Some remarks on tautological sheaves on Hilbert schemes of points on a surface

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Abstract

We study tautological sheaves on the Hilbert scheme of points on a smooth quasi-projective algebraic surface by means of the Bridgeland-King-Reid transform. We obtain Brion-Danila's Formulas for the derived direct image of tautological sheaves or their double tensor product for the Hilbert-Chow morphism; as an application we compute the cohomology of the Hilbert scheme with values in tautological sheaves or in their double tensor product, thus generalizing results obtained in [16] for tautological bundles.

1 Introduction

Let X be a smooth quasi-projective algebraic surface, $X^{[n]}$ be the Hilbert scheme of n points on X. If L is a line bundle on X, there is a natural rank n vector bundle $L^{[n]}$ on the Hilbert scheme associated to the line bundle L, called the tautological bundle. In [16] we studied tautological bundles and their tensor product on the Hilbert scheme with particular attention to their cohomology. The aim of this note is the generalization of some of the results of [16], extending them to *tautological sheaves*. The starting point of this work is the remark that the universal family $\Xi \subseteq X^{[n]} \times X$ of the Hilbert scheme is transverse to any coherent sheaf on X, and that we can consequently define tautological sheaves $F^{[n]}$ on the Hilbert scheme $X^{[n]}$ for an arbitray coherent sheaf F on X by means of the Fourier-Mukai functor $\Phi^{\mathcal{O}_\Xi}_{X \to X^{[n]}}$ of kernel $\mathcal{O}_\Xi$; the same remark implies moreover that the tautological functor is exact. We compute the image of the Bridgeland-King-Reid transform $\mathbf{\Phi} : \mathbf{D}^b(X^{[n]}) \longrightarrow \mathbf{D}^b_{\mathfrak{S}_n}(X^n)$ of a tautological sheaf and of a tensor product of them. For a tautological sheaf $F^{[n]}$ we obtain a complete answer in terms of an explicit $\mathfrak{S}_n$-equivariant complex $\mathcal{C}^\bullet_F \in \mathbf{D}^b_{\mathfrak{S}_n}(X^n)$, like in the case considered in [16]. For the tensor product of tautological bundles, to obtain a good characterization of the image along the lines of [16], we have to restrict ourselves to torsion free sheaves with pairwisely disjoint singular loci. In order to prove Brion-Danila's Formulas for the derived direct images of a tensor product of tautological sheaves for the Chow morphism, we interpret the invariants of the image of the Bridgeland-King-Reid equivalence in terms of an invariant Fourier-Mukai functor whose kernel is the sheaf of $\mathfrak{S}_n$-invariants of the structural sheaf $\mathcal{O}_{D(n,k)}$ of Haiman's polygraph $D(n,k)$. We get in this way general formulas for the the derived direct image of a tautological bundle or for a tensor product of two of them. As an application we can compute the cohomology of $X^{[n]}$ with value in any tautological sheaf or in a tensor product of tautological sheaves $E^{[n]} \otimes F^{[n]}$, if the sheaves E and F are torsion free with disjoint singular loci.

2 Tautological sheaves

Notation 2.1. Let X be a smooth quasi-projective algebraic surface. We will indicate with $X^{[n]}$ the Hilbert scheme of closed subschemes of length n on X, or, more briefly, the Hilbert scheme of n points on X, and with $\Xi \subseteq X^{[n]} \times X$ the universal subscheme; it is well know that the scheme Ξ is Cohen-Macauley, since it is flat and finite over the smooth variety $X^{[n]}$ (see [1, chapter 10, n. 7, §2,3]). Moreover we will denote with $S^nX := X^n/\mathfrak{S}_n$ the symmetric variety, with $\pi : X^n \longrightarrow S^nX$ the quotient morphism and with $\mu : X^{[n]} \longrightarrow S^nX$ the Hilbert-Chow morphism.

2.1 Tautological sheaves

Consider the projections: $X^{[n]} \xleftarrow{p_{X^{[n]}}} X^{[n]} \times X \xrightarrow{p_X} X$. Denote with $\Phi^{\mathcal{O}_\Xi}_{X \to X^{[n]}}$ the Fourier-Mukai functor: $\mathbf{R}p_{X^{[n]}*}(\mathcal{O}_\Xi \otimes^L_{\mathcal{O}_{X^{[n]} \times X}} p_X^*(-)) : \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(X^{[n]})$, from the bounded derived category of coherent sheaves on X to the bounded derived category of coherent sheaves on $X^{[n]}$, with kernel $\mathcal{O}_\Xi$. We will make use of the following preliminary results.

Lemma 2.2 ([16, Lemma A.2])**.** *Let A a be a regular noetherian local ring, $M_1, \dots, M_l$ finite Cohen-Macauley modules over A, such that* $\operatorname{codim}(M_1 \otimes \cdots \otimes M_l) = \sum_{i=1}^l \operatorname{codim}(M_i)$*. Then M_i, $i = 1, \dots, l$, are transverse, that is,* $\operatorname{Tor}_i^A(M_1, \dots, M_l) = 0$*, for all $i > 0$.*

Proposition 2.3. *Let F be a coherent sheaf on the smooth quasi-projective algebraic surface X. Then:*

$$\operatorname{Tor}_i^{\mathcal{O}_{X^{[n]} \times X}}(\mathcal{O}_\Xi, p_X^*F) = 0 \qquad \forall\, i > 0. \tag{2.1}$$

Proof. For any coherent sheaf F, it is easy to check that the sheaves $\mathcal{O}_\Xi$ and p_X^*F are transversely supported, that is, $\operatorname{codim}_{X^{[n]} \times X}(\mathcal{O}_\Xi \otimes_{\mathcal{O}_{X^{[n]} \times X}} p_X^*F) = \operatorname{codim}_{X^{[n]} \times X}(\mathcal{O}_\Xi) + \operatorname{codim}_{X^{[n]} \times X}(p_X^*F)$. If F is Cohen-Macaulay, then p_X^*F is Cohen Macauley, and then $\mathcal{O}_\Xi$ and p_X^*F are transverse by Lemma 2.2, since Ξ is Cohen-Macauley. Therefore (2.1) follows. Hence (2.1) is true for any 0-dimensional coherent sheaf, because any such sheaf is Cohen-Macaulay. If F is of dimension 1, it can be written as the extension[1]:

$$0 \longrightarrow F_0 \longrightarrow F \longrightarrow F_1 \longrightarrow 0$$

with F_0 of dimension 0 (hence Cohen-Macaulay) and F_1 of dimension 1 without immersed points (hence, again, Cohen-Macaulay); the long exact Tor-sequence gives the sought transversality. If F is of dimension 2, we can write the exact sequence:

$$0 \longrightarrow T \longrightarrow F \longrightarrow G \longrightarrow 0$$

[1] Such an extension can be obtained as follows. The statement is local; consequently, it suffices to prove such an extension when X is a smooth affine variety $X = \operatorname{Spec}(A)$. Let M be a finite generated module of dimension 1 over A. Let $\{0\} = \cap_{i \in I} N_i$ a primary decomposition for the zero module in M. Take I' the subset of I such that N_i is an embedded component if $i \in I'$ and take $J = I \setminus I'$. Remark that $J \neq \emptyset$. Define $M_1 := M / \cap_{j \in J} N_j$. It is easy now to check that M projects onto M_1 and the kernel M_0 of the projection $M \longrightarrow M_1$, when nonzero, is supported on closed points and hence of dimension zero. Finally, for any maximal ideal $\mathfrak{m}$ in A, the localized module $(M_1)_\mathfrak{m}$ is a module of dimension 1 and depth 1 over the noetherian regular local ring $A_\mathfrak{m}$, since of dimension 1 without embedded components; hence it is Cohen-Macauley.

where T is the torsion subsheaf (of dimension ≤ 1) and G is torsion-free. Therefore $\mathcal{O}_\Xi$ and T are transverse. To see that $\mathcal{O}_\Xi$ and G are transverse, write the short exact sequence:

$$0 \longrightarrow G \longrightarrow G^{**} \longrightarrow Q \longrightarrow 0 \, .$$

The the bidual G^{**} is reflexive: it is well known that a reflexive sheaf on a surface is locally free; moreover the sheaf Q is of dimension 0. The long exact Tor-sequence gives again the wanted transversality, first for G and then for F. □

Due to the previous proposition and to the fact that Ξ is finite over $X^{[n]}$, we can give the following definition.

Definition 2.4. Let X be a smooth quasi-projective algebraic surface and F be a coherent sheaf on X. The tautological sheaf $F^{[n]}$ on $X^{[n]}$, naturally associated to the coherent sheaf F on X, is the sheaf:

$$F^{[n]} := \Phi^{\mathcal{O}_\Xi}_{X \to X^{[n]}}(F) \, .$$

Remark 2.5. By virtue of Proposition 2.3, the Fourier-Mukai transform $\Phi^{\mathcal{O}_\Xi}_{X \to X^{[n]}}(F)$ of the coherent sheaf F appearing in the previous definition coincides with the sheaf $(p_{X^{[n]}})_*(\mathcal{O}_\Xi \otimes_{\mathcal{O}_{X^{[n]} \times X}} p_X^* F)$. If E is a vector bundle on X of rank r, since Ξ is flat and finite of degree n over $X^{[n]}$, then $E^{[n]}$ is the well known tautological bundle (of rank nr) over $X^{[n]}$, naturally associated to the vector bundle E on X.

As a corollary of Proposition 2.3, we have:

Corollary 2.6. *The functor* $[-]^{[n]} : \mathrm{Coh}(X) \longrightarrow \mathrm{Coh}(X^{[n]})$*, introduced in Definition 2.4, is exact.*

Remark 2.7. The functor $[-]^{[n]} := \Phi^{\mathcal{O}_\Xi}_{X \to X^{[n]}} : \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(X^{[n]})$, introduced in Definition 2.4, induces the functor in K-theory: $[-]^{[n]} : K(X) \longrightarrow K(X^{[n]})$, defined, for example, in [12].

