Complex Analysis

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Lecture 7. Oct 20, 2018

I guess I promised to talk about Riemann Surfaces. I've been hyping up the Riemann Roch theorem but it takes some time to build to the statement. Let me give you some candy:

Corollary 1. *An irreducible plane (2 variables) curve (algebraic variety of dimension 1) of degree d has genus*

$$g = \frac{(d-1)(d-2)}{2} - s,$$

where s is the number of singularities counted correctly.

So you can define elliptic curve as a "connected non-singular projective curve of genus 1". This means that such an elliptic curve can be embedded in projective space as a cubic. The converse was Seraphina's lecture, if I understood it correctly.

Let me try to build up to the statement of RR, at least. I'm following Terry Tao's 246c and moduloing out a lot of details. All Riemann surfaces today are compact.

Recall that a function $f\colon \mathbb{C} \to \mathbb{C}$ is meromorphic if it is basically holomorphic except at (at most) countably many points $\{z_1, z_2, \dots\}$ where f may have a pole, which means $1/f$, defined to be 0 at z_i, is holomorphic near the z_i.

This is saying that $f\colon \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ is a holomorphic map, where $\mathbb{C}$ and $\mathbb{C} \cup \{\infty\}$ are Riemann surfaces now, and "holomorphic" means that f composed with the charts give holomorphic maps in the usual sense; for the Riemann sphere we have

$$\begin{tikzcd} \mathbb{C} \arrow[r, "f"] & \mathbb{C} \cup \{\infty\} \arrow[d, bend right, "z"'] \arrow[d, bend left, "1/z"] \\ & \mathbb{C} \end{tikzcd}$$

and indeed we were saying that $z \circ f$ should be holomorphic at the $\mathbb{C}$-valued points, and when f is going to infinity then $1/z \circ f$ is holomorphic (we say $1/\infty = 0$).

For arbitrary Riemann surfaces I want to generalize: A function f is <u>meromorphic</u> on X if it is a holomorphic map $X \to \mathbb{C} \cup \{\infty\}$ that is not identically ∞.

Consider the free abelian group generated by the points of X. An element of this group, say $\sum_P c_P(P)$ with P ranging over a finite set of points in X, is called a <u>divisor</u>. The point is that this abstractifies "zeros and poles": if f is a nonzero meromorphic function on a Riemann surface X then it gives rise to a divisor $(f) := \sum_P \mathrm{ord}_P(f)(P)$, where $\mathrm{ord}_P(f)$ is the order of the zero, or negative the order of the pole, at P. Because X is compact this is a finite sum, so it is actually a divisor.

If a divisor is (f) for some f, then we say it is a <u>principal divisor</u>.

If $D = \sum_P a_P(P)$, then we denote the degree of D by $\deg D := \sum_P a_P$. Now if you have divisors D_1 and D_2 then you can define their sum $D_1 + D_2$ in the way you think it should be defined; you can also partial order divisors by saying $D_1 \geq D_2$ if

$$D_1 - D_2 = \sum_P ((d_1)_P - (d_2)_P)(P) \geq 0,$$

that is, if $(d_1)_P \geq (d_2)_P$ for all relevant P.

It is a mysterious black magic fact that principal divisors have degree 0 (for example, the identity map on the Riemann sphere has a zero of order 1 at the origin, but then it also has a pole of order 1 at infinity). One can check that $(fg) = (f) + (g)$ and $(f/g) = (f) - (g)$. So the principal divisors form a subgroup G of the group of divisors F_X, and two divisors are linearly equivalent if they differ by a principal divisor (if they project to the same coset in F_X/G). The group F_X/G is the divisor class group of X; if X is a nonsingular algebraic curve then this is the same as its Picard group! These are cool words.

Fix a divisor D. Define the set of functions $L(D) := \{f \text{ meromorphic}\colon (f) + D \geq 0\}$. Notice that if $f \in L(D)$ and $c \in \mathbb{C}$, then $cf \in L(D)$. Furthermore, if $f, g \in L(D)$, then $f + g \in L(D)$. So $L(D)$ is a vector space over $\mathbb{C}$. Say that D_1 and D_2 are linearly equivalent, so that $D_1 - D_2 = (f)$. Then the vector spaces $L(D_1)$ and $L(D_2)$ are isomorphic, via the map $g \mapsto fg$.

Let's reset for a bit. Take a meromorphic function f on X, where X has some charts $\varphi_\alpha \colon U_\alpha \to \mathbb{C}$. Then f gives rise to meromorphic functions $f_\alpha := z \circ f \circ \varphi_\alpha^{-1} \colon \varphi_\alpha(U_\alpha) \to \mathbb{C}$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\varphi_\alpha"'] & \mathbb{C} \arrow[d, bend right, "z"'] \arrow[d, bend left, "1/z"] \\
\varphi_\alpha(U_\alpha) & \mathbb{C}
\end{tikzcd}$$

such that for all α, β, we have

$$f_\beta(z) = f_\alpha(\varphi_\alpha \varphi_\beta^{-1}(z))$$

for all $z \in U_\alpha \cap U_\beta$. Taking the derivative of f is not so straightforward because now

$$f'_\beta(z) = (f'_\alpha(\varphi_\alpha \varphi_\beta^{-1}(z)) \circ (\varphi_\alpha \varphi_\beta^{-1})'(z)$$

so you can't just take the derivative at every chart and then push it up to a (well defined) map on X. But you can fix this:

A meromorphic 1-form on X is a collection of 1-forms $\omega_\alpha(z)dz$ for each chart $\varphi_\alpha \colon U_\alpha \to \mathbb{C}$, with $\omega_\alpha(z)$ meromorphic on $\varphi_\alpha(U_\alpha)$, so that

$$\omega_\beta(z) = \omega_\alpha(\varphi_\alpha \varphi_\beta^{-1}(z))(\varphi_\alpha \varphi_\beta^{-1})'(z).$$

for any $z \in \varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta)$. In this way, we can define the order of vanishing $\operatorname{ord}_P(\omega)$ of ω at a point $P \in X$; take a chart with $P \in U_\alpha$ and compute $\operatorname{ord}_P(\omega_\alpha)$. This is well defined. This means that we can define the divisor (ω).

Let ω_1, ω_2 be meromorphic 1-forms on X. There is a unique f so that $\omega_1 = f\omega_2$, and now $(\omega_1) = (f\omega_2) = (f) + (\omega_2)$, so (ω_1) and (ω_2) are linearly equivalent, and $L((\omega_1))$ and $L((\omega_2))$ are isomorphic. Up to linear equivalence, then (ω) is unique, and we call this the canonical divisor on X, and is denoted K.

Now Riemann Roch states:

Theorem 2. *Let X be a Riemann surface with genus g, and let D be a divisor on X. Then*

$$\dim(L(D)) - \dim(L(D - K)) = \deg(D) - g + 1.$$