Lemma 1. *Let Ω a non empty opens set of R^n. Then if $R \subseteq \Omega$, then there exists a rectangle R^ε such that $\Omega \supset R^\varepsilon \supset (R^\varepsilon)^0 \supset R$.*

Lemma 2. *Let $E \subseteq \mathbb{R}^n$ a bounded subset. Then:*

$$\overline{\mathrm{Vol}}(E) \geq (U)\int_{\mathbb{R}^n} \chi_E \qquad ; \qquad \underline{\mathrm{Vol}}(E) \leq (L)\int_{\mathbb{R}^n} \chi_E \ .$$

Proof. Take a rectangle $R \supseteq E$ containing E. By definition of $\overline{\mathrm{Vol}}(E)$ and $(U)\int_{\mathbb{R}^n} \chi_E$ For any $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$ there exist grids $\mathcal{G}_1$ and a $\mathcal{G}_2$ such that:

$$\overline{\mathrm{Vol}}(E) \leq V(\mathcal{G}_1, E) \lneq \overline{\mathrm{Vol}}(E) + \varepsilon$$
$$(U)\int_{\mathbb{R}^n} \chi_E \leq U(\chi_E, \mathcal{G}_2) \lneq (U)\int_{\mathbb{R}^n} \chi_E + \varepsilon \ .$$

Refining both $\mathcal{G}_1$ and $\mathcal{G}_2$ with a third grid $\mathcal{G}_3$ we get:

$$\overline{\mathrm{Vol}}(E) \leq V(\mathcal{G}_3, E) \lneq \overline{\mathrm{Vol}}(E) + \varepsilon$$
$$(U)\int_{\mathbb{R}^n} \chi_E \leq U(\chi_E, \mathcal{G}_3) \lneq (U)\int_{\mathbb{R}^n} \chi_E + \varepsilon \ .$$

Then

$$V(\mathcal{G}_3, E) - U(\chi_E, \mathcal{G}_3) - \varepsilon \leq \overline{\mathrm{Vol}}(E) - (U)\int_{\mathbb{R}^n} \chi_E$$

Now for any grid $\mathcal{G}$

$$V(\mathcal{G}, E) - U(\chi_E, \mathcal{G}) = \sum_{\substack{R_i \cap \bar{E} \neq \emptyset \\ R_i \cap E = \emptyset}} |R_i| \geq 0 \ .$$

Hence in the above inequality we have:

$$-\varepsilon \leq V(\mathcal{G}_3, E) - U(\chi_E, \mathcal{G}_3) - \varepsilon \leq \overline{\mathrm{Vol}}(E) - (U)\int_{\mathbb{R}^n} \chi_E \ .$$

Since this is true for any $\varepsilon > 0$, we get the statement of the lemma for outer volume - upper integral. The case of the inner volume – lower integral is completely analogous. □

Theorem 3. *Let $E \subseteq \mathbb{R}^n$ a bounded subset. Then:*

$$\overline{\mathrm{Vol}}(E) = (U)\int_{\mathbb{R}^n} \chi_E \qquad ; \qquad \underline{\mathrm{Vol}}(E) = (L)\int_{\mathbb{R}^n} \chi_E \ .$$

Proof. Step 1. Take a rectangle $R \supseteq E$ containing E. By definition of $\overline{\mathrm{Vol}}(E)$ and $(U)\int_{\mathbb{R}^n} \chi_E$ For any $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$ there exist grids $\mathcal{G}_1$ and a $\mathcal{G}_2$ such that:

$$\overline{\mathrm{Vol}}(E) \leq V(\mathcal{G}_1, E) \lneq \overline{\mathrm{Vol}}(E) + \varepsilon$$
$$(U)\int_{\mathbb{R}^n} \chi_E \leq U(\chi_E, \mathcal{G}_2) \lneq (U)\int_{\mathbb{R}^n} \chi_E + \varepsilon \ .$$

Refining both $\mathcal{G}_1$ and $\mathcal{G}_2$ with a third grid $\mathcal{G}_3$ we get:

$$\overline{\mathrm{Vol}}(E) \leq V(\mathcal{G}_3, E) \lneq \overline{\mathrm{Vol}}(E) + \varepsilon$$
$$(U)\int_{\mathbb{R}^n} \chi_E \leq U(\chi_E, \mathcal{G}_3) \lneq (U)\int_{\mathbb{R}^n} \chi_E + \varepsilon \ .$$