2.2 Tensor product of tautological sheaves

Consider the cartesian diagram:

$$\begin{array}{ccccc} \Xi(n,k) & \overset{j}{\hookrightarrow} & \Xi \times \cdots \times \Xi & \xrightarrow{p_X \times \cdots \times p_X} & X \times \cdots \times X \\ \downarrow t & \square & \downarrow p_{X^{[n]}} \times \cdots \times p_{X^{[n]}} & & \\ X^{[n]} & \overset{i}{\hookrightarrow} & X^{[n]} \times \cdots \times X^{[n]} & & \end{array} \tag{2.2}$$

where $\Xi(n,k)$ is the k-fold fibered product: $\Xi(n,k) := \Xi \times_{X^{[n]}} \cdots \times_{X^{[n]}} \Xi$ and j and i are the diagonal immersion. Remark that the projections t and $p_{X^{[n]}} \times \cdots \times p_{X^{[n]}}$ are flat and finite of degree n^k over $X^{[n]}$ and $X^{[n]} \times \cdots \times X^{[n]}$, respectively. Consider $F_1, \ldots, F_k$ coherent sheaves on X and their exterior tensor product[2] on $F_1 \boxtimes \cdots \boxtimes F_k$ on X^k.

[2] We recall that if X and Y are two algebraic varieties and if F and G are coherent sheaves on X and Y, respectively, the exterior tensor product $F \boxtimes G$ is the coherent sheaf on $X \times Y$ defined as: $F \boxtimes G := p_X^* F \otimes_{\mathcal{O}_{X \times Y}} p_Y^* G$, where p_X and p_Y are the projections of $X \times Y$ onto X and Y, respectively.

We will make use of the following Base Change Formula; it is an immediate consequence[3] of [6, Proposition 6.9.8].

Proposition 2.8. *Consider the fiber product:*

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle f'}\downarrow & \square & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{v} & Y \end{array}$$

where X,Y,X', Y' are noetherian k-schemes and f and f' are proper morphism. Let $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, and $\mathcal{G}$ a coherent $\mathcal{O}_{Y'}$-module. Then:

$$\mathbf{R}f'_*(\mathcal{F} \otimes^L_{\mathcal{O}_Y} \mathcal{G}) \simeq \mathbf{R}f_*\mathcal{F} \otimes^L_{\mathcal{O}_Y} \mathcal{G} \tag{2.3}$$

where $\mathcal{F} \otimes^L_{\mathcal{O}_Y} \mathcal{G}$ is naturally seen as an element of $\mathbf{D}^-(X')$.

Remark 2.9. 1. Suppose Y' is flat over Y, then X' is flat over X. If $\mathcal{G} = \mathcal{O}_{Y'}$, we get the known formula for Flat Base Change [9]: $\mathbf{R}f'_*(u^*\mathcal{F}) \simeq v^*\mathbf{R}f_*\mathcal{F}$.

2. If X is flat over Y and $\mathcal{G} = \mathcal{O}_{Y'}$, we get: $\mathbf{R}f'_*(\mathbf{L}u^*\mathcal{F}) \simeq \mathbf{L}v^*(\mathbf{R}f_*\mathcal{F})$.

Applying the Base Change Formula (2.3) to Diagram (2.2), remembering Proposition 2.3 and the fact that if F and G are two coherent sheaves on two algebraic varieties X and Y, respectively, then trivially $F \boxtimes^L G \simeq F \boxtimes G$ in $\mathbf{D}^-(X \times Y)$, we have:

$$\begin{aligned} F_1^{[n]} \otimes^L \cdots \otimes^L F_k^{[n]} &\simeq \mathbf{L}i^*(p_{X^{[n]}} \times \cdots \times p_{X^{[n]}})_*(p_X \times \cdots \times p_X)^* F_1 \boxtimes \cdots \boxtimes F_k \\ &\simeq t_*\mathbf{L}j^*(p_X \times \cdots \times p_X)^* F_1 \boxtimes \cdots \boxtimes F_k \\ &\simeq \Phi^{\mathcal{O}_{\Xi(n,k)}}_{X^k \to X^{[n]}}(F_1 \boxtimes \cdots \boxtimes F_k)\ . \end{aligned}$$

2.3 The Bridgeland-King-Reid equivalence

In this section we will transform tautological sheaves and their tensor products with the Bridgeland-King-Reid equivalence (see [2]) for the action of the symmetric group $\mathfrak{S}_n$ on the product X^n. We have first of all to introduce the isospectral Hilbert scheme. For every subset $J \subseteq \{1, \dots, n\}$, $|J| = p$, $p \geq 1$, denote with $p_J : X^n \longrightarrow X^J$ the projection onto the factors in J. If $|J| \geq 2$, we will indicate with Δ_J the pull-back of the small diagonal in X^J via the projection p_J.

Definition 2.10. The *isospectral Hilbert scheme* B^n is the reduced fibered product $B^n := (X^{[n]} \times_{S^nX} X^n)_{\mathrm{red}}$ of X^n and $X^{[n]}$ over the symmetric variety S^nX.

The isospectral Hilbert scheme plays a fundamental role in applying the Bridgeland-King-Reid theorem to the the action of the symmetric group $\mathfrak{S}_n$ to X^n. It has been defined and studied by Haiman in [7] and [8]. Haiman proves the following theorem.

[3]To get Proposition 2.8, it is just sufficient to set $S = Y$, $S' = Y'$ in [6, Proposition 6.9.8] and to express the result in the language of derived categories instead of spectral sequences.

Theorem 2.11. *The isospectral Hilbert scheme B^n is irreducible, normal, Cohen-Macauley and Gorenstein. Moreover it can be identified with the blow-up of X^n along the scheme-theoretic union of the pairwise diagonals Δ_{ij}, for all $1 \leq i < j \leq n$.*

By definition the isospectral Hilbert scheme B^n fits in the (non cartesian) commutative diagram:

$$\begin{array}{ccc} B^n & \xrightarrow{\;p\;} & X^n \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi} \\ X^{[n]} & \xrightarrow{\;\mu\;} & S^n X \end{array} \tag{2.4}$$

where p and μ are birational, π and q are finite, and q is flat.

Remark 2.12. Since $X^{[n]}$ is smooth and q is finite and surjective, the Cohen-Macauley property of B^n is equivalent to the flatness of the morphism q (see [4, Ex. 18.17]). Moreover B^n inherits the $\mathfrak{S}_n$-action. As a consequence B^n is a flat family of $\mathfrak{S}_n$-clusters, that is, $\mathfrak{S}_n$-invariant closed subschemes B^n_ξ of X^n, $\xi \in X^{[n]}$, such that $H^0(B^n_\xi) \simeq \mathbb{C}[\mathfrak{S}_n]$; hence $X^{[n]} \simeq B^n/\mathfrak{S}_n$. Haiman actually proves that B^n is the universal family of $\mathfrak{S}_n$-clusters and that $X^{[n]}$ can be identified with the Hilbert scheme of regular $\mathfrak{S}_n$-orbits $\mathrm{Hilb}^{\mathfrak{S}_n}(X^n)$ on X^n, in the sense of Nakamura (see [11], [10], [14], [7]). Diagram (2.4) is then exactly the picture of the derived McKay correspondence of Bridgeland-King-Reid theorem, whose hypothesis are now easily satisfied, since S^nX is Gorenstein and μ is semismall.

The Bridgeland-King-Reid theorem, in this context, states:

Theorem 2.13. *The Fourier-Mukai functor:*

$$\Phi^{\mathcal{O}_{B^n}}_{X^{[n]}\to X^n} : \mathbf{D}^b(X^{[n]}) \longrightarrow \mathbf{D}^b_{\mathfrak{S}_n}(X^n) \tag{2.5}$$

from the bounded derived category of coherent sheaves on $X^{[n]}$ to the bounded derived category of $\mathfrak{S}_n$-equivariant sheaves on X^n, is an equivalence.

In the sequel we will denote more briefly with $\mathbf{\Phi}$ the Fourier-Mukai transform $\Phi^{\mathcal{O}_{B^n}}_{X^{[n]}\to X^n}$ of Theorem 2.13 and we will also call it the Bridgeland-King-Reid equivalence[4]. We will indicate with $D_i \subseteq X^n \times X$ the partial diagonal $\Delta_{i,n+1}$ in $X^n \times X$ and with D the scheme theoretic union $D := \cup_{i=1}^n D_i$. We will finally denote with $\widetilde{D(n,k)}$ the k-fold fibered product $D \times_{X^n} \cdots \times_{X^n} D \subseteq X^n \times X^k$ of D over X^n and with $D(n,k)$ the reduced scheme underlying $\widetilde{D(n,k)}$: $D(n,k) := \widetilde{D(n,k)}_{\mathrm{red}}$. The scheme $D(n,k)$ is Haiman's *polygraph* (see [7], [8], [16]). Remark that $\widetilde{D(n,1)} = D$ is reduced, since it is a scheme theoretic union of smooth varieties, and hence $D(n,1) = \widetilde{D(n,1)} = D$. Denote with π_{X^n} and π_X the projections of $X^n \times X$ onto X^n and X, respectively; denote moreover with w the projection $w : X^n \times X^k \longrightarrow X^n$.

