Complex Analysis Final Presentation

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We discuss holomorphic and meromorphic functions on Riemann surfaces.

Definition 1. A **Riemann surface** X is a connected Hausdorff space with an atlas $\{(U_a, \varphi_a)\colon U_a \to \mathbb{C}\}$ of charts so that the transition maps $\{\varphi_b \circ \varphi_a^{-1}\colon \varphi_a(U_a \cap U_b) \to \varphi_b(U_a \cap U_b)\}$ are holomorphic. In this sense X is a 1-dimensional complex manifold. △

We emphasize that these transition maps are actually conformal, since the holomorphic transition map $\varphi_b \circ \varphi_a^{-1}$ is a bijection (the φ_* are homeomorphisms, in particular bijections) with inverse $\varphi_a \circ \varphi_b^{-1}$, which is holomorphic because $\varphi_a \circ \varphi_b^{-1}$ is a transition map. This will be useful later.

Definition 2. A map $f\colon X \to Y$ between Riemann surfaces is called **holomorphic** if composing with the charts give holomorphic maps in the usual sense, that is to say, given the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi_a"] & Y \arrow[d, "\varphi_b"] \\ \varphi_a(U_a) \subseteq \mathbb{C} \arrow[r, dashed, "f_{ab}"] & \varphi_b(U_b) \subseteq \mathbb{C} \end{tikzcd}$$

the dashed arrow is holomorphic for any a, b. Because notation is clunky, I lied slightly in the diagram above: really, f_{ab} is only defined on the subset $(\varphi_a \circ f^{-1})(U_b) \cap \varphi_a(U_a) \subseteq \varphi_a(U_a)$. △

Recall that a function $f\colon \mathbb{C} \to \mathbb{C}$ is meromorphic if it is holomorphic except at (at most) countably many points $\{z_1, z_2, \dots\}$ where f may have a pole, which means $1/f$, defined to be 0 at z_i, is holomorphic near the z_i. We remark that the z_i are necessarily discrete.

This is saying that $f\colon \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ is a holomorphic map of Riemann surfaces. To spell this out for the Riemann sphere, we have

$$\begin{tikzcd} \mathbb{C} \arrow[r, "f"] & \mathbb{C} \cup \{\infty\} \arrow[d, bend right, "z \mapsto z"'] \arrow[d, bend left, "z \mapsto 1/z"] \\ & \mathbb{C} \end{tikzcd}$$

and indeed we were saying that $(z \mapsto z) \circ f = f$ should be holomorphic at the $\mathbb{C}$-valued points, and when f is going to infinity then $(z \mapsto 1/z) \circ f = 1/f$ is holomorphic (we are using $1/\infty = 0$, which is really the statement that the chart $z \mapsto 1/z$ defined on $(\mathbb{C} \cup \{\infty\}) \setminus \{0\}$ sends the pole ∞ of the Riemann sphere to the point $0 \in \mathbb{C}$).

For arbitrary Riemann surfaces, this definition generalizes:

Definition 3. A function f is **meromorphic** on X if it is a holomorphic map $X \to \mathbb{C} \cup \{\infty\}$ that is not identically ∞. △

Let us fix some notation and make some easy observations.

Definition-Proposition 4. Let X be any Riemann surface.

1. The set of holomorphic functions $\mathcal{H}(X)$ is a complex vector space (the complex numbers sitting inside $\mathcal{H}(X)$ should be interpreted as constant functions). This vector space also forms a commutative ring (so one could call it a $\mathbb{C}$-algebra), in fact an integral domain, and hence has a field of fractions, which we denote $\operatorname{Frac} \mathcal{H}(X)$.

2. The set of meromorphic functions $\mathcal{M}(X)$ is also a complex vector space (as above, the complex numbers should be interpreted as constant functions). This vector space is also field, so I guess one could call it a field extension of $\mathbb{C}$.

