Research Statement

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1 Overview

My research focuses on problems related to nonlinear **elliptic partial differential equations**. In particular I have worked in free boundary problems, regularity results for viscosity solutions to some fully nonlinear elliptic equations, elliptic *integro-differential* equations, and also some problems in optimal design of composites.

Integro-differential equations arise naturally in the study of stochastic processes with jumps and have many applications to finance and physics (See for example [17] or [9]). They are also a natural generalization of elliptic partial differential equations. For elliptic PDEs, there is a well established theory of weak solutions and regularity results even in the fully nonlinear case [15]. Many of these results seem to hold for integro-differential equations too. In many cases the generalization is not straight forward but instead presents an interesting mathematical challenge. The study of integro-differential equations is a relatively new area in mathematics full of open problems that is attracting an increasing level of interest recently. I have worked on the obstacle problem for the fractional Laplacian [29], [10] and also on more general questions about regularity of solutions to integro-differential equations [27], [12].

From my interaction with Bob Kohn at the Courant Institute I became interested in problems of optimal design of composites. In this type of problem one tries to find the best way to combine two or more materials in order to obtain certain convenient properties for the composite. In some cases we can find a lower or upper bound for some effective property of the composite (the thermal conductivity, a sum of two conductivities, etc.), but in order to achieve this bound we must find a solution to a free boundary problem.

In the next sections I will describe what I have done and my future plans in each area of research. In section 2, I will talk about the obstacle problem for the fractional Laplacian and its relation to the *thin* obstacle problem (also called the Signorini problem). In section 3, I will describe my research related to general nonlinear integro-differential equations. Then, in section 4, I discuss my work in optimal design of composites.

2 The obstacle problem for the fractional Laplacian and the Signorini problem

In this section I describe my work related to the obstacle problem for the fractional laplacian and the thin obstacle problem (also called the Signorini problem). I describe these two problems together because they are strongly related. I have a few publications regarding these topics: [29], [11], [10] and [25]. I start with the obstacle problem for the fractional Laplacian, and at the end of the section I also describe my work with Manolis Milakis about the nonlinear Signorini problem.

In 2005 my PhD thesis examined the regularity of the solutions to the obstacle problem for the fractional Laplacian [29]. This problem has particular interest for financial mathematics because it arises in the pricing of American options in a market where the asset prices are driven by a discontinuous Levy process. Using very different methods from the ones in my thesis, we improved the result later in a joint work with Luis Caffarelli and Sandro Salsa [10] where we obtained an optimal regularity estimate for the solution and also a regularity result for the free boundary.

The fractional Laplacian is an operator which can be written in several forms. Given $s \in (0,1)$, it can be viewed as the pseudo-differential operator with symbol $|\xi|^{2s}$ and also as the *integro-differential* operator

$$(-\triangle)^s u(x) = C \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, \mathrm{d}y$$

for a constant C depending only on n and s. It is typically the first example of an integro-differential operator. Its role for integro-differential equations of order $2s$ can be compared to the role of the usual Laplacian with respect to elliptic PDEs.

Given a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$, the obstacle problem for the fractional Laplacian in $\mathbb{R}^n$ consists in finding the function u satisfying

$$\begin{aligned} u &\geq \varphi && \text{in } \mathbb{R}^n, \\ (-\triangle)^s u &\geq 0 && \text{in } \mathbb{R}^n, \\ (-\triangle)^s u(x) &= 0 && \text{for those } x \text{ such that } u(x) > \varphi(x), \\ \lim_{|x| \to +\infty} u(x) &= 0. \end{aligned} \tag{2.1}$$

This is a free boundary problem because the contact set $\{u = \varphi\}$ is not known a priori. The free boundary in this case is precisely $\partial\{u = \varphi\}$.