We aim to compute the image of the (derived) tensor product $F_1^{[n]} \otimes^L \cdots \otimes^L F_k^{[n]}$ of tautological sheaves on $X^{[n]}$ for the Bridgeland-King-Reid transform (2.5). By Haiman's vanishing

[4]Remark that, by Projection Formula and since B^n is a closed subscheme of the product $X^{[n]} \times X^n$, the transform $\mathbf{\Phi}$ coincides with the functor $\mathbf{R}p_* \circ q^*$, in the notations of Diagram (2.4).

theorem [8, Thm 5.1], the composition:

$$\mathbf{\Phi}(F_1^{[n]} \otimes^L \cdots \otimes^L F_k^{[n]}) = \Phi^{\mathcal{O}_{B^n}}_{X^{[n]} \to X^n} \circ \Phi^{\mathcal{O}_{\Xi(n,k)}}_{X^k \to X^{[n]}}(F_1 \boxtimes \cdots \boxtimes F_k) \tag{2.6}$$

$$= \Phi^{\mathcal{O}_{D(n,k)}}_{X^k \to X^n}(F_1 \boxtimes \cdots \boxtimes F_k) \tag{2.7}$$

can be computed (see [16, Eq. (2.2)]) as a Fourier-Mukai functor with kernel the structural sheaf of the polygraph $D(n,k)$.

The case $k = 1$: the complex $\mathcal{C}^\bullet_F$. We extend the definition of the complex of equivariant sheaves $\mathcal{C}^\bullet_F$, given in [16] for a line bundle, for a coherent sheaf F on X. For every subset $J \subseteq \{1, \ldots, n\}$, $|J| = p \geq 2$, denote with F_J the sheaf $p_J^*(j_{J*}F)$, where j_J is the diagonal immersion of X into X^J. Define $\mathcal{C}^0_F \simeq \oplus_{i=1}^n p_i^* F$ and $\mathcal{C}^p_F := \bigoplus_{|J|=p+1} F_J$, for $p \geq 1$. The differentials $\partial^p_F : \mathcal{C}^p_F \longrightarrow \mathcal{C}^{p+1}_F$ are defined by: $\partial^p_F(x)_J := \sum_{i \in J} \varepsilon_{i,J} x_{J \setminus \{i\}}|_{\Delta_J}$ where x is a local section of $\mathcal{C}^p_F$ and where $\varepsilon_{i,J}$ is the sign: $\varepsilon_{i,J} := (-1)^{\sharp\{b \in J | b < i\}}$. The complex $(\mathcal{C}^\bullet_F, \partial^\bullet_F)$ is endowed with the following $\mathfrak{S}_n$-action. Let $\sigma \in \mathfrak{S}_n$ and $\sigma_* : X^n \longrightarrow X^n$ the permutation of the factors given by: $\sigma_*(x_1, \ldots, x_n) \longrightarrow (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$. We have $\sigma_*(\Delta_J) \simeq \Delta_{\sigma(J)}$. Therefore we can give $\mathcal{C}^\bullet_F$ the natural $\mathfrak{S}_n$-linearization: $(\sigma.x)_J := \varepsilon_{\sigma,J} \sigma_* x_{\sigma^{-1}(J)}$, where $\varepsilon_{\sigma,J}$ is the signature of the only permutation $\tau \in \mathfrak{S}_n$ such that $\sigma^{-1}\tau$ is strictly increasing.

We now recall the following result from [16]. Let $I \subseteq \{1, \ldots, l\}$. We will indicate with $\overline{I}$ the complementary of I in $\{1, \ldots, l\}$. If $M_1, \ldots, M_l$ are modules on a ring A, we will indicate with $M_I = \otimes_{i \in I} M_i$.

Proposition 2.14 ([16, Proposition A.3])**.** *Let $(A, \mathfrak{m})$ be a noetherian regular local ring and M_i, $i = 1, \ldots, l$, finite Cohen-Macauley modules on A. Consider the exact sequences:*

$$0 \longrightarrow N_i \longrightarrow E_i \longrightarrow M_i \longrightarrow 0$$

where E_i are free A-modules. Let $K^\bullet_i$ be the complex (in degree 0 and 1): $K^\bullet_i := E_i \longrightarrow M_i \longrightarrow 0$. Suppose that $\operatorname{codim}(M_1 \otimes \cdots \otimes M_l) = \sum_{i=1}^l \operatorname{codim} M_i$*. Then the complex $K^\bullet := \otimes_i K^\bullet_i$:*

$$0 \longrightarrow \otimes_{i=1}^l E_i \longrightarrow \oplus_{i=1}^l M_i \otimes E_{\overline{\{i\}}} \longrightarrow$$
$$\longrightarrow \oplus_{|I|=2} M_I \otimes E_{\overline{I}} \longrightarrow \ldots \longrightarrow \otimes_{i=1}^l M_i \longrightarrow 0$$

is a right resolution of the module $\otimes_{i=1}^l N_i$. Moreover $\operatorname{Tor}_i(N_1, \ldots, N_l) = 0$ *for all $i > 0$.*

Remark 2.15. For any non-empty multi-index $\emptyset \neq I \subseteq \{1, \ldots, n\}$, define D_I as being the scheme-theoretic intersection: $D_I := \cap_{i \in I} D_i$. The fact that D is a scheme-theoretic union of *transverse smooth subvarieties* D_i of the smooth variety $X^n \times X$ allows us, applying the preceding proposition to the sequences $0 \longrightarrow I_{D_i} \longrightarrow \mathcal{O}_{X^n \times X} \longrightarrow \mathcal{O}_{D_i} \longrightarrow 0$, $i = 1, \ldots, n$, to resolve the structural sheaf $\mathcal{O}_D$ with a Čech-type right resolution $\mathcal{K}^\bullet$ defined by $\mathcal{K}^p := \oplus_{|I|=p+1} \mathcal{O}_{D_I}$, $\partial^p(x)_J := \sum_{i \in J} \varepsilon_{i,J} x_{J \setminus \{i\}}\big|_{D_J}$ (see [16, appendix A] for details). Each term of the complex $\mathcal{K}^\bullet$ is flat over X (via the projection $\pi_X : X^n \times X \longrightarrow X$) because, for all $\emptyset \neq I \subseteq \{1, \ldots, n\}$, the schemes D_I are trivially flat over X (via the projection π_X). Consider now the fourth quadrant hypertor spectral sequence $E_1^{p,q} = \operatorname{Tor}^{\mathcal{O}_X}_{-q}(\mathcal{K}^p, G)$, converging to $\operatorname{Tor}^{\mathcal{O}_X}_{-p-q}(\mathcal{O}_D, G)$, where G is a quasi-coherent sheaf over X. It clearly degenerates at level

$E_2^{p,q}$, because $E_1^{p,q} = 0$ if $q \neq 0$. This implies that $\mathrm{Tor}_l^{\mathcal{O}_X}(\mathcal{O}_D, G) = 0$ for $l > 0$, and for all G, yielding that the sheaf $\mathcal{O}_D$ is flat over X (via the projection π_X).

Theorem 2.16. *Let X a smooth quasi-projective algebraic surface and F a coherent sheaf on X. Let $F^{[n]}$ the tautological sheaf on the Hilbert scheme $X^{[n]}$ associated to F. The image $\mathbf{\Phi}(F^{[n]})$ of the tautological sheaf $F^{[n]}$ for the Bridgeland-King-Reid equivalence $\mathbf{\Phi}$ is concentrated in degree 0 and is quasi-isomorphic in $\mathbf{D}^b_{\mathfrak{S}_n}(X^n)$ to the complex $(\mathcal{C}^\bullet_F, \partial^\bullet_F)$:*

$$\mathbf{\Phi}(F^{[n]}) \simeq p_* q^* F^{[n]} \simeq \mathcal{C}^\bullet_F \; .$$

Proof. By Equation (2.7) the Bridgeland-King-Reid transform of $F^{[n]}$ is simply:

$$\mathbf{\Phi}(F^{[n]}) \simeq \Phi^{\mathcal{O}_D}_{X \to X^n}(F) \simeq \pi_{X^n *}(\mathcal{O}_D \otimes^L_{\mathcal{O}_{X^n \times X}} \pi_X^* F) \; .$$

We now have, by Remark 2.15: $\mathcal{O}_D \otimes^L_{\mathcal{O}_{X^n \times X}} \pi_X^* F \simeq \mathcal{O}_D \otimes^L_{\mathcal{O}_X} F = \mathcal{O}_D \otimes_{\mathcal{O}_X} F$, because D is flat over X. This yields

$$\mathbf{\Phi}(F^{[n]}) \simeq \Phi^{\mathcal{O}_D}_{X \to X^n}(F) \simeq \pi_{X^n *}(\mathcal{O}_D \otimes_{\mathcal{O}_{X^n \times X}} \pi_X^* F) \simeq \pi_{X^n *}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_{X^n \times X}} \pi_X^* F) \; .$$

Now one proceeds like in the case of a line bundle (see [16, Theorem 2.2.2]), proving that $\pi_{X^n *}(\mathcal{K}^p \otimes_{\mathcal{O}_{X^n \times X}} \pi_X^* F) \simeq \mathcal{C}^p_F$ for all p. □

The case $k > 1$. In the case $k > 1$ consider the cartesian diagram:

$$\begin{array}{ccccc}
\widetilde{D(n,k)} & \overset{j}{\hookrightarrow} & \underbrace{D \times \cdots \times D}_{k-\text{times}} & \xrightarrow{\pi_X \times \cdots \times \pi_X} & \underbrace{X \times \cdots \times X}_{k-\text{times}} \\
\downarrow w & \square & \downarrow \pi_{X^n} \times \cdots \times \pi_{X^n} & & \\
X^n & \overset{i}{\hookrightarrow} & \underbrace{X^n \times \cdots \times X^n}_{k-\text{times}} & &
\end{array} \tag{2.8}$$

Remark 2.17. Remark that

$$\mathbf{L}(\pi_X \times \cdots \times \pi_X)^*(F_1 \boxtimes \cdots \boxtimes F_k) = \mathbf{L}\pi_X^* F_1 \boxtimes^L \cdots \boxtimes^L \mathbf{L}\pi_X^* F_k = \pi_X^* F_1 \boxtimes \cdots \boxtimes \pi_X^* F_k$$

because $\mathcal{O}_D$ is flat over X (via the projection π_X), by Remark 2.15.