Then, by the preceding lemma, and by the preceding inequalities, we get:

$$0 \le \overline{\mathrm{Vol}}(E) - (U)\int_{\mathbb{R}^n} \chi_E \le V(\mathcal{G}_3, E) - U(\chi_E, \mathcal{G}_3) + \varepsilon \ .$$

Moreover for any refinement $\mathcal{H}$ of $\mathcal{G}_3$ we have:

$$\overline{\mathrm{Vol}}(E) \le V(\mathcal{H}, E) \le V(\mathcal{G}_3, E) \lneq \overline{\mathrm{Vol}}(E) + \varepsilon$$

$$(U)\int_{\mathbb{R}^n} \chi_E \le U(\chi_E, \mathcal{H}) \le U(\chi_E, \mathcal{G}_3) \lneq (U)\int_{\mathbb{R}^n} \chi_E + \varepsilon \ ,$$

and hence:

$$0 \le \overline{\mathrm{Vol}}(E) - (U)\int_{\mathbb{R}^n} \chi_E \le V(\mathcal{H}, E) - U(\chi_E, \mathcal{H}) + \varepsilon \ . \tag{0.1}$$

We just have to prove now that we can find a refinement $\mathcal{H}$ of of $\mathcal{G}_3$ such that $V(\mathcal{H}, E) - U(\chi_E, \mathcal{H}) \le \varepsilon$.

Step 2. For any grid $\mathcal{G}$ on R, we have

$$\begin{aligned} V(\mathcal{G}, E) - U(\chi_E, \mathcal{G}) &= \sum_{R_i \cap \bar{E} \ne \emptyset} |R_i| - \sum_{R_i \cap E \ne \emptyset} |R_i| \\ &= \sum_{\substack{R_i \cap \bar{E} \ne \emptyset \\ R_i \cap E = \emptyset}} |R_i| \end{aligned}$$

But now we have:

$$\mathcal{S}(\mathcal{G}, E) := \{R_i \in \mathcal{G} \ | R_i \cap \bar{E} \ne \emptyset \ , R_i \cap E = \emptyset\} = \{R_i \in \mathcal{G} \mid \emptyset \ne R_i \cap \bar{E} = R_i \cap \partial E \subseteq \partial R_i \cap \partial E \ , R_i \cap E = \emptyset\}$$

This means if R_i intersects $\bar{E}$, but it does not intersect E, then R_i intersects $\bar{E}$ only at the boundary of E (which is obvious, since R_i does not intersect E) and contemporarily at its own boundary ∂R_i. The inclusion "$\supseteq$" is obvious, while the inclusion "$\subseteq$" can be proved as follows: if $x \in R_i \cap \bar{E}$, and $R_i \cap E = \emptyset$, then $x \in \partial E$ (since $x \in \bar{E} \setminus E$). But if $x \in R_i^\circ$, then R_i° would be a neighbourhood of x, and since $x \in \partial E$, any neighbourhood of x intersects E, Then we would have $R_i \cap E \supseteq R_i^\circ \cap E \ne \emptyset$, which is a contradiction. Then $R_i \cap \bar{E} \subseteq R_i \cap \partial E \subseteq \partial R_i \cap \partial E$.

Step 3. Suppose now that $R = \prod_{i=1}^n [a_i, b_i]$, $a_i < b_i$ and the grid $\mathcal{G}_3$ is given by partitions: $\mathcal{P}_i(\mathcal{G}_3) = \{a_i, x_1^i, \dots, x_{k_i}^i, b_i\}$. Define the grid $\mathcal{H}_\delta$ by partitions

$$\mathcal{P}_i(\mathcal{H}_\delta) = \mathcal{P}_i(\mathcal{G}_3) \cup \{a_i + \delta, x_1^i - \delta, x_1^i + \delta, \dots, x_{k_i}^i - \delta, x_{k_i}^i + \delta, b_i - \delta\} \ .$$

