Abstract. Given a finite subgroup $G$ of the mapping class group of a surface $S$, the Nielsen realization problem asks whether $G$ can be realized as a finite group of homeomorphisms of $S$. In 1983, Kerckhoff showed that for $S$ a finite-type surface, any finite subgroup $G$ may be realized as a group of isometries of some hyperbolic metric on $S$. We extend Kerckhoff’s result to orientable, infinite-type surfaces. As applications, we classify torsion elements in the mapping class group of the plane minus a Cantor set, and also show that topological groups containing sequences of torsion elements limiting to the identity do not embed continuously into the mapping class group of $S$.

In 1932, Nielsen asked whether finite subgroups of mapping class groups act on surfaces. In 1983, Kerckhoff [Ker83] gave the following strong affirmative answer.

**Theorem 1** (Kerckhoff, Theorem 5 [Ker83]). Let $S$ be a finite-type surface with negative Euler characteristic. Any finite subgroup of the mapping class group of $S$ may be realized as a group of isometries of some hyperbolic metric on $S$.

Let $S$ be a surface. We distinguish two kinds of surfaces, saying $S$ is a finite-type surface if its fundamental group is finitely generated, and is an infinite-type surface otherwise. Recently there has been a surge of interest in infinite-type surfaces and their mapping class groups. We refer the interested reader to a recent survey by Aramayona and Vlamis [AV20].

The main purpose of this paper is to extend Kerckhoff’s result to the infinite-type case.

**Theorem 2.** Let $S$ be an orientable, infinite-type surface. Any finite subgroup of the mapping class group of $S$ may be realized as a group of isometries of some hyperbolic metric on $S$.

Let us briefly sketch the idea of the proof. Let $G$ be a finite subgroup of the mapping class group of $S$. The mapping class group acts on the Teichmüller space of $S$, denoted $T(S)$, which parameterizes hyperbolic structures on $S$ up to isotopy. Kerckhoff defines a $G$-invariant map $\ell_G: T(S) \to \mathbb{R}_+$. The map sends a hyperbolic structure to the sum of the geodesic lengths of a certain finite $G$-invariant collection of simple closed curves on $S$. When $S$ is of finite type, Kerckhoff proves that the map $\ell_G$ attains a unique minimum. This minimum is fixed by $G$, yielding an action of $G$ by isometries of the corresponding hyperbolic structure on $S$.

In the infinite-type setting, all of the tools in Kerckhoff’s proof are available to us, but the sum defining $\ell_G$ diverges. Instead, we find an exhaustion of $S$ by connected, $G$-invariant (homotopy classes of) finite-type subsurfaces $S_0 \subset S_1 \subset \cdots$. Kerckhoff’s theorem applies to each piece $S_k \setminus S_{k-1}$, and we show how to assemble the pieces to give a hyperbolic structure on $S$ and an action of $G$ by isometries. It would be interesting to know if Kerckhoff’s method of proof could be applied more directly.

We remark that if $\Sigma$ is a finite-type subsurface of $S$, the hyperbolic metric on $S$ in the theorem is chosen so that $\Sigma$ inherits a hyperbolic metric of finite volume. In other words,
each isolated end of $S$ is given a metric modeled on the pseudosphere rather than a flared annulus.

**Corollary 3.** If $S$ is an orientable, infinite-type surface with nonempty compact boundary, the relative mapping class group fixing the boundary pointwise is torsion-free.

As an application, the Nielsen realization theorem allows us to classify torsion elements in the mapping class group of the plane minus a Cantor set; see Theorem 6.

Equip the full homeomorphism group of $S$ with the compact-open topology, and the mapping class group of $S$ with the quotient topology. As another application, we have the following theorem.

**Theorem 4.** If $G$ is a topological group containing a sequence of nontrivial finite order elements limiting to the identity, then $G$ does not embed (as a topological group) in the mapping class group of $S$.

