norm

A ⊂ X subspace, α ∈ H₂(X, A; ℤ). (S, ∂S) → (X, A) represents α, the Gromov norm

\[ \|α\| := \inf_{[S]=nα} \frac{-2\chi^{-}(S)}{n}. \]

When (X, A) = (M³, ∂M),

\[ \|α\|_{Th} := \inf_{[S]=α} -\chi^{-}(S) = \frac{1}{2} \|α\|. \]

Remove “embedded” by Gabai–Thurston

Example

K ⊂ S³ knot, X = S³ \ N(K), A = ∂X, γ = K = ∂Σ, then there is a unique α ∈ H₂(X, A) with ∂α = [γ]. We have

\[ \text{scl}_G(γ) = \frac{1}{4} \|α\| = \frac{1}{2} \|α\|_{Th} = g(K) - \frac{1}{2} \in \mathbb{Q} \]

extremal

0 = H₂(X) → H₂(X, A) → H₁(A) → H₁(X)
**Definition**

An admissible surface $(S, f)$ is *extremal* for $g$ if

$$\text{scl}_G(g) = \inf_{(S, f)} \frac{-\chi^-(S)}{2 \cdot n}$$

Note: extremal $\xrightarrow{\text{finite cover}}$ extremal  

Existence $\implies$ $\text{scl}_G(g) \in \mathbb{Q}$

**Proposition**

If $(S, f)$ is extremal then $f_* : \pi_1(S) \to G$ is *injective*.

Proof: Compression+LERF
Computations and rationality

**Theorem (Calegari’08)**

\[ G = F_n, \text{ there is a linear programming algorithm to compute } \text{scl}_G(g) \]
and produces *extremal surfaces* (in particular \( \text{scl}_G(g) \in \mathbb{Q} \)).

**Gromov’s Question**

Does every one-ended word-hyperbolic group \( G \) contain a \( \pi_1(S) \)?
- Kahn–Marković’09: yes if \( G = \pi_1(M), M^3 \) closed hyperbolic.

**Application (surface subgroups)**

Calegari’08: \( g \in [F_n, F_n], G = DF_n(g) \) has a surface subgroup.
Main theorem

**Theorem (C.’19)**

$G$ a graph of groups, such that for each vertex group $G_v$ f.i.

1. $\text{scl}_{G_v} \equiv 0$; and (e.g. $G_v$ amenable, $\text{SL}_3 \mathbb{Z}$)
2. $\text{Im}(G_e \to G_v)$ are central and mutually commensurable;

Then there is a linear programming algorithm to compute $\text{scl}_G(g)$, which is rational.

**Examples**

$G = \text{BS}(M, L) := \langle a, t \mid a^M = ta^L t^{-1} \rangle$;

(Clay–Forester–Louwsma’13):
alternating words in $\text{BS}(M, L)$;

(C.’16) $G = \ast_\lambda G_\lambda$ with $\text{scl}_{G_\lambda} \equiv 0$;

(Susse’13) $G = \mathbb{Z}^m \ast_{\mathbb{Z}^k} \mathbb{Z}^n$. 
Proof ideas

• **Step 1:** Obtain *simple normal form* of (relative) admissible surfaces. Each is made of simple “Lego” pieces.

• **Step 2:** Use linear programming to find the best combination of “Lego” pieces.
Relative admissible surfaces

- **relative** admissible surface: allow extra boundary curves in vertex spaces.

\[ X \]
\[ G = A \ast B \]

**Lemma:** \( \text{scl}_G(g) = \inf_{(S,f) \in \text{adm}} \frac{-\chi^{-}(S)}{2 \cdot n} \geq \inf_{(S,f) \in \text{rel adm}} \frac{-\chi^{-}(S)}{2 \cdot n} \).

\[ "=\] if \( \text{scl}_{G_v} \equiv 0 \)
Simple normal form

- Edge spaces cut $S$ into subsurfaces supported in vertex spaces.
- Simplify each component into a disk or annuli.
Simple normal form

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- Disk iff winding number $= 0$. 
minimize $\frac{-\chi(S)}{2n} \iff$ maximize $\#\text{disk pieces}/n$

max: $\sum x_D$
subj: gluing & normalizing conditions
Fix a suitable \( D \in \mathbb{Z}_+ \).

- A piece is \textit{disk-like} if its winding number is divisible by \( D \).
- \( \hat{\chi}(S) = \chi(S) + \# \{ \text{disk-like but not disk} \} \).

\[
\Rightarrow \inf_{S \text{ rel adm}} \frac{-\hat{\chi}(S)}{2n(S)} \leq \inf_{S \text{ rel adm}} \frac{-\chi(S)}{2n(S)} = \text{scl}(\gamma).
\]

**Key Lemma:** There is a suitable \( D = D(\gamma) \) such that “=” holds: for any relative admissible \( S \) and any \( \epsilon > 0 \), there is another \( \hat{S} \) satisfying

\[
\frac{-\chi(\hat{S})}{2n(\hat{S})} < \frac{-\hat{\chi}(S)}{2n(S)} + \epsilon.
\]
Asymptotic promotion

- there is $N = N(\gamma)$ such that if $dm^N \ell^N \mid w(C)$, $(M = dm, L = d\ell)$ then $C$ can be promoted to a disk asymptotically.
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