Take δ so small that $x_j^i + \delta < x_{j+1}^i - \delta$ for any i and j, and $a_i + \delta < x_1^i - \delta$, $x_{k_i}^i + \delta < b_i - \delta$. The grid $\mathcal{H}_\delta$ refines $\mathcal{G}_3$. Set $\mathcal{G}_3 = \{R_i\}$. Set $\mathcal{H}_\delta = \{Q_i\}$. There are two kinds of Q_i;

$$\mathcal{H}_\delta = \mathcal{A}_\delta \coprod \mathcal{B}_\delta \ ,$$

where $\mathcal{A}_\delta = \{Q_i^\delta \in \mathcal{H}_\delta \mid Q_i^\delta \cap \partial R_j \ne \emptyset \ \exists R_j \in \mathcal{G}_3\}$; these are exactly the rectangles Q_i^δ that have at least one of their sides containing a point in $\mathcal{P}_i(\mathcal{G}_3)$. The set $\mathcal{B}_\delta$ is defined as $\{L_i^\delta \in \mathcal{H}_\delta \mid L_i \subseteq R_j^\circ \ \exists R_j \in \mathcal{G}_3\}$. These rectangles L_i are characterized by the property that all their sides do not contain any point in $\mathcal{P}_i(\mathcal{G}_3)$. Remark for any $R_i \in \mathcal{G}_3$, every side of R_i remains divided in 3 parts: hence the whole R_i remains divided in 3^n rectangles, of which only one in the interior, and hence in $\mathcal{B}_\delta$, while the remaining $3^n - 1$ touch the boundary ∂R_i, and hence are in $\mathcal{A}_\delta$. Then we have: $R_i = L_i^\delta \cup Q_{i_1}^\delta \cup \dots \cup Q_{i_{3^n-1}}^\delta$, where $L_i \in \mathcal{B}_\delta$, $Q_j^\delta \in \mathcal{A}_\delta$. Remark moreover that the function $|R_i| - |L_i^\delta|$ is continuous in δ and is 0 for $\delta = 0$.

This means that for any $\varepsilon' > 0$ there exists $\delta > 0$ such that $|R_i| - |L_i^\delta| \lneq \varepsilon'$. Moreover the function $|R_i| - |L_i^\delta|$ is strictly increasing in $[0, l]$, where l is lower than any of the sides of R_i.

Step 4. The idea is now clear: we want to use a partition $\mathcal{H}_\delta$ to get a good estimate in (0.1). We get

$$V(\mathcal{H}_\delta, E) - U(\chi_E, \mathcal{H}_\delta) = \sum_{T_i \in S(\mathcal{H}_\delta, E)} |T_i| \ .$$

Now we have:

$$S(\mathcal{H}_\delta, E) = (S(\mathcal{H}_\delta, E) \cap \mathcal{A}_\delta) \coprod (S(\mathcal{H}_\delta, E) \cap \mathcal{B}_\delta) \ .$$

We show now that if δ is sufficietly small $(S(\mathcal{H}_\delta, E) \cap \mathcal{B}_\delta) = \emptyset$. It is clear that if $L_i^\delta \in (S(\mathcal{H}_\delta, E) \cap \mathcal{B}_\delta)$ then $L_i^{\delta'} \notin (S(\mathcal{H}_\delta, E) \cap \mathcal{B}_\delta)$ for any $\delta > \delta' > 0$, because $(L_i^{\delta'})^\circ \supset L_i^\delta$, and then if L_i^δ intersects E, then $L_i^{\delta'}$ would intersect E ($(L_i^{\delta'})^\circ$) would be a neighbourhood of any point in L_i^δ. So decreasing δ we are sure that elements that could be in $(S(\mathcal{H}_\delta, E) \cap \mathcal{B}_\delta)$ would disappear.

The problem is that we could create new elements in $(S(\mathcal{H}_\delta, E) \cap \mathcal{B}_\delta)$ from a rectangle L_i^δ that was not in $(S(\mathcal{H}_\delta, E) \cap \mathcal{B}_\delta)$. But now if $L_i^\delta \notin (S(\mathcal{H}_\delta, E) \cap \mathcal{B}_\delta)$ this means in particular that $L_i^\delta \cap E \neq \emptyset$, or that $L_i^\delta \subseteq \bar{E}^c$, and now $\bar{E}^c$ is open. For rectangles L_i^δ such that $L_i^\delta \cap E \neq \emptyset$, there is no problems, since decreasing δ, they will continue to intersect E, and then they will remain out of $(S(\mathcal{H}_\delta, E) \cap \mathcal{B}_\delta)$. For the others L_i^δ, such that $L_i^\delta \subseteq \bar{E}^c$, by lemma 1, there exists $\eta_i < \delta$ such that for any $\eta_i < \delta' < \delta$ such that $L_i^{\delta'} \subseteq \bar{E}^c$.