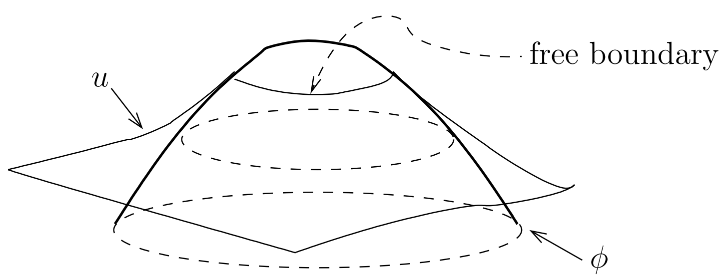

Figure 1: Obstacle problem

The obstacle problem arises from the following stochastic control model. We have a stochastic process X_t in $\mathbb{R}^n$ and some given function $\varphi : \mathbb{R}^n \to \mathbb{R}$. We want to find the optimal stopping strategy to maximize the expected value of φ at the end point. For any initial point $x \in \mathbb{R}^n$ we consider the function $u(x)$ defined as

$$u(x) = \sup_{\text{all stopping times } \tau} \mathbb{E}(\varphi(x + X_\tau)).$$

In the context of financial mathematics, this equation appears as a model for pricing perpetual American basket options. The function φ represents the option's payoff. The asset prices evolve by the stochastic process X_t, and the stopping time refers to the optimal time for exercising the option.

If in the problem above we consider the stochastic processes X_t to be plain Brownian motion, then the function u will satisfy the classic obstacle problem in the whole space $\mathbb{R}^n$. When we consider other Levy processes instead, the Laplacian is replaced by an integro-differential equation.

In the classical case ($s = 1$) the regularity of both the solution and the free boundary are well understood (see [14]). For the square root of the laplacian $(-\triangle)^{1/2}$, the problem turns out to be equivalent to the Signorini problem (or thin obstacle problem). In 1979, Caffarelli showed that the solution of the Signorini problem is $C^{1,\alpha}$ for a small enough α [13]. More recently, the optimal regularity was shown to be $C^{1,1/2}$ [2] and also the regularity of free boundary was studied [3]. It is worth saying that studying the regularity of the thin obstacle problem is much harder than in the usual obstacle problem. So it was expected that the general case for the fractional Laplacian would present significant difficulties.

In my PhD thesis [29], I proved that for any $s \in (0, 1)$ the solution u is in the class $C^{1,\alpha}$ for every $\alpha < s$. There are examples that show that the regularity of u is at most $C^{1,s}$, so this result was very close to being optimal. This work was done entirely using methods from potential analysis.

Later we found a way to relate the fractional Laplacian, for every $s \in (0, 1)$, to a Dirichlet to Neumann operator for a degenerate elliptic PDE in the upper half space [11]. In this way we could rewrite the obstacle problem (2.1) as a thin obstacle problem for a degenerate elliptic equation. We obtained the following equivalent formulation

$$\begin{aligned}
\operatorname{div}(r^a \nabla u(x,r)) &= 0 && \text{in } \mathbb{R}^n \times [0,+\infty), \\
u(x,0) &\geq \varphi(x) && \text{in } \mathbb{R}^n, \\
\lim_{r\to 0^+} r^a u_r(x,r) &= 0 && \text{for those points } x \in \mathbb{R}^n \text{ where } u(x,0) > \varphi(x), \\
\lim_{r\to 0^+} r^a u_r(x,r) &\leq 0 && \text{for every } x \in \mathbb{R}^n.
\end{aligned}$$

The most clear advantage is that we now have a local PDE and we can use the usual methods for regularity theory of free boundary problems. We obtained optimal regularity results for the solution and also for the free boundary of the obstacle problem for any $s \in (0, 1)$ in [10] using tools like monotonicity formulas and characterization of blowup profiles. The intuition behind the results in [11] and [10] is interesting. The equation that we consider comes from writing the Laplacian in cylindrical coordinates in a fractional number of dimensions. Even though $\mathbb{R}^a$ is not well defined if a is not an integer number, this intuition leads us to a well defined (degenerate elliptic) equation that shares many properties with the Laplace equation.

The construction of the fractional Laplacian as a Dirichlet to Neumann operator for a degenerate elliptic equation in the upper half space [11] is interesting in itself. It allows us to study the fractional Laplace equation as if it was a local equation (at the expense of adding one dimension, and degenerate ellipticity). It is natural to wonder what other nonlocal operators we can obtain by considering different PDEs in the upper half space extension problem. This seems to be a very challenging question. We have thought about this problem, but so far I must admit we have made no progress. Our proof for the fractional Laplacian relies on the symmetry properties of the operator.