Notation 2.18. We indicate now with $D^j \subseteq X^n \times X^k$ the subscheme $D^j := (\mathrm{id} \times p_j)^{-1}(D)$ and with $D^j_I \subset X^n \times X^k$ the subscheme $D^j_I := (\mathrm{id} \times p_j)^{-1}(D_I)$, where $p_j : X^k \longrightarrow X$ is the projection on the j-th factor. Consider now the resolution $\mathcal{K}^\bullet$ of the sheaf $\mathcal{O}_D$ on $X^n \times X$ of Remark 2.15; by flatness of the map $\mathrm{id} \times p_j$ the sheaf $\mathcal{O}_{D^j}$ is resolved by the complex $(\mathrm{id} \times p_j)^* \mathcal{K}^\bullet$, which will be denoted more briefly by $\mathcal{K}^\bullet_j$. In the following we will sometimes consider D^j_I and $\mathcal{K}^\bullet_j$ as objects over X, via the projection $X^n \times X^k \longrightarrow X^k \xrightarrow{p_j} X$.

Remark 2.19. By definition of the non reduced polygraph $\widetilde{D(n,k)}$, if $F_1, \ldots, F_k$ are coherent sheaves on X, by Base Change Formula (2.3) applied to the fibered product $\widetilde{D(n,k)}$ and to the sheaves $\mathcal{O}_{D^1} \otimes_{\mathcal{O}_X} F_1, \ldots, \mathcal{O}_{D^k} \otimes_{\mathcal{O}_X} F_k$, we get the quasi-isomorphism:

$$\begin{aligned}
w_*[(\mathcal{K}^\bullet_1 \otimes_{\mathcal{O}_X} F_1) \otimes^L_{\mathcal{O}_{X^n}} & \cdots \otimes^L_{\mathcal{O}_{X^n}} (\mathcal{K}^\bullet_k \otimes_{\mathcal{O}_X} F_k)] \\
& \simeq w_*[(\mathcal{O}_{D^1} \otimes_{\mathcal{O}_X} F_1) \otimes^L_{\mathcal{O}_{X^n}} \cdots \otimes^L_{\mathcal{O}_{X^n}} (\mathcal{O}_{D^k} \otimes_{\mathcal{O}_X} F_k)] \\
& \simeq \pi_{X^n *}(\mathcal{O}_{D^1} \otimes_{\mathcal{O}_X} F_1) \otimes^L_{\mathcal{O}_{X^n}} \cdots \otimes^L_{\mathcal{O}_{X^n}} \pi_{X^n *}(\mathcal{O}_{D^k} \otimes_{\mathcal{O}_X} F_k) \\
& \simeq \mathcal{C}^\bullet_{F_1} \otimes^L_{\mathcal{O}_{X^n}} \cdots \otimes^L_{\mathcal{O}_{X^n}} \mathcal{C}^\bullet_{F_k} \; .
\end{aligned} \tag{2.9}$$

Now, there exists a canonical morphism:

$$(\mathcal{O}_{D^1} \otimes_{\mathcal{O}_X} F_1) \otimes^L_{\mathcal{O}_{X^n}} \cdots \otimes^L_{\mathcal{O}_{X^n}} (\mathcal{O}_{D^k} \otimes_{\mathcal{O}_X} F_k) \longrightarrow \mathcal{O}_{\widetilde{D(n,k)}} \otimes^L_{\mathcal{O}_{X^k}} (F_1 \boxtimes \cdots \boxtimes F_k) ,$$

and hence a natural morphism

$$a : (\mathcal{O}_{D^1} \otimes_{\mathcal{O}_X} F_1) \otimes^L_{\mathcal{O}_{X^n}} \cdots \otimes^L_{\mathcal{O}_{X^n}} (\mathcal{O}_{D^k} \otimes_{\mathcal{O}_X} F_k) \longrightarrow \mathcal{O}_{D(n,k)} \otimes^L_{\mathcal{O}_{X^k}} (F_1 \boxtimes \cdots \boxtimes F_k) .$$

Pushing forward by w_*, and remembering Remark 2.19, we get a natural morphism in the derived category $\mathbf{D}^b_{\mathfrak{S}_n}(X^n)$:

$$\alpha := w_* a : \mathcal{C}^\bullet_{F_1} \otimes^L \cdots \otimes^L \mathcal{C}^\bullet_{F_k} \longrightarrow \mathbf{\Phi}(F_1^{[n]} \otimes^L \cdots \otimes^L F_k^{[n]}) \tag{2.10}$$

In all generality the comparison between the sought image $\mathbf{\Phi}(F_1^{[n]} \otimes^L \cdots \otimes^L F_k^{[n]})$ and the hyperderived tensor product $\mathcal{C}^\bullet_{F_1} \otimes^L \cdots \otimes^L \mathcal{C}^\bullet_{F_k}$ can't go much further.

Lemma 2.20. *Let F_1, ..., F_k coherent sheaves on X. Consider the statements:*

(i) $\mathrm{Tor}_i^{\mathcal{O}_{X^{[n]}}}(F_1^{[n]}, \ldots, F_k^{[n]}) = 0$, $\forall i > 0$;

(ii) $\mathrm{Tor}_i^{\mathcal{O}_{X^k}}(\mathcal{O}_{D(n,k)}, F_1 \boxtimes \cdots \boxtimes F_k) = 0$, $\forall i > 0$;

(iii) $\mathbf{\Phi}(F_1^{[n]} \otimes^L \cdots \otimes^L F_k^{[n]})$ *is concentrated in degree* 0.

Then we have the implications: $(i) \implies (ii) \iff (iii)$.

Proof. From the quasi-isomorphism:

$$\mathbf{\Phi}(F_1^{[n]} \otimes^L \cdots \otimes^L F_k^{[n]}) \simeq w_*(\mathcal{O}_{D(n,k)} \otimes^L_{\mathcal{O}_{X^k}} F_1 \boxtimes \cdots \boxtimes F_k) .$$

we get that (ii) is equivalent to (iii). Moreover if (i) holds, then, by the preceding quasi-isomorphism, $\mathbf{\Phi}(F_1^{[n]} \otimes^L \cdots \otimes^L F_k^{[n]})$ is a complex concentrated at the same time in degree ≥ 0 and ≤ 0, then (iii) holds. □

Proposition 2.21. *Let X be a smooth quasi-projective algebraic surface and let F_1, ..., F_k be coherent sheaves on X such that the tautological sheaves $F_1^{[n]}$, ..., $F_k^{[n]}$ are transverse. Then the mapping cone of the natural morphism α (2.10) is acyclic in degree > 0; this means that $R^i p_* q^*(F_1^{[n]} \otimes \cdots F_k^{[n]}) = 0$ for all $i > 0$ and in degree 0 we have the epimorphism: $p_* q^*(F_1^{[n]}) \otimes \cdots \otimes p_* q^*(F_k^{[n]}) \longrightarrow p_* q^*(F_1^{[n]} \otimes \cdots \otimes F_k^{[n]})$.*

Proof. The fact that the mapping cone of the morphism α is acyclic in degree > 0 can be proved exactly in the same way as in the case of a line bundle (see [16, Theorem 2.3.1]): it relies on the transversality of $F_1^{[n]}$, ..., $F_k^{[n]}$ and on Haiman's vanishing theorem, using Lemma 2.3. □

Lemma 2.22. *Let X a smooth quasi-projective algebraic surface. Let Q_i, $i = 1, \ldots, l$ coherent sheaves of dimension 0. Then for each i the tautological sheaf $Q_i^{[n]}$ is Cohen-Macauley of codimension 2. If $\mathrm{Supp}\, Q_i \cap \mathrm{Supp}\, Q_j = \emptyset$ for all $i \neq j$ then $\mathrm{codim}\, Q_1^{[n]} \otimes \cdots \otimes Q_l^{[n]} = 2l$; hence $\mathrm{Tor}_i^{\mathcal{O}_{X^{[n]}}}(Q_1^{[n]}, \ldots, Q_l^{[n]}) = 0$ for all $i > 0$.*

Proof. The sheaves Q_j are 0-dimensional and hence Cohen-Macauley; by Auslander–Buchsbaum Formula [13, Thm 19.1], they can be resolved locally (on adequate affine open sets $U \subseteq X$) by locally free resolutions

$$0 \longrightarrow R_2^i \longrightarrow R_1^i \longrightarrow R_0^i \longrightarrow Q_i\big|_U \longrightarrow 0\,.$$

The open subsets of the form $U^{[n]}$, where U is an affine open subset of X, cover the Hilbert scheme $X^{[n]}$, by [16, lemma 1.4.3]. The following diagram is now commutative:

$$\begin{array}{ccc} \mathbf{D}^b(X) & \xrightarrow{[-]^{[n]}} & \mathbf{D}^b(X^{[n]}) \\ \downarrow & & \downarrow \\ \mathbf{D}^b(U) & \xrightarrow{[-]^{[n]}} & \mathbf{D}^b(U^{[n]}) \end{array}$$

where the vertical arrows are restriction functors. Hence, for any tautological sheaf $F^{[n]}$, its restriction to $U^{[n]}$ is the tautological sheaf associated to $F\big|_U$:

$$F^{[n]}\big|_{U^{[n]}} = (F\big|_U)^{[n]}\,.$$

Applying the functor $[-]^{[n]}$ (relative to U) to the above resolutions of $Q_i\big|_U$ we get locally free resolutions $P_i^\bullet$ of $Q_i^{[n]}$ on $U^{[n]}$ of at most length 2:

$$P_i^\bullet := 0 \longrightarrow (R_2^i)^{[n]} \longrightarrow (R_1^i)^{[n]} \longrightarrow (R_0^i)^{[n]} \longrightarrow Q_i^{[n]}\big|_{U^{[n]}} \longrightarrow 0\,.$$

