Exercise 1. Prove the following linear properties for integrals.

- Let \( f, g \in \text{Ri}(\mathbb{R}^n) \). Prove that if \( \alpha, \beta \in \mathbb{R} \) then \( \alpha f + \beta g \in \text{Ri}(\mathbb{R}^n) \) and that:
  \[
  \int_{\mathbb{R}^n} (\alpha f + \beta g) = \alpha \int_{\mathbb{R}^n} f + \beta \int_{\mathbb{R}^n} g .
  \]
  (Hint: Prove first that for any grid \( G \) and for any \( R_i \in G \), \( M_i(\alpha f) = \alpha M_i(f) \) and \( m_i(\alpha f) = \alpha m_i(f) \). Deduce that \( Lf(G) = \alpha Lf(G) \); \( Uf(G) = \alpha Uf(G) \). Prove moreover that \( U(f + g, G) \leq U(f, G) + U(g, G) \); \( L(f + g, G) \geq L(f, G) + L(g, G) \).)

- Deduce that if \( f, g : D \rightarrow \mathbb{R} \) are integrable on a Jordan region \( E \subseteq D \) then \( \alpha f + \beta g \), for \( \alpha, \beta \in \mathbb{R}^n \) is integrable on \( E \) and
  \[
  \int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g .
  \]

- Let \( f, g : \mathbb{R}^n \rightarrow \mathbb{R} \), two functions in \( \text{Ri}(\mathbb{R}^n) \), with \( f \leq g \). Prove that
  \[
  \int_{\mathbb{R}^n} f \leq \int_{\mathbb{R}^n} g .
  \]

- Prove that if \( f, g : D \rightarrow \mathbb{R} \) are bounded functions, \( E \) is a Jordan region \( E \subseteq D \), with \( f, g \) integrable on \( E \), and if \( f \big|_E \leq g \big|_E \) then
  \[
  \int_E f \leq \int_E g .
  \]

- Prove that if \( E_1 \) and \( E_2 \) are non overlapping Jordan regions, for any bounded function \( f : D \rightarrow \mathbb{R} \), \( D \supset E_1 \cup E_2 \), \( f \) integrable on \( E_1 \) and \( E_2 \) then \( f \) is integrable on \( E_1 \cup E_2 \) and that
  \[
  \int_{E_1 \cup E_2} f = \int_{E_1} f + \int_{E_2} f .
  \]
  (Hint: show first that \( f(E_1 \cup E_2) = f(E_1) + f(E_2) - f(E_1 \cap E_2) \))

- If \( E_1 \) and \( E_2 \) are any two Jordan regions, and \( f : D \rightarrow \mathbb{R}^n \) is nonnegative and bounded, \( D \supset E_1 \cup E_2 \), prove that
  \[
  \int_{E_1 \cap E_2} f \leq \min\{\int_{E_1} f, \int_{E_2} f\} \leq \max\{\int_{E_1} f, \int_{E_2} f\} \leq \int_{E_1 \cup E_2} f \leq \int_{E_1} f + \int_{E_2} f .
  \]

After reading Theorems 12.25 and 12.26 from Wade, Exercises, 5, 6, 10 from Wade, chapter 12.2.