Our results so far have focused in the fractional Laplacian because it is the simplest operator of fractional order. For the applications, and specifically for those in finance, the most interesting integro-differential operators have kernels that decay exponentially instead of polynomially. Intuitively, the regularity of the solutions u should depend only on the behavior of the kernel close to the origin. This is captured by a result in my PhD thesis (which was not included in the published version [29]) that says essentially that the same regularity results hold for any integro-differential equation with a kernel K such that $K(y) - c|y|^{-n-2s}$ is in $L^1(\mathbb{R}^n)$. This allows us to consider kernels with exponential decay or even compact support. It seems, however, that the condition is too strong because it does not allow kernels like $K(y) = |y|^{-n-2s_1} + |y|^{-n-2s_1}$ (the sum of two fractional Laplacians). Moreover, in applications the integro-differential terms are often mixed with diffusion and drift terms. There is still much work to do in explaining how all the different terms in the equation interact.

The parabolic version of this problem is also very important. In finance for example, it corresponds to pricing an American option with maturity $T < +\infty$ rather than a perpetual option. So far all our results are for the stationary equation, but we are planning to study the parabolic case in the near future.

As it is explained above, the obstacle problem for the fractional Laplacian is closely related to the thin obstacle problem or Signorini problem. Working in this direction, jointly with Manolis Milakis, I recently obtained a regularity result for the solutions to the Signorini problem for a fully nonlinear elliptic equation [25]. The Signorini problem models the shape of an elastic membrane lying on top of a very thin obstacle. It is also used to model the saline concentration on one side of a semipermeable membrane, and in optimal control of temperature across a surface, all these applications are explained in [19]. For a general fully nonlinear equation it can be stated as

$$\begin{aligned} F(D^2u(X)) &= 0 \qquad && X \in B_1^+ \text{ (the half ball in } n \text{ dimensions)} \\ \max(u_y(x,0), \varphi(x) - u(x,0)) &= 0 \qquad && x \in B_1^* \text{ (the ball in } n-1 \text{ dimensions)} \end{aligned}$$

where F is a convex uniformly elliptic equation with ellipticity constants $0 < \lambda < \Lambda$. The technical condition of F being convex seems to be essential for regularity results and it is somehow related to the fact that the boundary condition on B_1^* is convex. The optimal regularity in this case is an open problem; I believe the optimal regularity class depends on the ellipticity constants λ and Λ.

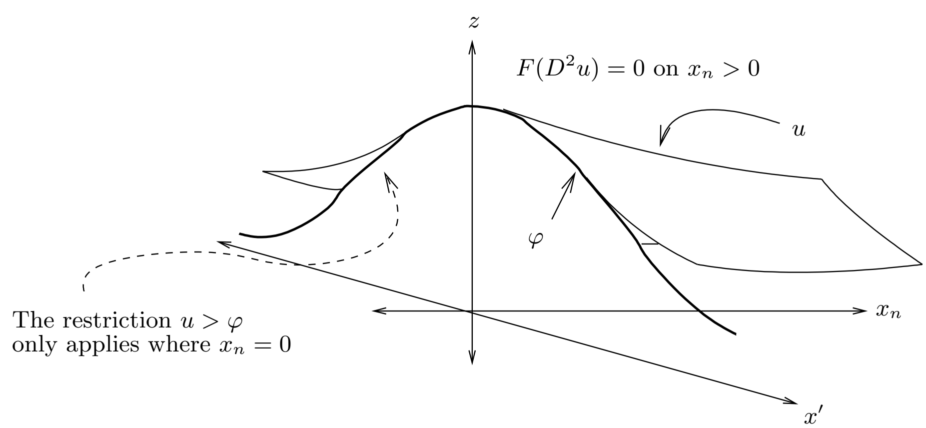

Figure 2: thin obstacle problem

In the paper [25] we used several ideas taken from the classical paper by Luis Caffarelli in the Signorini problem [13]. The main difficulty was that we needed regularity results up to the boundary for a fully nonlinear equation with Neumann boundary data, so we developed the results we needed in a separate previous paper [24]. In dealing with a fully nonlinear equation we had to understand the boundary condition in the viscosity sense; this led to additional technical difficulties.