The support $\operatorname{Supp} Q_i^{[n]}$ is of codimension 2, hence by Auslander-Buchsbaum Formula the sheaf $Q_i^{[n]}$ is Cohen-Macauley. Denote with Z_i the support of Q_i and consider $W := \operatorname{Supp} Q_1^{[n]} \otimes \cdots \otimes Q_l^{[n]} = \operatorname{Supp} Q_1^{[n]} \cap \cdots \cap \operatorname{Supp} Q_l^{[n]} = \{\xi \in X^{[n]} \mid \xi \cap Z_i \neq \emptyset \ \forall i\}$. We will now prove that $\operatorname{codim}_{X^{[n]}} W = 2l$. If $l > n$ then it is clear that $W = \emptyset$; if $l \leq n$ and $\xi \in W$ then $\mu(\xi) - \sum_{i=1}^{l} p_i \geq 0$ where $p_i \in Z_i \cap \xi$, $i = 1, \ldots, l$, are l distinct points; hence W is contained in the finite union of the subschemes $\mu^{-1}(p_1 + \cdots + p_l + S^{n-l}X)$, where $(p_1, \ldots, p_l) \in Z_1 \times \cdots \times Z_l$. If λ is a partition of $k \in \mathbb{N}^*$, denote with $S^k_\lambda X$ the stratum of $S^k X$ given by 0-cycles of the form $\sum_i \lambda_i x_i$. Each closed subscheme $\mu^{-1}(p_1 + \cdots + p_l + S^{n-l}X)$ is the finite union : $\mu^{-1}(p_1 + \cdots + p_l + S^{n-l}X) = \cup_\lambda \mu^{-1}(p_1 + \cdots + p_l + S^{n-l}_\lambda X)$, where $\lambda = \lambda_1 + \cdots + \lambda_h$ is a partition of $n-l$. Hence $\dim W = \max_\lambda \{\dim \mu^{-1}(p_1 + \cdots + p_l + S^{n-l}_\lambda X)\}$, where λ runs among the partitions of $n-l$. We can write $S^{n-l}_\lambda X = \cup_I V_I$, where $I \subseteq \{1, \ldots, l\}, 0 \leq |I| \leq \min\{l, n-l\}$ and $V_I = \{\eta \in S^{n-l}_\lambda X \mid \operatorname{ord}_{p_j} \eta \geq 1 \ \forall j \in I\}$, where we indicated with $\operatorname{ord}_{p_j} \eta$ the order of the cycle η at the point p_j. Now $\dim V_I = h - |I|$ and $\dim \mu^{-1}(p_1 + \cdots + p_l + \eta) = \sum_{i=1}^h (\lambda_i - 1) + |I| = n - l + |I|$, for each $\eta = \sum_{i=1}^h \lambda_i x_i \in V_I$, by [5]; hence $\dim \mu^{-1}(p_1 + \cdots + p_l + V_I) = n - l + h \leq 2(n-l)$. Therefore $\operatorname{codim}_{X^{[n]}} \mu^{-1}(p_1 + \cdots + p_l + S^{n-l}X) \geq 2l$. Hence $2l \leq \operatorname{codim} Q_1^{[n]} \otimes \cdots \otimes Q_l^{[n]} \leq 2l$. The result now follows from Lemma 2.2. ☐

Lemma 2.23. *Let F_i, $i = 1, \ldots, k$, be torsion free coherent sheaves on the smooth quasi-projective algebraic surface X. Suppose that the singular loci* $\operatorname{Sing} F_i$ *are pairwisely disjoint. Then $F_1^{[n]}$, ..., $F_k^{[n]}$ are transverse and their tensor product $F_1^{[n]} \otimes \cdots \otimes F_k^{[n]}$ is torsion free.*

Proof. The sheaf F_i is resolved by the complex $K_i^\bullet = F_i^{**} \longrightarrow Q_i \longrightarrow 0$, where F_i^{**} is locally free and $\dim Q_i = 0$. Since the functor $[-]^{[n]}$ is exact, the tautological sheaves $F_i^{[n]}$ are resolved by the complexes $(K_i^\bullet)^{[n]}$; this proves that $F_i^{[n]}$ is torsion free if F_i is. Since $\operatorname{Sing} F_i = \operatorname{Supp} Q_i$, $i = 1, \dots, k$, are pairwisely disjoint, by the previous lemma $\operatorname{codim} Q_1^{[n]} \otimes \cdots \otimes Q_k^{[n]} = 2k$. Hence, by Proposition 2.14, the tensor product of complexes $(K_1^\bullet)^{[n]} \otimes \cdots \otimes (K_k^\bullet)^{[n]}$ resolves the sheaf $F_1^{[n]} \otimes \cdots \otimes F_k^{[n]}$, which is torsion free, because of the injection in the locally free $F_1^{[n]} \otimes \cdots \otimes F_k^{[n]} \hookrightarrow (F_1^{**})^{[n]} \otimes \cdots \otimes (F_k^{**})^{[n]}$; moreover the sheaves $F_1^{[n]}, \dots, F_k^{[n]}$ are transverse. $\square$

We can now strengthen Proposition 2.21.

Theorem 2.24. *Let F_i, $i = 1, \dots, k$, be torsion free coherent sheaves on a smooth quasi-projective surface X, with pairwisely disjoint singular loci $\operatorname{Sing} F_i$. Then the image $\mathbf{\Phi}(F_1^{[n]} \otimes \cdots \otimes F_k^{[n]})$ of the tensor product of tautological sheaves $F_i^{[n]}$ is concentrated in degree 0 and is quasi isomorphic to the sheaf $p_* q^*(F_1^{[n]} \otimes \cdots \otimes F_k^{[n]})$, which is torsion free; this sheaf is isomorphic to the quotient of $\otimes_{i=1}^k p_* q^*(F_i^{[n]})$ by the torsion subsheaf and can be realized as the quotient modulo torsion $E_\infty^{0,0}/\mathrm{tors}$, of the term $E_\infty^{0,0}$ of the hyperderived spectral sequence*

$$E_1^{p,q}(F_1, \dots, F_k) := \bigoplus_{i_1 + \cdots + i_k = p} \operatorname{Tor}_{-q}(\mathcal{C}_{F_1}^{i_1}, \dots, \mathcal{C}_{F_k}^{i_k})$$

associated to the hyperderived tensor product $\mathcal{C}_{F_1}^\bullet \otimes^L \cdots \otimes^L \mathcal{C}_{F_k}^\bullet$ and abutting to the cohomology $H^{p+q}(\mathcal{C}_{F_1}^\bullet \otimes^L \cdots \otimes^L \mathcal{C}_{F_k}^\bullet)$.

Proof. Once we know that the tensor product $F_1^{[n]} \otimes \cdots \otimes F_k^{[n]}$ is torsion free, then $q^*(F_1^{[n]} \otimes \cdots \otimes F_k^{[n]})$ is torsion free (because q is flat[5]). As a consequence $p_* q^*(F_1^{[n]} \otimes \cdots \otimes F_k^{[n]})$ is torsion free as $p_* \mathcal{O}_{B^n}$-module, but $p_* \mathcal{O}_{B^n} \simeq \mathcal{O}_{X^n}$, by [16, Theorem 1.7.3]. Hence $p_* q^*(F_1^{[n]} \otimes \cdots \otimes F_k^{[n]})$ is torsion free as $\mathcal{O}_{X^n}$-module. The proof now goes on easily as in the case of a line bundle (see [16, Thm 2.3.1 and Cor. 2.3.2]). $\square$

3 Invariants of Polygraphs

The aim of this section is to compute the $\mathfrak{S}_n$-invariants[6] of the image $\mathbf{\Phi}(F_1^{[n]} \otimes \cdots \otimes F_k^{[n]})$ of the tensor product of tautological sheaves $F_1^{[n]} \otimes \cdots \otimes F_k^{[n]}$. Since $X^{[n]} \simeq B^n/\mathfrak{S}_n$ and since (see [16, Proposition 1.7.2])

$$\mathbf{\Phi}(F_1^{[n]} \otimes \cdots \otimes F_k^{[n]})^{\mathfrak{S}_n} := \pi_*^{\mathfrak{S}_n} \circ \mathbf{\Phi}(F_1^{[n]} \otimes \cdots \otimes F_k^{[n]}) = \mathbf{R}\mu_*(F_1^{[n]} \otimes \cdots \otimes F_k^{[n]}) \,,$$

this will provide general Brion-Danila Formulas for the derived direct image of $F_1^{[n]} \otimes \cdots \otimes F_k^{[n]}$ via the Hilbert-Chow morphism. Instead of using results from Proposition 2.21 or Theorem 2.24, we will take a slightly different point of view, computing the invariants $\mathbf{\Phi}(F_1^{[n]} \otimes \cdots \otimes F_k^{[n]})^{\mathfrak{S}_n}$ as the image of a Fourier-Mukai functor with kernel the invariants of the structural sheaf of polygraph $\mathcal{O}_{D(n,k)}^{\mathfrak{S}_n}$ on $S^n X \times X^k$: in this way we can obtain somewhat more general results than directly applying Proposition 2.21.

[5] The fact that q is flat implies that, for any coherent sheaf G on $X^{[n]}$, $q^* G^{**} \simeq (q^* G)^{**}$ (see, for example [17, Prop. 21]), whence, if G is torsion free, the injection $q^* G \hookrightarrow (q^* G)^{**}$.

[6] We recall that if G is a finite group and if F is a G-equivariant coherent sheaf on an algebraic variety X, upon which G acts, the sheaf of invariants F^G of the sheaf F is by definition the sheaf $F^G := \pi_*^G F$, where $\pi : X \longrightarrow X/G$ is the quotient projection and π_*^G is the G-invariant push forward $\pi_*^G := [-]^G \circ \pi_*$, that is, the push forward followed by the functor of G-invariants.