3. A function $f \in \mathcal{M}(X)$ is said to have a pole (respectively, a zero) of order k at $x \in X$ if for some chart $U_a \ni x$, the holomorphic map $(z \mapsto 1/z) \circ f \circ \varphi_a = 1/f(\varphi_a(z))$ (respectively, the holomorphic map $(z \mapsto z) \circ f \circ \varphi_a = f(\varphi_a(z)))$ has a zero of order k as usual maps $\varphi_a(U_a) \to \mathbb{C}$. This is independent of choice of chart precisely because the transition maps are *conformal* (as opposed to just holomorphic), as we now explain. Indeed, if $U_b \ni x$ as well, then

$$\underbrace{(z \mapsto 1/z) \circ f \circ \varphi_b}_{1/f(\varphi_b(z))} = \underbrace{(z \mapsto 1/z) \circ f \circ \varphi_a}_{1/f(\varphi_a(z))} \circ (\varphi_a^{-1} \circ \varphi_b)$$
$$\underbrace{(z \mapsto z) \circ f \circ \varphi_b}_{f(\varphi_b(z))} = \underbrace{(z \mapsto z) \circ f \circ \varphi_a}_{f(\varphi_a(z))} \circ (\varphi_a^{-1} \circ \varphi_b)$$

and composing with a conformal map cannot change the order of a zero or a pole (for example because conformal maps have nonvanishing derivative).

4. Functions on X analytically continue, because on each chart we have analytic continuation, and the charts cover X.

△

The last item is a powerful observation, as it immediately implies

Proposition 5. *Holomorphic functions on compact Riemann surfaces X are constant.*

Proof. Because X is compact $|f|$ obtains a maximum, say at x; by the maximum modulus principle on a chart containing x we see that f is constant on the chart, and by analytic continuation f must be constant on all of X. □

Let $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ with $\tau \in \mathbb{C} \setminus \mathbb{R}$ denote a lattice in $\mathbb{C}$. Then the torus $\mathbb{C}/\Lambda$ is a (compact) Riemann surface, so the theory of holomorphic functions on $\mathbb{C}/\Lambda$ is degenerate. Meromorphic functions on $\mathbb{C}/\Lambda$ are precisely elliptic functions, as in Chapter 9 of Stein/Shakarchi, and their theory is much richer, as the next observation shows.

Observation 6. There exists a nonconstant meromorphic function $f \in \mathcal{M}(\mathbb{C}/\Lambda)$. For any nonconstant meromorphic function f, the field $\mathcal{M}(\mathbb{C}/\Lambda)$ is an algebraic extension of the field $\mathbb{C}(f)$, that is to say, given any $f, g \in \mathcal{M}(\mathbb{C}/\Lambda)$, there is a polynomial $P \in \mathbb{C}[x, y]$ so that $P(f, g)$ is identically 0 on $\mathbb{C}/\Lambda$. △

Let us see why this is the case. Associated to Λ is a distinguished (nonconstant) elliptic function, called the Weierstrass p function, denoted $\wp$. Theorem 9.1.8 in Stein/Shakarchi says that $\mathcal{M}(\mathbb{C}/\Lambda) = \mathbb{C}(\wp, \wp')$, where $\wp'$ is the derivative of $\wp$ (one might need to do some gymnastics to ensure everything works, e.g. take the derivative of $\wp$ on the $\mathbb{C}$-valued points, obtain a series representation of $\wp'$, and use analytic continuation).

Note that $(\wp')^2$ is a cubic polynomial in $\wp$, as Theorem 9.1.7 states. So $\mathcal{M}(\mathbb{C}/\Lambda)$ is a finite dimensional extension, and hence an algebraic extension, of $\mathbb{C}(\wp)$ (alternatively, it's an algebraic extension of $\mathbb{C}(\wp')$). Standard algebraic nonsense implies that $\mathcal{M}(\mathbb{C}/\Lambda)$ is an algebraic extension of $\mathbb{C}(f)$ for *any* nonconstant $f \in \mathcal{M}(\mathbb{C}/\Lambda)$:

Definition-Proposition 7. The **transcendence degree** of a field extension M/K is the maximum number of algebraically independent elements of M over K, denoted $\operatorname{trdeg}(M/K)$. Transcendence degree is additive with respect to towers of extensions, that is to say, if $M/L/K$ is a tower of extensions, then

$$\operatorname{trdeg}(M/K) = \operatorname{trdeg}(M/L) + \operatorname{trdeg}(L/K).$$

△

Since $\operatorname{trdeg}(\mathcal{M}(\mathbb{C}/\Lambda)/\mathbb{C}(\wp)) = 0$ and $\operatorname{trdeg}(\mathbb{C}(\wp)/\mathbb{C}) = 1$, it follows that $\operatorname{trdeg}(\mathcal{M}(\mathbb{C}/\Lambda)/\mathbb{C}) = 1$. Any nonconstant $f \in \mathcal{M}(\mathbb{C}/\Lambda)$ is transcendental over $\mathbb{C}$ because meromorphic functions can only have countably many zeros, so we have the tower of inclusions

$$\overbrace{\underbrace{\mathbb{C} \subseteq \mathbb{C}(f)}_{\text{trdeg } 1} \subseteq \mathcal{M}(\mathbb{C}/\Lambda)}^{\text{trdeg } 1}.$$

This means that $\mathcal{M}(\mathbb{C}/\Lambda)/\mathbb{C}(f)$ is an algebraic extension.