3 Nonlinear integro-differential equations.

In this section I describe my latest work with Luis Caffarelli about the regularity of integro-differential equations. Our idea is to generalize most of the results presented in [15] for elliptic partial differential equations to nonlocal equations. This is a relatively new area of research that is attracting increasing attention lately. We hope (and expect) we will see many new developments in this direction in the next few years.

The fractional Laplacian is a linear integro-differential operator. An example of a nonlinear integro-differential operator could be the maximum of two linear ones, or an *inf-sup* of a family of linear operators. These are the operators that naturally appear in stochastic control problems. Luis Caffarelli and I are currently working on extending the classical regularity results for nonlinear elliptic PDEs to nonlinear elliptic integro-differential equations. We are able to obtain a result analogous to the Alexandroff-Bakelman-Pucci estimates, the Harnack inequality and $C^{1,\alpha}$ estimates for a fairly general class of nonlinear equations [12].

The generator of an n-dimensional Lèvy process is given by a linear operator with the general form

$$Lu(x) = \sum_{ij} a_{ij}(x)\partial_{ij}u + \sum_i b_i(x)\partial_i u + \int_{\mathbb{R}^n} (u(x+y) - u(x) - \nabla u(x) \cdot y) \; \chi_{B_1}(y) \; \mathrm{d}\mu_x(y). \quad (3.1)$$

The first term corresponds to the diffusion, the second to the drift, and the third to the jump part.

Nonlinear integro-differential equations appear in stochastic control problems [30]. If in a stochastic game a player is allowed to choose from different strategies at every step in order to maximize the expected value of some function at the first exit point of a domain, a convex nonlinear equation emerges with an operator of the form

$$Iu(x) = \sup_\alpha L_\alpha u(x) \quad (3.2)$$

In a competitive game with two or more players, more complicated equations appear. We can obtain equations of the type

$$Iu(x) = \inf_\beta \sup_\alpha L_{\alpha\beta} u(x) \quad (3.3)$$

The suitable notion of weak solution for this type of equation is a viscosity solution [18]. There are some recent papers that deal with the issues of uniqueness and existence of solutions [5], [4]. The equation with the maximum of two linear operators is studied in [1] with a slightly different approach.

My specialty is the regularity of the solutions to integro-differential equations. A general rule in PDEs is that the highest order term in the equation tends to control the regularity (of course this is not *always* true). Instead of equations with a nondegenerate second order part, I prefer to focus on purely *jump* equations (equations without diffusion or drift term) so that the regularization comes from the integro-differential part and not from the second order part of the equation.

A few years ago I obtained a Hölder regularity result for solutions to a linear integro-differential equation with coefficients that could be very discontinuous with respect to x and y. The result could be seen as a fractional-order analogue to the Krylov-Safonov Harnack inequality [22]. There were some previous probabilistic results in this direction [7], [6], [31], but my result had slightly different assumptions and a purely analytic proof. From my point of view, all these estimates have the serious weakness that they blow up as the order of the equation goes to 2. In other words, the Krylov-Safonov Harnack inequality does not follow from them. This is somewhat unnatural because we know that the Harnack inequality holds for elliptic equations of second order and we need a uniform estimate so that the theory of elliptic partial differential and integro-differential equations looks *unified*. My recent work with Luis Caffarelli [12] provides the first proof of such a uniform estimate.

Let me say more about [12], in which Caffarelli and I study nonlinear integro differential equations. It is interesting to observe that the operators like (3.2) or (3.3) have the following property:

$$\inf_\alpha L_\alpha v(x) \leq I(u+v)(x) - Iu(x) \leq \sup_\alpha L_\alpha v(x) \ .$$