Take now $\delta > \delta' > \max\{\eta_i\}$, for all i. Then we have that the two conditions are satisfied and $(S(\mathcal{H}_{\delta'}, E) \cap \mathcal{B}_{\delta'}) = \emptyset$.

Step 6. For this grid $\mathcal{H}_{\delta'}$, refining $\mathcal{G}_3$ we have:

$$\begin{aligned} V(\mathcal{H}_{\delta'}, E) - U(\chi_E, \mathcal{H}_{\delta'}) &= \sum_{T_i \in S(\mathcal{H}_{\delta'}, E)} |T_i| \\ &\leq \sum_{Q_i^\delta \in S(\mathcal{H}_{\delta'}, E) \cap \mathcal{A}_\delta} |Q_i^\delta| \\ &\leq \sum_i |R_i| - |L_i^{\delta'}| \leq \varepsilon' C \end{aligned}$$

where C is the number of rectangles R_i. If in step 3, ε' was chosen smaller than ε/C, we now have $V(\mathcal{H}_{\delta'}, E) - U(\chi_E, \mathcal{H}_{\delta'}) \leq \varepsilon$, as small as we want. This estimate gives:

$$0 \leq \overline{\mathrm{Vol}}(E) - (U) \int_{\mathbb{R}^n} \chi_E \leq V(\mathcal{H}_{\delta'}, E) - U(\chi_E, \mathcal{H}_{\delta'}) + \varepsilon \leq 2\varepsilon \ .$$

This proves the theorem for the equality outer volume - upper integral.

For the statement on inner volume – lower integral the proof is completely analogous: with the same notations as above:

1) we can take a grid $\mathcal{G}_3$ such that

$$0 \leq (L) \int_{\mathbb{R}^n} \chi_E - \underline{\mathrm{Vol}}(E) \leq L(\chi_E, \mathcal{G}_3) - v(\mathcal{G}_3, E) + \varepsilon \ .$$

2) We remark that for any grid $\mathcal{G}$ we have

$$L(\chi_E, \mathcal{G}) - v(\mathcal{G}, E) = \sum_{\substack{R_i \subseteq E \\ R_i \cap \partial E \neq \emptyset}} |R_i|$$

3) We remark that if $R_i \subseteq E$, $R_i \cap \partial E \neq \emptyset$, then $R_i \cap \partial E \subseteq \partial R_i \cap \partial E$. Setting $R(\mathcal{G}, E) = \{R_i \in \mathcal{G} \mid R_i \subseteq E\ ,\ R_i \cap \partial E \neq \emptyset\}$
4) Take $\mathcal{H}_\delta$, refinement of $\mathcal{G}_3$, for δ small, and decompose it in $\mathcal{A}_\delta$ and $\mathcal{B}_\delta$. Then $R(\mathcal{H}_\delta, E) = (R(\mathcal{H}_\delta, E) \cap \mathcal{A}_\delta) \coprod (R(\mathcal{H}_\delta, E) \cap \mathcal{B}_\delta)$.
5) We can then choose $\delta' < \delta$ such that $R(\mathcal{H}_{\delta'}, E) \cap \mathcal{B}_{\delta'} = \emptyset$.
6) Then

$$L(\chi_E, \mathcal{H}_{\delta'}) - v(\mathcal{H}_{\delta'}, E) = \sum_{Q_i \in R(\mathcal{H}_\delta, E) \cap \mathcal{A}_{\delta'}} |Q_i| \leq \sum_{Q_i \in \mathcal{A}_{\delta'}} |Q_i| \leq \sum_i |R_i| - |L_i^{\delta'}| \leq \varepsilon' C \leq \varepsilon$$

if ε' is chosen sufficiently small.
Then, since $\mathcal{H}_{\delta'}$ is a refinement of $\mathcal{G}_3$, we get:

$$0 \leq (L) \int_{\mathbb{R}^n} \chi_E - \underline{\mathrm{Vol}}(E) \leq L(\chi_E, \mathcal{H}_{\delta'}) - v(\mathcal{H}_{\delta'}, E) + \varepsilon \leq 2\varepsilon$$

□

Corollary 4. *Let $E \subseteq \mathbb{R}^n$ a bounded subset of $\mathbb{R}^n$. Then E is a Jordan region if and only if $\chi_E \in \mathrm{Ri}(\mathbb{R}^n)$. In this case*

$$\mathrm{Vol}(E) = \int_{\mathbb{R}^n} \chi_E = \int_E 1\ .$$