Exercise 1. Let $V$ a vector space over a field $F$. Show that $(-1) \cdot v = -v$ for any $v \in V$.

Exercise 2. Prove that the following sets are vector spaces over the specified field.

- $\{(x, y, z) \in \mathbb{R}^3 \mid x - 2y + z = 0\}$ (over $\mathbb{R}$)
- $\{f : A \rightarrow \mathbb{Q} \mid f \text{ function} \}$ where $A$ is a nonempty set. (over $\mathbb{Q}$).
- $\mathbb{Q}[x]$ (polynomials with coefficients in $\mathbb{Q}$).
- $\{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ differentiable}, f'(0) = 0, f(0) = 0\}$ (over $\mathbb{R}$)
- $F$, a field (over a subfield $E \subseteq F$) (Think to $\mathbb{R}$ over $\mathbb{Q}$).

Exercise 3. Say if the following sets of vectors are linearly dependent, linearly independent, a spanning set, or a basis (over the indicated field).

- $V = \mathbb{R}^2$ (over $\mathbb{R}^2$); $v_1 = (1, 3), v_2 = (2, 6)$;
- $V = \mathbb{R}^2$ (over $\mathbb{R}^2$); $v_1 = (3, 5), v_2 = (-1, 6)$;
- $V = \mathbb{Q}^3$ (over $\mathbb{Q}$); $v_1 = (1, 0, 3); v_2 = (0, -1, 3) v_3 = (2, 3 - 3)$;
- $V = \mathbb{Q}^3$ (over $\mathbb{Q}$); $v_1 = (1, 0, 3); v_2 = (0, -1, 3) v_3 = (2, 3 - 3); v_4 = (0, 1, 0)$
- $V = \mathbb{Q}^3$ (over $\mathbb{Q}$); $v_1 = (1, 0, 3); v_2 = (0, -1, 3)$
- $V = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ differentiable}, f'(0) = 0, f(0) = 0\}$ (over $\mathbb{R}$) $v_1 = x^2; v_2 = e^x - 1 - x; v_3 = \cos x - 1$
- $V = \mathbb{R}^2$ (over $\mathbb{Q}$) $v_1 = (1, 2), v_2 = (\sqrt{2}, 2\sqrt{2})$;

Exercise 4. Let $V$ and $W$ two vector spaces over a field $F$. Let $f : V \rightarrow W$ a map between them such that $f(\lambda v_1 + \mu v_2) = \lambda f(v_1) + \mu f(v_2)$ for all $v_1, v_2 \in V$ and for all $\lambda$ and $\mu$ in $F$ (such a map between two vector spaces is called linear (or $F$-linear)). Show that for such a map $f(0) = 0$ and $f(-v) = -f(v)$. Show that ker $f := \{v \in V \mid f(v) = 0\}$ and im $f := \{w \in W \mid w = f(v) \exists v \in V\}$ are vector spaces over $F$. Prove moreover that $f$ is injective if and only if ker $f = \{0\}$ and $f$ is surjective if and only if im $f = W$. 