This is the key hypothesis for our regularity theory. Indeed, we define the extremal operators

$$M^- u(x) = \inf_\alpha L_\alpha v(x)$$
$$M^+ u(x) = \sup_\alpha L_\alpha v(x)$$

for any family of linear operators L_α. Then the type of nonlinear integral operators for which our results apply are the ones such that

$$M^- v(x) \leq I(u+v)(x) - Iu(x) \leq M^+ u(x) \ . \tag{3.4}$$

This is a very rich family of nonlinear equations, potentially richer than the ones from stochastic control like (3.3). The ellipticity of the equations appears in the choice of the family $\{L_\alpha\}_\alpha$. For example note that if M^+ and M^- were the classical Pucci extremal operators, (3.4) forces I to be a second order elliptic (local) partial differential operator. For elliptic equations of order σ, a natural choice of the family $\{L_\alpha\}$ is to take the purely integro-differential operators with symmetric kernels comparable to those of the fractional Laplacian:

$$\left\{ L : Lu = PV \int_{\mathbb{R}^n} (u(x+y) - u(x)) \frac{K(y)}{|y|^{n+\sigma}} \, \mathrm{d}y \quad \text{with } \lambda \leq K(y) \leq \Lambda \text{ and } K(y) = K(-y) \right\}$$

In this case, the extremal operators have a simple formula

$$M^- u(x) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\lambda(u(x+y) + u(x-y) - 2u(x))^+ + \Lambda(u(x+y) + u(x-y) - 2u(x))^-}{|y|^{n+\sigma}} \, \mathrm{d}y$$
$$M^+ u(x) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\Lambda(u(x+y) + u(x-y) - 2u(x))^+ + \lambda(u(x+y) + u(x-y) - 2u(x))^-}{|y|^{n+\sigma}} \, \mathrm{d}y$$

where a^+ and a^- stand for the positive and negative part of a ($a = a^+ - a^-$).

The main results in [12] are:

An nonlocal ABP estimate. The precise statement for integro-differential equations is very technical. It provides an estimate of the maximum of a function u such that $u \leq 0$ outside of the ball B_1 and $M^+ u \geq f$ in B_1. The classical ABP estimate is in terms of the integral of $|f|^n$ in the contact set between u and its convex envelope. We recover that estimates if we make the order of the equation $\sigma \to 2$. However, for given $\sigma < 2$, our estimate is in terms of a Riemann sum of that integral.

The Harnack inequality. We prove that if u is a nonnegative function such that $M^+ u \geq -C_0$ and $M^- u \leq C_0$ in B_1, then

$$\sup_{B_{1/2}} u \leq C(\inf_{B_{1/2}} u + C_0)$$

where the constant C depends on a lower bound for the order σ, but remains uniformly bounded as $\sigma \to 2$.

The $C^{1,\alpha}$ regularity for nonlinear integro-differential equations. If a bounded function u is a solution to $Iu = 0$ in the ball B_1, for some operator I satisfying (3.4), we prove $u \in C^{1,\alpha}(B_{1/2})$ and get an estimate

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C(\|u\|_{L^\infty} + I_0) \ .$$

where $I_0 = I(0)$ is the value of I applied to the constant function 0.

Again, it is important to point out that the constant C does not blow up as the degree of the equation goes to 2. Therefore, the $C^{1,\alpha}$ estimates for second order fully nonlinear elliptic equations can be recovered from this result.

The are many remaining problems in the regularity theory of integro-differential equations. Our results in [12] are only for translation invariant equations; we plan to extend the results for the variable coefficient case. Our plan is to apply the techniques from [16] to the integro-differential setting.

One of the most important problems for nonlinear integro-differential equations is to find an analog of Evan-Krylov theorem [20], [21]. Namely, that the solutions to the Hamilton-Jacobi-Belmann equation are classical. We already have some ideas in this direction; they seem to solve the problem under additional hypothesis on the family of kernels. This is another direction for further work.

4 Optimal composite materials.

In this section I describe my research related to optimal design of composites. I got into this topic through my discussions with Bob Kohn at the Courant Institute. I will start with a general description of the area and I will briefly explain how free boundary problems arise naturally. My main contribution is [28], where I studied the problem of the optimal design of a composite in order to maximize the sum of two different effective conductivities. I will explain my result and some further ideas.