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In the remainder of this introduction, let us make the statement of Theorem 2 more precise. The mapping class group of $S$, denoted $\text{Map}(S)$, is the group $\pi_0(\text{Homeo}(S))$ of homotopy classes of homeomorphisms $f : S \to S$. If $S$ has nonempty boundary $\partial S$, let $\text{Map}(S, \partial S)$ denote the group of homotopy classes of homeomorphisms $f : S \to S$ where we require all homeomorphisms and homotopies to fix the boundary $\partial S$ pointwise.

A classification of infinite-type surfaces was given by Kerékjártó [Ker23] and Richards [Ric63]. We recall that each surface $S$ has a space of ends, defined as usual as an inverse limit $\lim \leftarrow \pi_0(S \setminus K)$ as $K$ ranges over the compact subsets of $S$. If we fix a compact exhaustion $K_0 \subset K_1 \subset \cdots$ of $S$, an end is represented as a sequence $U_0 \supset U_1 \supset \cdots$, where each $U_i$ is a connected component of $S \setminus K_i$. The end is isolated if all but finitely many of the $U_i$ have one end, and planar if all but finitely many of the $U_i$ are planar. These properties do not depend on the choice of compact exhaustion. Finite-type surfaces have finitely many isolated planar ends.

By a hyperbolic structure on a surface $S$, we mean $S$ equipped with a complete Riemannian metric of constant curvature $-1$ satisfying the following two conditions. (i) We require that each isolated planar end of $S$ is modeled on the pseudosphere, and following Kerckhoff [Ker83, Section 4], (ii) we require that each boundary curve is a geodesic with length 1. In the case where $S$ is of finite type, the condition on ends is satisfied by insisting that the metric has finite volume. We remark that the condition that boundary curves have length 1 is slightly nonstandard but useful. In addition, forcing this condition on our finite-type exhaustion of $S$ places the hyperbolic structure on $S$ in the component containing the “thick part” of Teichmüller space.

1. Finding an Invariant Exhaustion

Fix $S$ an orientable, connected, infinite-type surface, and $G$ a nontrivial finite subgroup of $\text{Map}(S)$. The goal of this section is to prove the following proposition.

**Proposition 5.** There exists an exhaustion of $S$ by connected, finite-type subsurfaces $\emptyset = S_0 \subset S_1 \subset \cdots$ such that for each $k \geq 1$, $S_k$ is $G$-invariant up to homotopy, and each component of $S_k \setminus S_{k-1}$ has negative Euler characteristic.
This proposition is crucial to our proof of Theorem 2.

**Proof.** First, fix an arbitrary exhaustion \( \{ K_i \} \) of \( S \) by connected, finite-type subsurfaces, e.g. coming from a pants decomposition of \( S \). Let \( X \) be a subset of \( \text{Diff}(S) \) containing a single representative from each mapping class in \( G \). We will proceed by induction on the length of a chain of subsurfaces \( S_0 \subset S_1 \subset \cdots \subset S_n \). The base case \( n = 0 \) is automatic.

Since \( S \) is of infinite-type, there exists some essential, simple closed curve \( \alpha \) on \( S \setminus S_n \). Recall that a simple closed curve is *essential* if it does not bound a disk nor is homotopic to a boundary curve or a puncture. Moreover, there exists some term \( K_{i_n} \) of the exhaustion containing both the \( X \)-orbit of \( \alpha \) and \( S_n \), since \( X \) is finite. To the end of creating a \( G \)-invariant exhaustion, consider

\[
K = \bigcap_{f \in X} f(K_{i_n}).
\]

Note that \( K \) contains the \( X \)-orbit of \( \alpha \) as well as \( S_n \). By performing a preliminary isotopy, we may assume the \( f(K_{i_n}) \) pairwise intersect transversely, and thus \( K \) is itself a subsurface. Since \( f(K) = K \) for each \( f \in X \) by construction, we may take \( S_{n+1} \) to be \( K \), together with the finitely many components of \( S \setminus K \) that contain no essential, simple closed curve. If any component of \( S_{n+1} \setminus S_n \) is an annulus, by induction we may assume that the component of \( S \setminus S_n \) it determines contains an essential simple closed curve. Thus there is no loss in replacing \( S_{n+1} \) by an isotopic subsurface so that no component of \( S_{n+1} \setminus S_n \) is an annulus.