3.1 Equivariant Fourier-Mukai functors

Let X and Y two varieties equipped with the actions of two finite groups G and H. Then $G\times H$ acts on the product $X\times Y$ via the diagonal action. The projections π_X and π_Y are equivariant with respect to the projections $G\times H \longrightarrow G$ and $G\times H \longrightarrow H$, respectively. As a consequence of general facts (see [15]), they define functors: $\pi_X^* : \mathrm{Coh}_G(X) \longrightarrow \mathrm{Coh}_{G\times H}(X\times Y)$ and ${\pi_Y}_*^G : \mathrm{Coh}_{G\times H}(X\times Y) \longrightarrow \mathrm{Coh}_H(Y)$, which can be derived, defining equivariant pull-back and push-forwards: $\pi_X^* = \mathbf{L}\pi_X^* : \mathbf{D}_G^-(X) \longrightarrow \mathbf{D}_{G\times H}^-(X\times Y)$ and $\mathbf{R}{\pi_Y}_*^G : \mathbf{D}_{G\times H}^-(X\times Y) \longrightarrow \mathbf{D}_H^-(Y)$. The bifunctor $-\otimes-$ passes as well on the G-equivariant level, hence, deriving it, we get a bifunctor: $-\otimes^L - : \mathbf{D}_{G\times H}^-(X\times Y)\times \mathbf{D}_{G\times H}^-(X\times Y) \longrightarrow \mathbf{D}_{G\times H}^-(X\times Y)$. The choice of a kernel $P\in \mathbf{D}_{G\times H}^-(X\times Y)$ defines a functor : $-\otimes^L P : \mathbf{D}_{G\times H}^-(X\times Y) \longrightarrow \mathbf{D}_{G\times H}^-(X\times Y)$. The composition of these functors defines the equivariant Fourier-Mukai functor with kernel P:

$$\Phi_{X\to Y}^{P} := \mathbf{R}{\pi_Y}_*^G(\pi_X^*(-)\otimes^L P) : \mathbf{D}_G^-(X) \longrightarrow \mathbf{D}_H^-(Y)\ .$$

Let us consider the diagram:

$$\begin{array}{ccccc} X & \xleftarrow{\pi_X} & X\times Y & \xrightarrow{\pi_Y} & Y \\ \downarrow u & & \downarrow u\times v & & \downarrow v \\ X/G & \xleftarrow{\pi_{X/G}} & X/G\times Y/H & \xrightarrow{\pi_{Y/H}} & Y/H \end{array}$$

Taking the invariants of P by $G\times H$, we get a kernel $P^{G\times H}\in \mathbf{D}^-(X/G\times Y/H)$ and consequently an associated Fourier-Mukai functor:

$$\Phi_{X/G\to Y/H}^{P^{G\times H}} : \mathbf{D}^-(X/G) \longrightarrow \mathbf{D}^-(Y/H)\ .$$

This new functor is linked with the previous by the relation:

Proposition 3.1.

$$\Phi_{X/G\to Y/H}^{P^{G\times H}} = v_*^H \circ \Phi_{X\to Y}^{P} \circ \mathbf{L}u^*\ .$$

Proof. If $F\in \mathbf{D}^-(X/G)$ we have:

$$\begin{aligned} v_*^H \circ \Phi_{X\to Y}^P \circ \mathbf{L}u^* F &= v_*^H \circ \mathbf{R}{\pi_Y}_*^G(\pi_X^*(\mathbf{L}u^*F)\otimes_{\mathcal{O}_{X\times Y}}^L P) \\ &= v_*^H \circ \mathbf{R}{\pi_Y}_*^G\Big(\mathbf{L}(u\times v)^*\pi_{X/G}^*(F)\otimes_{\mathcal{O}_{X\times Y}}^L P\Big) \\ &= \mathbf{R}{\pi_{Y/H}}_*(u\times v)_*^{G\times H}\Big(\mathbf{L}(u\times v)^*\pi_{X/G}^*(F)\otimes_{\mathcal{O}_{X\times Y}}^L P\Big) \\ &= \mathbf{R}{\pi_{Y/H}}_*\Big[\pi_{X/G}^*(F)\otimes_{\mathcal{O}_{X/G\times Y/G}}^L (u\times v)_*P\Big]^{G\times H} \\ &= \mathbf{R}{\pi_{Y/H}}_*\Big[\pi_{X/G}^*(F)\otimes_{\mathcal{O}_{X/G\times Y/G}}^L (u\times v)_*^{G\times H}P\Big] \\ &= \Phi_{X/G\to Y/G}^{P^{G\times H}}(F) \end{aligned}$$

where we used Projection Formula in the fourth equality and the fact that $G\times H$ acts trivially on $\pi_{X/G}^*F$ in the fifth equality. □

Applying the proposition to our situation we get immediately:

Corollary 3.2. *Let X a smooth quasi-projective algebraic surface and $F_1, \dots, F_k$ coherent sheaves on X. Then*

$$\mathbf{\Phi}(F_1^{[n]} \otimes^L \cdots \otimes^L F_k^{[n]})^{\mathfrak{S}_n} := \pi_*^{\mathfrak{S}_n} \circ \Phi^{\mathcal{O}_{D(n,k)}}_{X^k \to X^n}(F_1 \boxtimes \cdots \boxtimes F_k) = \Phi^{\mathcal{O}^{\mathfrak{S}_n \times \{1\}}_{D(n,k)}}_{X^k \to S^n X}(F_1 \boxtimes \cdots \boxtimes F_k) \; .$$

As a consequence, in order to compute the invariants $\mathbf{\Phi}(F_1^{[n]} \otimes^L \cdots \otimes^L F_k^{[n]})^{\mathfrak{S}_n}$, the main point is the computation of the invariants $\mathcal{O}^{\mathfrak{S}_n \times \{1\}}_{D(n,k)} := (\pi \times \mathrm{id})_*^{\mathfrak{S}_n \times \{1\}} \mathcal{O}_{D(n,k)} \in \mathrm{Coh}(S^n X \times X^k)$ of the structural sheaf of the polygraph $D(n,k)$. This will be done in the next section for $k \leq 2$.

3.2 Invariants of the polygraph

Remark 3.3. Let $f : X \longrightarrow Y$ be a proper morphism of smooth algebraic varieties; let moreover $Z \subseteq X$ be a reduced subscheme of X such that $f\big|_Z : Z \longrightarrow Y$ is finite and surjective. Now if $P^\bullet$ is a complex of coherent sheaves on X, supported on Z, then $f_* \mathcal{H}^p(P^\bullet) = \mathcal{H}^p(f_* P^\bullet)$ and moreover[7] $\mathcal{H}^p(P^\bullet) = 0$ if and only if $\mathcal{H}^p(f_* P^\bullet) = 0$.

Consider now the hyperderived spectral sequence

$$\mathbb{E}_1^{p,q}(F_1, \dots, F_k) := \bigoplus_{i_1 + \cdots + i_k = p} \mathrm{Tor}_{-q}^{\mathcal{O}_{X^n \times X^k}}(\mathcal{K}_1^{i_1} \otimes_{\mathcal{O}_X} F_1, \dots, \mathcal{K}_k^{i_k} \otimes_{\mathcal{O}_X} F_k)$$

associated to the derived tensor product $(\mathcal{K}_1^\bullet \otimes_{\mathcal{O}_X} F_1) \otimes^L_{\mathcal{O}_{X^n}} \cdots \otimes^L_{\mathcal{O}_{X^n}} (\mathcal{K}_k^\bullet \otimes^L_{\mathcal{O}_X} F_k) \in \mathbf{D}^b_{\mathfrak{S}_n \times \{1\}}(X^n \times X^k)$ and abutting to its $p+q$-cohomology. Rewriting the quasi-isomorphism (2.9) in term of this spectral sequence, and remembering that the first complex is w_*- acyclic, since supported in $D(n,k)$, which is finite over X^n, we get:

$$w_* \mathbb{E}_1^{p,q}(F_1, \dots, F_k) \simeq E_1^{p,q}(F_1, \dots, F_k) \; ,$$

and explicitating the terms $\mathbb{E}_1^{p,q}$ and $E_1^{p,q}$, we have, for any $i_1, \dots, i_k$:

$$w_* \mathrm{Tor}_{-q}(\mathcal{K}_1^{i_1} \otimes_{\mathcal{O}_X} F_1, \dots, \mathcal{K}_k^{i_k} \otimes_{\mathcal{O}_X} F_k) \simeq \mathrm{Tor}_{-q}(\mathcal{C}^{i_1}_{F_1}, \dots, \mathcal{C}^{i_k}_{F_k}) \; . \tag{3.1}$$

Consider the diagram:

$$\begin{array}{ccc} X^n \times X^k & \xrightarrow{\pi \times \mathrm{id}} & S^n X \times X^k \\ \Big\downarrow w & & \Big\downarrow \nu \\ X^n & \xrightarrow{\pi} & S^n X \end{array} \tag{3.2}$$

Consider moreover the spectral sequence on $S^n X \times X^k$:

$$\mathfrak{E}_1^{p,q}(F_1, \dots, F_k) := (\pi \times \mathrm{id})_*^{\mathfrak{S}_n \times \{1\}} \mathbb{E}_1^{p,q}(F_1, \dots, F_k) \; ,$$

supported on $D(n,k)/\mathfrak{S}_n \times \{1\}$. Then we have:

$$\nu_* \mathfrak{E}_1^{p,q}(F_1, \dots, F_k) = \pi_*^{\mathfrak{S}_n} E_1^{p,q}(F_1, \dots, F_k) \; . \tag{3.3}$$

Indicate with $\mathcal{E}_1^{p,q}(F_1, \dots, F_k)$ the spectral sequence (3.3). As a consequence of Remark 3.3, the spectral sequences $\mathbb{E}_1^{p,q}$ and $\mathfrak{E}_1^{p,q}$ behave exactly as the spectral sequences $E_1^{p,q}$ and $\mathcal{E}_1^{p,q}$, respectively. As an immediate corollary of this fact and of Theorem 2.24 (or of [16, Thm 2.3.1 and Cor. 2.3.2]), we have:

[7] Here we denote with $\mathcal{H}^p(P^\bullet)$ the sheaf of p-cohomology of the complex $P^\bullet$.