It is an important issue in engineering to design composite materials with advantageus effective properties. Consider for example composites made of two or more specific materials with certain thermal conductivities. If the mixing of the two materials is done at the microscopic scale, we may want to know the conductivity properties of the composite at the macroscopic scale. Moreover, we may want to find the best possible way to arrange the two materials microscopically in order to maximize or minimize the macroscopic conductivity. The same problem can be stated for the other properties of the materials, like electric resistivity or linear elasticity.

In mathematics, this kind of problems are studied within the framework of homogenization. A standard reference is the book of Milton [26]. For simplicity, let us discuss the case in which we mix two materials and we arrange them periodically. The *macroscopic* conductivity will depend on the conductivities a_1 and a_2 of each material and the microscopic structure. Since we assume that the microstructure is periodic, we can define the periodic function $a(x)$ as the conductivity at each point. Without loss of generality we can assume its period is the unit cube Q. Let us say that $Q = A_1 \cup A_2$, where $a(x) = a_1$ in A_1 and $a(x) = a_2$ in A_2. The equation for the evolution of temperature in the mix would be

$$u_t = \operatorname{div}(a(x/\varepsilon)\nabla u).$$

A smaller value of ε corresponds to a finer structure. Homogenization theory says that as $\varepsilon \to 0$, the solution u converges to the solution of a constant coefficient equation

$$u_t = \operatorname{div}(A_{\text{eff}}\nabla u).$$

The effective conductivity A_{eff} is in general a matrix. Only if the structure has extra symmetries we can guarantee that $A_{\text{eff}} = a_{\text{eff}} I$ for some scalar a_{eff}.

A small value of ε corresponds to a very fine composite. If ε is very small, the composite would behave as if it were a homogeneous material with thermal conductivity A_{eff} in the macroscopic scale. That is why it is important to be able to determine the value of A_{eff} in practice. Moreover,

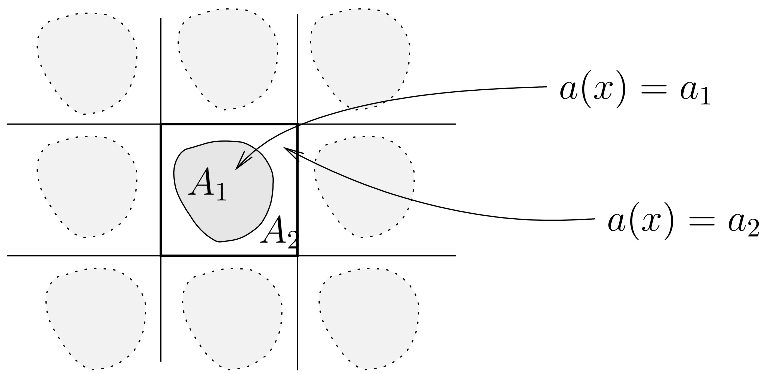

Figure 3: A microscopic periodic structure

in engineering it is important to know the optimal way to design the composites in order to achieve certain properties of A_{eff}. For example, the aim could be to make A_{eff} a scalar as large as possible, or as small as possible.

In general, the matrix A_{eff} can be computed by the following minimization

$$\langle A_{\text{eff}}\, e, e\rangle = \min_{u \in H^1_{\text{per}}(Q)} \int_Q a(x)|e + \nabla u|^2 \, dx. \tag{4.1}$$

The corresponding function u for which the minimum is achieved is the solution of the *cell* problem, and it will be the periodic solution of the elliptic PDE:

$$\operatorname{div}(a(x)(e + \nabla u)) = 0 \tag{4.2}$$

An important problem is to find sets A_1 and A_2 with a prescribed measure inside Q and symmetry properties such that the effective conductivity a_{eff} is maximized or minimized. In a recent result [23], it was shown that the sets A_1 and A_2 must be chosen to be the contact sets of certain periodic obstacle problem, showing an interesting link between composite design and free boundary problems.

The problem I studied in [28] involves maximizing the sum of two different conductivities. Let us say that one material in the mix is a good thermal conductor but a bad electric conductor, and the other material has the opposite properties. We want to find the optimal configuration of the sets A_1 and A_2 in order to maximize the sum of the effective thermal and electric conductivity of the mix. In this problem the two conductivities are competing and it is stated in a way that the problem is symmetric with respect to exchanging the phases. A nice geometric structure is not surprising under this setting.