Choosing the essential simple loop \( \alpha \) appropriately at each step, the inductive construction above gives an exhaustion of \( S \). \( \square \)

## 2. Nielsen Realization

We continue with the notation from the previous section. Choose a hyperbolic structure on \( S \) so that each \( S_i \) has finite volume and each component of \( \partial S_i \) is a geodesic with length 1. Write \( P_i = S_i \setminus S_{i-1} \). Since each \( S_i \) is \( G \)-invariant up to isotopy, so is each \( P_i \). Each \( g \in G \) gives a well-defined mapping class in \( \text{Map}(P_i) \) by choosing an arbitrary homotopy between \( P_i \) and \( x_g(P_i) \), where \( x_g \) is a diffeomorphism on \( S \) representing \( g \). Thus each \( g \in G \) defines a mapping class \( \rho(g) \) in \( \text{Map}(P_i) \). The inclusion \( G \to \text{Map}(P_i) \) is actually injective and follows from the proof below (see e.g. Proposition 8). Our proof does not depend on this fact.

**Proof of Theorem 2.** According to Kerckhoff, there is another hyperbolic structure on each \( P_i \) with respect to which the image of \( G \) in \( \text{Map}(P_i) \) may be realized as a group of isometries [Ker83, Theorem 5, discussion following Theorem 4]. Note that each boundary curve of each \( P_i \) remains geodesic with length 1.

Let us say a few words about the case where \( P_i \) is disconnected. Let \( C \) be a component of \( P_i \), and write \( H \) for the stabilizer of \( C \) in \( G \). By the above, there is a hyperbolic structure on \( C \) with an action of \( H \) by isometries. Choose a set of coset representatives for \( G/H \). If \( C' \) is a component of the orbit of \( C \) distinct from \( C \) choose a representative in \( X \) taking \( C' \) to \( C \), and give \( C' \) a new hyperbolic structure by pulling back the metric from \( C \). Proceeding orbit by orbit, this yields a realization of \( G \) on \( P_i \).

We now have a hyperbolic structure for each \( P_i \) with an action of \( G \) by isometries. Gluing up the \( P_i \) via the identifications coming from \( S \) yields a hyperbolic structure on \( S \), but we need to do so respecting the action of \( G \). As above, we glue the boundary curves shared by
$P_1$ and $S_{i-1}$ orbit by orbit. It suffices to show that for each boundary curve $c$ in $P_1 \cap S_{i-1}$ there is a gluing that identifies the two circle actions on $c$ by its stabilizer $\text{Stab}(c)$. Note that each $g \in \text{Stab}(c)$ acts on $c$ by rigid rotations, and with the (opposite) orientations induced from $P_1$ and $S_{i-1}$, it suffices to show that the angles of rotations for the two actions differ by a negative sign. Indeed, on the $P_1$ (resp. $S_{i-1}$) side, the angle is the rotation number of the $g$ action on the circle of geodesic rays on $P_1$ (resp. $S_1$) starting from and perpendicular to $c$. This action is similar to the one on the conical circle (see e.g. [BW18b]), and only depends on the mapping class $g$. Thus the two angles differ by a negative sign due to the opposite orientations.

As a result, we have an isometric action of $G$ on each $P_1$ that respects the gluing, yielding a hyperbolic structure and an isometric action of $G$ on $S$.

3. Classification of torsion elements

If $S$ is a surface of finite genus, then by the classification theorem [Ric63] it is realized as $\Sigma - E$, where $\Sigma$ is the closed surface with the same genus as $S$ and $E$ is a totally disconnected closed subset of $\Sigma$ homeomorphic to the space of ends. In this case, Theorem 2 implies that any finite subgroup $G$ of $\text{Map}(S)$ is realized by some $G$-action on $\Sigma$ by homeomorphisms preserving $E$. This is because $\text{Homeo}^+(\Sigma - E) \cong \text{Homeo}^+(\Sigma, E)$, where the latter denotes orientation-preserving homeomorphisms of $\Sigma$ preserving $E$.