Corollary 3.4. *The structural sheaf of the polygraph $\mathcal{O}_{D(n,k)}$ can be identified with the term $\mathbb{E}_\infty^{0,0}(\mathcal{O}_X,\dots,\mathcal{O}_X)$ of the hyperderived spectral sequence associated to the derived tensor product $\mathcal{K}_1^\bullet \otimes^L_{\mathcal{O}_{X^n}} \cdots \otimes^L_{\mathcal{O}_{X^n}} \mathcal{K}_k^\bullet$; its $\mathfrak{S}_n \times \{1\}$ -invariants $\mathcal{O}_{D(n,k)}^{\mathfrak{S}_n\times\{1\}}$ can be identified with the term $\mathfrak{E}_\infty^{0,0}(\mathcal{O}_X,\dots,\mathcal{O}_X)$ of the spectral sequence $\mathfrak{E}_1^{p,q}(\mathcal{O}_X,\dots,\mathcal{O}_X)$*

Corollary 3.5. *We have $\mathcal{O}_D^{\mathfrak{S}_n\times\{1\}} \simeq (\oplus_{i=1}^n \mathcal{O}_{D_i})^{\mathfrak{S}_n\times\{1\}}$ on $S^nX\times X$.*

Proof. From $\mathcal{O}_D \simeq \mathcal{K}^\bullet$ we get $\mathcal{O}_D^{\mathfrak{S}_n\times\{1\}} \simeq (\mathcal{K}^\bullet)^{\mathfrak{S}_n\times\{1\}}$. One can prove by a direct computation that $(\mathcal{K}^p)^{\mathfrak{S}_n\times\{1\}}=0$, for $p>0$, whence the result. Otherwise, using Diagram (3.2) for $k\geq 1$, the lemma follows applying ν_*: $\nu_*\mathcal{O}_D^{\mathfrak{S}_n\times\{1\}} = \mathbf{R}\mu_*\mathcal{O}_X^{[n]} = (\oplus_i p_i^*\mathcal{O}_X)^{\mathfrak{S}_n} \simeq \nu_*[(\mathcal{K}^\bullet)^{\mathfrak{S}_n\times\{1\}})]$, where we used the known Brion-Danila Formula (see [3]) in the middle isomorphism. □

Corollary 3.6. *We have the short exact sequence on $S^nX\times X^2$:*

$$0 \longrightarrow \mathcal{O}_{D(n,2)}^{\mathfrak{S}_n\times\{1\}} \longrightarrow (\mathcal{K}_1^0\otimes\mathcal{K}_2^0)^{\mathfrak{S}_n\times\{1\}} \longrightarrow (\mathcal{K}_1^0\otimes\mathcal{K}_2^1)^{\mathfrak{S}_n\times\{1\}} \longrightarrow 0\,.$$

Proof. We proved in [16] that the sheaf $\mathcal{E}_\infty^{0,0}(\mathcal{O}_X,\mathcal{O}_X)\simeq \nu_*\mathcal{O}_{D(n,2)}^{\mathfrak{S}_n\times\{1\}}$ is quasi-isomorphic to the two-term complex: $0 \longrightarrow (\mathcal{C}^0_{\mathcal{O}_X}\otimes\mathcal{C}^0_{\mathcal{O}_X})^{\mathfrak{S}_n} \longrightarrow (\mathcal{C}^0_{\mathcal{O}_X}\otimes\mathcal{C}^1_{\mathcal{O}_X})^{\mathfrak{S}_n} \longrightarrow 0$. This complex is exactly the complex $0 \longrightarrow \nu_*(\mathcal{K}_1^0\otimes\mathcal{K}_2^0)^{\mathfrak{S}_n\times\{1\}} \longrightarrow \nu_*(\mathcal{K}_1^0\otimes\mathcal{K}_2^1)^{\mathfrak{S}_n\times\{1\}} \longrightarrow 0$. Moreover $\mathcal{O}_{D(n,k)}^{\mathfrak{S}_n\times\{1\}} \simeq \mathfrak{E}_\infty^{0,0}(\mathcal{O}_X,\mathcal{O}_X)$ is in the kernel of the map: $(\mathcal{K}_1^0\otimes\mathcal{K}_2^0)^{\mathfrak{S}_n\times\{1\}} \longrightarrow (\mathcal{K}_1^0\otimes\mathcal{K}_2^1)^{\mathfrak{S}_n\times\{1\}}$. Since the terms $(\mathcal{K}_1^0\otimes\mathcal{K}_2^0)^{\mathfrak{S}_n\times\{1\}}$ and $(\mathcal{K}_1^0\otimes\mathcal{K}_2^1)^{\mathfrak{S}_n\times\{1\}}$ are supported in $D(n,k)/\mathfrak{S}_n\times\{1\}$ and since ν, restricted to $D(n,k)/\mathfrak{S}_n\times\{1\}$, is finite, we can conclude by Remark 3.3. □

4 Brion-Danila Formulas and cohomology

In this last section we will use the computations of the invariants of the polygraphs $D(n,k)$, for $k\leq 2$, performed in the previous section, in order to find general Brion-Danila's Formulas for the derived direct image of a tautological sheaf and of a tensor product of two of them.

Theorem 4.1. *Let F a coherent sheaf on the smooth quasi-projective surface X. Then $\mathbf{R}\mu_*F^{[n]} \simeq (\oplus_i p_i^*F)^{\mathfrak{S}_n}$.*

Proof. We have

$$\begin{aligned}\mathbf{R}\mu_*F^{[n]} &\simeq \nu_*(\mathcal{O}_D^{\mathfrak{S}_n\times\{1\}}\otimes^L_{\mathcal{O}_X} F) \simeq \nu_*((\oplus_{i=1}^n\mathcal{O}_{D_i})^{\mathfrak{S}_n\times\{1\}}\otimes^L_{\mathcal{O}_X} F)\\ &= \nu_*(\oplus_{i=1}^n \mathcal{O}_{D_i}\otimes_{\mathcal{O}_X} F)^{\mathfrak{S}_n\times\{1\}} = (\oplus_{i=1}^n p_i^*F)^{\mathfrak{S}_n}\end{aligned}$$

where we used the flatness of D_i over X and the fact that $\mathfrak{S}_n$ does not act on F. □

Theorem 4.2. *Let E and F be coherent sheaves on the smooth quasi-projective surface X. Then we have the long exact sequence on S^nX:*

$$\begin{aligned}0 \longrightarrow R^{-2}\mu_*(E^{[n]}\otimes^L F^{[n]}) &\longrightarrow (\oplus_i \mathrm{Tor}_2(p_i^*E,p_i^*F))^{\mathfrak{S}_n} \longrightarrow (\oplus_{i\in J,|J|=2}\mathrm{Tor}_2(p_i^*E,F_J))^{\mathfrak{S}_n}\\ \longrightarrow R^{-1}\mu_*(E^{[n]}\otimes^L F^{[n]}) &\longrightarrow (\oplus_i \mathrm{Tor}_1(p_i^*E,p_i^*F))^{\mathfrak{S}_n} \longrightarrow (\oplus_{i\in J,|J|=2}\mathrm{Tor}_1(p_i^*E,F_J))^{\mathfrak{S}_n} \longrightarrow\\ \longrightarrow R^0\mu_*(E^{[n]}\otimes^L F^{[n]}) &\longrightarrow (\mathcal{C}_E^0\otimes\mathcal{C}_F^0)^{\mathfrak{S}_n} \longrightarrow (\mathcal{C}_E^0\otimes\mathcal{C}_F^1)^{\mathfrak{S}_n} \longrightarrow 0\end{aligned}$$

As a consequence $R^i\mu_*(E^{[n]}\otimes^L F^{[n]})=0$ if $i\notin[-2,0]$.

Proof. Denote now with $\mathcal{P}^\bullet$ the complex resolving the sheaf $\mathcal{O}_{D(n,2)}^{\mathfrak{S}_n\times\{1\}}$:

$$\mathcal{P}^\bullet \simeq 0 \longrightarrow (\mathcal{K}_1^0\otimes\mathcal{K}_2^0)^{\mathfrak{S}_n\times\{1\}} \longrightarrow (\mathcal{K}_1^0\otimes\mathcal{K}_2^1)^{\mathfrak{S}_n\times\{1\}} \longrightarrow 0\ ,$$

concentrated in degree 0 and 1. Let $\tilde{\mathcal{P}}^\bullet$ the complex on $X^n\times X^2$ defined by: $\tilde{\mathcal{P}}^\bullet := 0 \longrightarrow \mathcal{K}_1^0\otimes\mathcal{K}_2^0 \longrightarrow \mathcal{K}_1^0\otimes\mathcal{K}_2^1 \longrightarrow 0$. We have $\mathcal{P}^\bullet = (\tilde{\mathcal{P}}^\bullet)^{\mathfrak{S}_n\times\{1\}}$. By Projection Formula we have:

$$\begin{aligned}\mathcal{P}^\bullet\otimes^L_{\mathcal{O}_{X^2}}(E\boxtimes F) &= (\pi\times\mathrm{id})_*^{\mathfrak{S}_n\times\{1\}}(\tilde{\mathcal{P}}^\bullet)\otimes^L_{\mathcal{O}_{X^2}}(E\boxtimes F)\\ &= (\pi\times\mathrm{id})_*^{\mathfrak{S}_n\times\{1\}}\left[\tilde{\mathcal{P}}^\bullet\otimes^L_{\mathcal{O}_{X^2}}(E\boxtimes F)\right]\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{R}\mu_*(E^{[n]}\otimes^L F^{[n]})\simeq\nu_*(\mathcal{P}^\bullet\otimes^L_{\mathcal{O}_{X^2}}E\boxtimes F) &= \nu_*(\pi\times\mathrm{id})_*^{\mathfrak{S}_n\times\{1\}}\left[\tilde{\mathcal{P}}^\bullet\otimes^L_{\mathcal{O}_{X^2}}(E\boxtimes F)\right]\\ &= \pi_*^{\mathfrak{S}_n}w_*\left[\tilde{\mathcal{P}}^\bullet\otimes^L_{\mathcal{O}_{X^2}}(E\boxtimes F)\right]\end{aligned}$$