An upper bound for the sum of two conductivities can be obtained from Bergman's cross property bounds [8], however it is not clear if the bound is achievable. In [32], Torquato, Hyun and Donev studied the problem using numerical computations and found evidence that in three dimensional space there may be a cubically symmetric triply periodic structure that achieves Bergman's bound. From the results of their computation, the sum of the two conductivities for a structure whose two phases are separated by a periodic minimal surface, like for example the Schwartz P surface, agrees with the upper bound within three digits of precision.

I got involved in this problem by trying to prove that the Schwartz P surface was indeed the optimal structure. In [28], I found a new proof of the relevant case of Bergman's cross-property bound. The important feature of this proof, besides its simplicity, is that it provides explicit

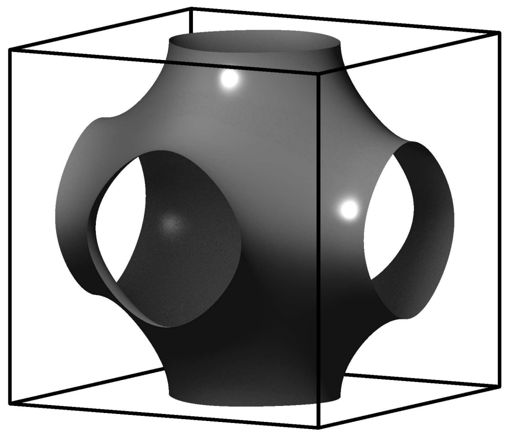

Figure 4: The Schwartz P surface

conditions for a pair of sets A_1, A_2 so that the bound is achieved. The condition comes in the form of a free boundary problem. In fact, A_1 and A_2 are an optimal choice if the periodic solution to the equation

$$\triangle p = \begin{cases} 1 & \text{in } A_1 \\ -1 & \text{in } A_2 \end{cases}$$

satisfies the following conditions at the boundary

$$D^2 p = \begin{cases} M(x_0) + \nu \otimes \nu & \text{in } A_1 \text{ close to } \partial A_1 \\ M(x_0) - \nu \otimes \nu & \text{in } A_2 \text{ close to } \partial A_1 \end{cases}$$

where ν is the unit normal to the boundary $\partial A_1 = \partial A_2$ and $M(x_0)$ is some matrix depending on each point x_0 on the boundary such that $M(x_0) \cdot \nu = 0$.

This extra condition on the boundary ∂A_1 makes the equation for p overdetermined, and so it is best understood as a free boundary problem. We conclude that if Bergman's bound is achievable then the optimal microstructure is given by the two phases of this free boundary problem.

Given a division of the unit cube in two sets A_1 and A_2, using the equation above we can check if it is optimal. It is not so simple to do this numerically because the free boundary condition is of second order, the solution q is not C^2 accross the free boundary, and thus it would require a very precise and careful computation. However, when the interface has mean curvature zero, it can be shown that the free boundary condition implies that both p and p_ν are constant on the free boundary. This is a very simple condition to check numerically. Using a standard finite element scheme, we checked that the Schwartz P surface does not satisfy the optimality conditions, and (contrary to the conjecture) it cannot achieve equality in Bergman's bound.

I believe the method used in [28] to reprove the upper bound for the sum of two conductivities is interesting in itself. The idea of the proof is to find an upper bound for tr A_{eff} by reducing the set of test functions u in (4.1) to be $e \cdot \nabla p$ for some potential function p (the same for every e). The minimum value obtained for this reduced set of test functions becomes *magically* independent of the choice of A_1 and A_2. Thus, the upper bound is attained when the solutions to the cell problem happen to be the directional derivatives of a potential function. The free boundary problem above is exactly the set of conditions that (a rescaled) p must satisfy in order for its directional derivatives to solve the cell problem.

The same idea of looking at gradients of potentials can be used to obtain a simple proof of the classical Hashin-Shtrikman bounds using only elementary vector calculus. I hope this idea will be applicable to other problems involving the design and analysis of composites with extremal properties. The conditions for a configuration to achieve the bounds always leads to a free boundary problem in terms of the potential.

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