In particular, one can use Theorem 2 to classify torsion elements. Here we focus on an example, the case of $S = \mathbb{R}^2 - K$, where $K$ is a Cantor set. In this situation, the mapping class group acts faithfully on the conical circle $S^1_C$, consisting of geodesics (for a fixed complete hyperbolic metric on $S$) emanating from $\infty$; see e.g. [BW18a, CC20]. Thus each mapping class $g$ has a rotation number, which can be read off from its action on a special subset of $S^1_C$, namely the short rays, which are proper simple geodesics connecting $\infty$ to some point in the Cantor set.

**Theorem 6.** Let $S = \mathbb{R}^2 - K$, where $K$ is a Cantor set. For each $n \geq 2$, elements in $\text{Map}(S)$ of order $n$ fall into $2\varphi(n)$ conjugacy classes, which are distinguished by the rotation number and whether the element fixes exactly one point in $K$ or none. Here $\varphi(n)$ is the number of positive integers up to $n$ that are coprime to $n$.

**Proof.** Let $g \in \text{Map}(S)$ be an element of order $n$. Then the action of $g$ on the conical circle $S^1_C$ has rotation number $m/n \mod \mathbb{Z}$ for some $m$ coprime to $n$. By Theorem 2, we can realize $g$ as some $\tilde{g} \in \text{Homeo}^+(S^2, K \cup \{\infty\})$ of order $n$. It is known that any finite order homeomorphism on $S^2$ is conjugate to a rigid rotation [Zim12] and the quotient $S^2/\tilde{g}$ is still homeomorphic to $S^2$. Considering the rotation number, we conclude that $\tilde{g}$ is conjugate to a rigid rotation by $2m\pi/n$, and there is exactly one fixed point $p \in S^2$ other than $\infty$.

We can put a Cantor set on $S^2$ invariant under a rigid rotation on $S^2$ by an angle of $2m\pi/n$ fixing $\infty$ and $p$ for any $1 \leq m \leq n$ coprime to $n$. We may or may not include the fixed point $p$ in the Cantor set. Apparently this gives $2\varphi(n)$ different conjugacy classes in $\text{Map}(S)$ by looking at the rotation number and whether the fixed point $p$ lies in the Cantor set.

Conversely, suppose we have two homeomorphisms $\tilde{g}_i$ on $S^2$ as above fixing $p_i \neq \infty$ such that either both $p_i \in K$ or $p_i \notin K$, $i = 1, 2$. Suppose further that they have the same rotation number $m/n \mod \mathbb{Z}$. Let $q_i : S^2 \to S^2/\tilde{g}_i$ be the quotient map. Then $q_i(K)$ is still a Cantor set. There is a homeomorphism $h : S^2/\tilde{g}_1 \to S^2/\tilde{g}_2$ taking $q_1(K)$ to $q_2(K)$. 

Moreover, in the case \( p_i \in K_1 \), we can choose \( h \) so that \( h(q_1(p_1)) = q_2(p_2) \). Then the map 
\[
h \circ q_1 : S^2 \setminus \{\infty, p_1\} \to (S^2/\tilde{g}_2) \setminus \{q_2(\infty), q_2(p_2)\}
\]
lifts to \( S^2 \setminus \{\infty, p_2\} \), which extends uniquely to a map \( \tilde{h} : S^2 \to S^2 \). The map \( \tilde{h} \) preserves the Cantor set \( K \), satisfies \( \tilde{h}(\infty) = \infty \), \( \tilde{h}(p_1) = p_2 \), and fits into the following commutative diagram.

\[
\begin{array}{ccc}
S^2 & \xrightarrow{h} & S^2 \\
\downarrow{q_1} & & \downarrow{q_2} \\
S^2/\tilde{g}_1 & \xrightarrow{\tilde{h}} & S^2/\tilde{g}_2
\end{array}
\]