Contrast Observation 6 with the following observation, which is a rephrasing of things we have already seen in class:

Observation 8. Given a (possibly infinite) set of points $\{a_1, a_2, \dots\} \in \mathbb{C}$ with no points of accumulation and equally many complex numbers $\{b_1, b_2, \dots\}$, there exists a holomorphic function f with $f(a_i) = b_i$. Furthermore, $\mathcal{M}(\mathbb{C}) = \operatorname{Frac} \mathcal{H}(\mathbb{C})$. $\triangle$

Indeed, we showed that given a (possibly infinite) set of points $\{a_1, a_2, \dots\} \in \mathbb{C}$ with no points of accumulation and equally many other complex numbers $\{b_1, b_2, \dots\}$, there exists a holomorphic function f with $f(a_i) = b_i$ (Exercise 5.6.17 in Stein/Shakarchi). On that same set, we also showed that every meromorphic function in $\mathbb{C}$ is the quotient of two entire functions (this is Exercise 5.6.15).

These two observations are special cases of the following (very deep) theorem, which is a nontrivial corollary of Riemann-Roch. As far as I can tell these statements were only proven after the introduction of algebraic geometry, and fall out naturally out of that theory, even though these are basic questions about Riemann surfaces (which were indeed studied since the time of Riemann).

Theorem 9. *We have the following dichotomy for $\mathcal{H}(X)$ and $\mathcal{M}(X)$ on Riemann surfaces:*

1. *Let X be a compact Riemann surface. Then $\mathcal{H}(X) = \mathbb{C}$. Given a set of points $\{a_1, a_2, \dots, a_n\} \in X$, and complex numbers $\{b_1, b_2, \dots, b_n\}$, there exists a meromorphic function $f \in \mathcal{M}(X)$ with $f(a_i) = b_i$. For any nonconstant meromorphic function f, the field $\mathcal{M}(X)$ is an algebraic extension of the field $\mathbb{C}(f)$, that is to say, given any $f, g \in \mathcal{M}(X)$, there is a polynomial $P \in \mathbb{C}[x, y]$ so that $P(f, g)$ is identically 0 on X.*

2. *Let X be a noncompact Riemann surface. Given a (possibly infinite) set of points $\{a_1, a_2, \dots\} \in X$ with no points of accumulation, and equally many complex numbers $\{b_1, b_2, \dots\}$, there exists a holomorphic function f with $f(a_i) = b_i$. Furthermore, $\mathcal{M}(X) = \operatorname{Frac} \mathcal{H}(X)$.*

Let us gather which parts of this theorem we have already seen. That compact Riemann surfaces X have $\mathcal{H}(X) = \mathbb{C}$ is just Proposition 5. Furthermore, when X is a torus we have seen that $\mathcal{M}(X)$ is a finite dimensional extension of $\mathbb{C}(\wp)$, and an algebraic extension of $\mathbb{C}(f)$ for any nonconstant $f \in \mathcal{M}(X)$..

When $X = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere, Theorem 3.3.4 in Stein/Shakarchi asserts that $\mathcal{M}(X) = \mathbb{C}(z)$. Thus given a finite set of points in X, we can pick a chart that contains them all; this chart is homeomorphic to an open subset U of $\mathbb{C}$ and we can Lagrange interpolate on U to get the prescribed function values. Polynomials always be extended onto the Riemann sphere since either $\lim_{|z| \to \infty} |f(z)|$ is infinity, or f is constant.

When $X = \mathbb{C}$, this theorem is precisely Observation 8.

The references we used are as follows. Up until Proposition 5, we mostly follow Terry Tao's 246c notes 1. Part 1 of Theorem 9 is Corollary 14.13 in Otto Forster's Lectures on Riemann Surfaces and Exercise 26 in Terry Tao's 246c notes 1. Part 2 of Theorem 9 is Theorem 26.7 and Exercise 26.3 of Otto Forster's Lectures on Riemann Surfaces.