We compute now $w_*[\tilde{\mathcal{P}}^\bullet\otimes^L_{\mathcal{O}_{X^2}}(E\boxtimes F)]$ via the spectral sequence: $E_1^{p,q}=w_*\mathrm{Tor}_{-q}^{\mathcal{O}_{X^2}}(\tilde{\mathcal{P}}^p,E\boxtimes F)$, abutting to $w_*\mathrm{Tor}_{-p-q}^{\mathcal{O}_{X^2}}(\tilde{\mathcal{P}}^\bullet,E\boxtimes F)$. By (3.1) we have:

$$w_*\mathrm{Tor}_{-q}^{\mathcal{O}_{X^2}}(\tilde{\mathcal{P}}^0,E\boxtimes F)=\mathrm{Tor}_{-q}(\mathcal{C}_E^0,\mathcal{C}_F^0)\quad,\quad w_*\mathrm{Tor}_{-q}^{\mathcal{O}_{X^2}}(\tilde{\mathcal{P}}^1,E\boxtimes F)=\mathrm{Tor}_{-q}(\mathcal{C}_E^0,\mathcal{C}_F^1)\ .$$

Remark now that if $q<0$ $\mathrm{Tor}_{-q}(\mathcal{C}_E^0,\mathcal{C}_F^0)=\oplus_{i=1}^n\mathrm{Tor}_{-q}(p_i^*E,p_i^*F)$, and that $\mathrm{Tor}_{-q}(\mathcal{C}_E^0,\mathcal{C}_F^1)=\oplus_{i,|J|=2,i\in J}\mathrm{Tor}_{-q}(p_i^*E,F_J)$. Since $E_1^{p,q}$ is a two columns spectral sequence, the result follows. □

Remark 4.3. Remark that, in the long exact sequence of the previous proposition, $\mathrm{Tor}_q(p_i^*E,p_i^*F)=p_i^*\mathrm{Tor}_q(E,F)$, $\mathrm{Tor}_q(p_i^*E,F_J)\simeq\mathrm{Tor}_q(E,F)_J$. Indeed, for $\mathrm{Tor}_q(p_i^*E,p_i^*F)$: $p_i^*E\otimes^L_{\mathcal{O}_{X^n}}p_i^*F=w_*[(\mathcal{O}_{D_1^i}\otimes_{\mathcal{O}_{X^n\times X^2}}\mathcal{O}_{D_2^i})\otimes^L_{\mathcal{O}_{X^n\times X^2}}(E\boxtimes F)]$, and indicating with Δ the diagonal in X^2:

$$\begin{aligned}(\mathcal{O}_{D_1^i}\otimes_{\mathcal{O}_{X^n\times X^2}}\mathcal{O}_{D_2^i})\otimes^L_{\mathcal{O}_{X^n\times X^2}}(E\boxtimes F) &= \mathcal{O}_{\Delta_{i,n+1,n+2}}\otimes^L_{\mathcal{O}_{X^2}}(E\boxtimes F)\\ &= \mathcal{O}_{\Delta_{i,n+1,n+2}}\otimes^L_{\mathcal{O}_\Delta}(\mathcal{O}_\Delta\otimes^L_{\mathcal{O}_{X^2}}(E\boxtimes F))\\ &= \mathcal{O}_{\Delta_{i,n+1,n+2}}\otimes_{\mathcal{O}_\Delta}(E\otimes^L_{\mathcal{O}_X}F)\end{aligned}$$

because $\Delta_{i,n+1,n+2}\subseteq X^n\times X^2$ is flat over Δ. The case $\mathrm{Tor}_q(E_i,F_J)\simeq\mathrm{Tor}_q(E,F)_J$ is analogous.

Corollary 4.4. *Let E and F transverse coherent sheaves on X. Then $R^i\mu_*(E^{[n]}\otimes^L F^{[n]})=0$ if $i\neq 0$ and we have:*

$$0\longrightarrow R^0\mu_*(E^{[n]}\otimes^L F^{[n]})\longrightarrow(\mathcal{C}_E^0\otimes\mathcal{C}_F^0)^{\mathfrak{S}_n}\longrightarrow(\mathcal{C}_E^0\otimes\mathcal{C}_F^1)^{\mathfrak{S}_n}\longrightarrow 0\ .$$

In particular, the statement holds if E and F are torsion free; if moreover they have disjoint singular loci, we have $R^0\mu_*(E^{[n]}\otimes^L F^{[n]})=\mu_*(E^{[n]}\otimes F^{[n]})$.

Proof. The first statement is immediate. If E and F are torsion free, then $\mathrm{Tor}_i(E,F)=0$, for $i>0$. Indeed $E\simeq R_1^\bullet:=0\longrightarrow E^{**}\twoheadrightarrow Q_E\longrightarrow 0$ and $F\simeq R_2^\bullet:=0\longrightarrow F^{**}\twoheadrightarrow Q_F$, where Q_E and Q_F are the cokernels of the natural injections $E\hookrightarrow E^{**}$ and $F\hookrightarrow F^{**}$, respectively. Then $\mathrm{Tor}_i(E,F)$ can be computed via the fourth-quadrant spectral sequence $E_1^{p,q}=\oplus_{i_1+i_2=p}\mathrm{Tor}_{-q}(R_1^{i_1},R_2^{i_2})$. Now $E_1^{p,q}=0$ if $p>2$ or if $p=0,1$ and $q<0$. Moreover $E_1^{2,q}=\mathrm{Tor}_{-q}(Q_E,Q_F)=0$ if $q<-2$. Hence $\mathrm{Tor}_i(E,F)=0$, for $i>0$. The last statement follows from Lemma 2.23. □

Applying Theorem 4.1 and Corollary 4.4 we get the following results on the cohomology.

Corollary 4.5. *Let F a coherent sheaf on the smooth quasi-projective algebraic surface X. Then $H^*(X^{[n]}, F^{[n]}) \simeq H^*(F) \otimes S^{n-1}H^*(\mathcal{O}_X)$.*

Corollary 4.6. *Let E and F two transverse coherent sheaves on the smooth quasi-projective surface X. Then the hypercohomology of the Hilbert scheme of n points on X with values in the complex $E^{[n]} \otimes^L_{\mathcal{O}_{X^{[n]}}} F^{[n]}$ is given by:*

$$\mathbf{H}^*(X^{[n]}, E^{[n]} \otimes^L_{\mathcal{O}_{X^{[n]}}} F^{[n]}) \simeq H^*(E) \otimes H^*(F) \otimes S^{n-2}H^*(\mathcal{O}_X) \bigotimes H^*(E \otimes F) \otimes \mathcal{J} ,$$

where $\mathcal{J}$ is the kernel of the map: $S^{n-1}H^*(\mathcal{O}_X) \longrightarrow S^{n-2}H^*(\mathcal{O}_X)$ induced by the map $S^{n-2}X \longrightarrow S^{n-1}X$, defined by $y \longmapsto y + a$, where a is a fixed point in X. In particular, the statement is true if E and F are torsion free; if moreover the singular loci of E and F are disjoint, the formula gives the cohomology $H^*(X^{[n]}, E^{[n]} \otimes F^{[n]})$.

Proof. The proof follows from Corollary 4.4 in the same way as done in [16] for line bundles. □

References

[1] Nicolas Bourbaki. *Éléments de Mathématiques, Algèbre Commutative, Chapitre 10.* Masson, Paris, 1998.

[2] Tom Bridgeland, Alastair King, and Miles Reid. The McKay correspondence as an equivalence of derived categories. *J. Amer. Math. Soc.*, 14(3):535–554 (electronic), 2001.

[3] Gentiana Danila. Sur la cohomologie d'un fibré tautologique sur le schéma de Hilbert d'une surface. *J. Algebraic Geom.*, 10(2):247–280, 2001.

[4] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[5] Geir Ellingsrud and Manfred Lehn. Irreducibility of the punctual quotient scheme of a surface. *Ark. Mat.*, 37(2):245–254, 1999.

[6] Alexander Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. II. *Inst. Hautes Études Sci. Publ. Math.*, (17):91, 1963.

[7] Mark Haiman. Hilbert schemes, polygraphs and the Macdonald positivity conjecture. *J. Amer. Math. Soc.*, 14(4):941–1006 (electronic), 2001.

[8] Mark Haiman. Vanishing theorems and character formulas for the Hilbert scheme of points in the plane. *Invent. Math.*, 149(2):371–407, 2002.

[9] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[10] Y. Ito and I. Nakamura. Hilbert schemes and simple singularities. In *New trends in algebraic geometry (Warwick, 1996)*, volume 264 of *London Math. Soc. Lecture Note Ser.*, pages 151–233. Cambridge Univ. Press, Cambridge, 1999.

[11] Yukari Ito and Iku Nakamura. McKay correspondence and Hilbert schemes. *Proc. Japan Acad. Ser. A Math. Sci.*, 72(7):135–138, 1996.

[12] Manfred Lehn. Chern classes of tautological sheaves on Hilbert schemes of points on surfaces. *Invent. Math.*, 136(1):157–207, 1999.

[13] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[14] Iku Nakamura. Hilbert schemes of abelian group orbits. *J. Algebraic Geom.*, 10(4):757–779, 2001.

[15] David Ploog. Equivariant autoequivalences for finite group actions. *Advances in Mathematics, to appear;*. arxiv: math/0508625.

[16] Luca Scala. Cohomology of the Hilbert scheme of points on a surface with values in representations of tautological bundles. preprint, math/arXiv:0710.3072v2, 2007.

[17] Jean-Pierre Serre. Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier, Grenoble*, 6:1–42, 1955–1956.

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