For any \( x_0 \in S^2 \setminus \{\infty, p_1\} \), let \( x_j = \tilde{g}_1^{j} x_0 \) for \( 0 \leq j \leq n - 1 \). Fix a short ray \( r \) that passes through \( x_0 \) but not any \( x_j \) for \( j \neq 0 \) such that \( \{\tilde{g}_1^j r\}_{j=1}^{n-1} \) are disjoint (except at \( \infty \)). Such a ray can obtained for instance by lifting a short ray on \( S^2/\tilde{g}_1 \). Then there is a permutation \( \sigma \) on \( \{0, 1, \ldots, n-1\} \) such that \( \tilde{h}(\tilde{g}_1^j x) = \tilde{g}_2^{\sigma(j)} \tilde{h}(x) \) for all \( x \) on \( r \) and all \( 0 \leq j \leq n-1 \). Since \( \tilde{g}_1 \) and \( \tilde{g}_2 \) have the same rotation number and \( h \) maps \( \{\tilde{g}_1^j r\}_{j=1}^{n-1} \) to \( \{\tilde{g}_2^{\sigma(j)} \tilde{h}(r)\}_{j=1}^{n-1} \) preserving their circular order on the conical circle \( S_C^1 \), we must have \( \sigma = 1 \) and \( h\tilde{g}_1(x_0) = \tilde{g}_2 \tilde{h}(x_0) \). Since \( x_0 \) is arbitrary, we conclude that \( g_1 \) and \( g_2 \) are conjugate by the image of \( \tilde{h} \) in \( \text{Map}(S) \). \( \square \)

## 4. Using Torsion to Obstruct Embeddings

If \( S \) is an orientable, infinite-type surface, \( \text{Map}(S) \) has a natural nontrivial topology called the permutation topology which agrees with the quotient topology inherited from \( \text{Homeo}(S) \). Let \( \mathcal{C}(S) \) denote the set of isotopy classes of essential, simple closed curves in \( S \). The set \( \mathcal{C}(S) \) is countable, and it follows from work of Hernández Hernández–Morales–Valdez [HHMV19] that the action of \( \text{Map}(S) \) on \( \mathcal{C}(S) \) is faithful. A neighborhood basis of the identity in \( \text{Map}(S) \) is given by the sets

\[
\bigcap_{c \in C} \text{Stab}(c),
\]

where \( C \) ranges over the finite subsets of \( \mathcal{C}(S) \). The action of \( \text{Map}(S) \) on \( \mathcal{C}(S) \) exhibits \( \text{Map}(S) \) as a closed subgroup of \( \text{Sym}(\mathbb{N}) \), the group of bijections of a countable set, again given the permutation topology [Vla19, Corollary 6]. A natural question to ask is whether \( \text{Sym}(\mathbb{N}) \) embeds in any big mapping class group. The main result of this section is that there is no embedding.

**Theorem 7.** If \( G \) is a topological group containing a sequence of nontrivial torsion elements limiting to the identity, then \( G \) does not embed (as a topological group) in \( \text{Map}(S) \), for \( S \) any orientable surface.

Examples of such groups \( G \) include \( \text{Sym}(\mathbb{N}) \), \( \text{Out}(\pi_1(S)) \) when \( S \) is of infinite type, the automorphism group of a rooted tree, and \( \text{Homeo}(S^1) \). Theorem 7 follows from the following proposition, which exhibits an open neighborhood of the identity in \( \text{Map}(S) \) that contains no nontrivial torsion elements.

**Proposition 8.** Let \( \gamma_1, \gamma_2, \gamma_3 \) be essential, simple closed curves that cobound a pair of pants. Any finite order element of

\[
\text{Stab}[\gamma_1] \cap \text{Stab}[\gamma_2] \cap \text{Stab}[\gamma_3]
\]
in the action of $\text{Map}(S)$ on $\mathcal{C}(S)$ is the identity.

Proof. Suppose $f \in \text{Map}(S)$ has finite order, and that $f$ preserves the isotopy class of each $\gamma_i$ as in the statement. By Theorem 2, there is a hyperbolic metric on $S$ such that $f$ may be represented by an isometry $\varphi : S \to S$. The isometry $\varphi$ restricts to an isometry of the pair of pants $P$ bounded by the geodesic representatives of the curves $\gamma_i$. Since $f$ fixes the isotopy class of each $\gamma_i$, the isometry $\varphi$ must fix each geodesic representative setwise. But this implies $\varphi$ restricts to the identity on $P$, from which it follows that $\varphi : S \to S$ is the identity. □

References


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