Math 6670. Algebraic Geometry

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This course is an introduction to schemes in algebraic geometry. Please let me know if you spot any mistakes! These notes are ~~probably~~ definitely quite far from being typo-free. Most things in [blue font square brackets] are personal comments. Most things in [red font square brackets] are (important) announcements. I should thank David Mehrle for his notes; they inspired me to do this. If you're interested, he once took notes for a previous version of this class.

Theorem numbering unchanged since May 7, 2019. Last compiled February 6, 2020.

Contents

1 Preamble

1.1 Jan 22, 2019

We'll get a detailed syllabus on Thursday; due to the snowstorm we'll mostly give a big picture overview today. There is a Toric Varieties course that will be coordinated with this course happening at TR 10:10 in MLT 207; he encourages us to attend.

This course will be slightly different from what has been done in the past (usually, there are varieties and then schemes later). Starting Thursday we'll be talking about schemes. Unfortunately it's fairly technically demanding – it's important to know the theory of manifolds, vector/line bundles, number theory, representation theory, commutative algebra, category theory...

The classical objects of study in algebraic geometry is an algebraic variety (on the other hand, the working algebraic geometer doesn't often think about these) over a field k. Roughly, algebraic varieties are the set of solutions to a system of polynomial equations with coefficients in k.

But this course is about schemes, which is a surprisingly modern construction. We'll talk about some natural contexts where schemes pop up, and why you should care even if you're not an algebraic geometer. It'll be inherently somewhat unrigorous today.

Another first approximation of an algebraic variety is that it is an "algebraic manifold". Importantly, varieties don't need to be smooth. We want to bootstrap our understanding of manifolds onto varieties and consequently the theory of schemes.

Definition 1.1.1. Given a field k and polynomials $f_1, \ldots, f_k \in k[X_1, \ldots, X_n]$, define

$$V(f_1, \ldots, f_k) \colon \{x \in k^n \colon f_i(x) = 0 \,\forall 1 \leq i \leq k\}.$$

[I might denote $1 \leq i \leq k$ by $i \in [k]$ out of habit].

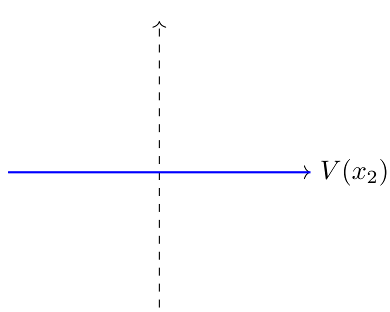
An affine variety over k is a set of the form $V(f_1, \ldots, f_k)$ for some choice of polynomials f_i. Morphisms of affine varieties are given by polynomials in the coefficients. A variety is a space that can be covered by finitely many affine varieties; a general morphism between varieties is a map such that when restricted to affine varieties in this covering is a morphism of affine varieties. △

Your intuition for these should come from manifolds (charts, atlases).

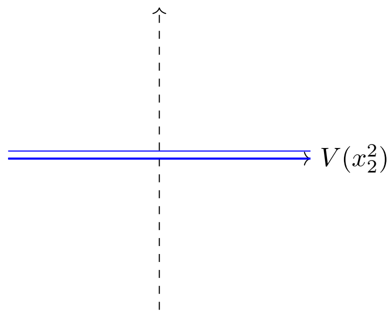
Example 1.1.2. If $k = 0$, that is, there are no equations, then we get $k^n = \mathbb{A}^n_k$, which we sometimes call affine n-space over k. △

The "affine" is referring to the fact that we do not choose a distinguished base point (ie. origin) when we work with the space.

Example 1.1.3. In $k[x_1, x_2]$, with $f_1 = x_2$, we have

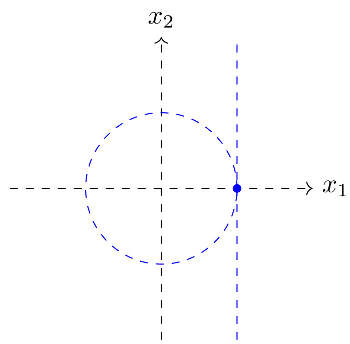


On the other hand, if $f_1 = x_2^2$, then we still have the same topological picture



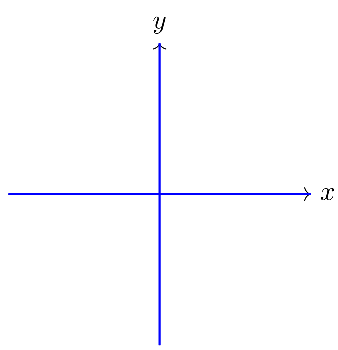
though somehow we want to account for the multiplicity of x_2^2, that is, there needs to be two lines (algebraically), even though topologically the picture is the same. This is one aspect where working with scheme explains more than working with varieties.

If we have $f_1 = \{x_1^2 + x_2^2 = 1\}$ and $f_2 = \{x_2 = 1\}$, then we have the picture



Schemes remember more than just the blue point – it also remembers that the point is the intersection of the circle and the line.

If instead we have $f_1 = \{xy = 0\}$ then we can also talk about the spaces



which is never a manifold. In this sense we are really gaining something (we can study spaces such as the one above), even though we are losing some nice structure from the theory of manifolds. △

In scheme theory there is a very basic perspective switch: instead of focusing on the set of points of an affine variety, look at the ring of polynomial functions on it. This was controversial in the beginning. Hopefully after taking this class you'll also accept this perspective switch.

Example 1.1.4. The ring of polynomial functions on $k^n = \mathbb{A}_k^n$ is $k[x_1, \dots, x_n]$. △

Example 1.1.5. The ring of polynomial functions on the circle over $\mathbb{R}$, that is, $V(f_1)$ for $f_1 = \{x^2+y^2 = 1\}$, is given by $\mathbb{R}[x, y]/\langle x^2 + y^2 - 1\rangle$. △

In many cases, the ring of polynomial functions on a variety in k^n cut out by $f_1, \ldots, f_k$, is precisely $k[x_1, \ldots, x_n]/\langle f_1, \ldots, f_k\rangle$. But the ring of polynomials in k^2 cut out by $y^2 = 0$ is $k[x, y]/\langle y\rangle$, not $k[x, y]/\langle y^2\rangle$.

As a justificiation for Grothendieck's perspective switch, we can recover both the set of points and the topological structure from this ring of functions.

Here's another sketchy defintion:

Definition 1.1.6. An affine scheme, denoted $\operatorname{Spec}(A)$, is the data of a ring A in a sense. A morphism of affine schemes $\operatorname{Spec}(A) \to \operatorname{Spec}(B)$ is a ring homomorphism $B \to A$ (that is, Spec is a contravariant functor). △

It's slightly misleading to think about $\operatorname{Spec}(A)$ as just the set of prime ideals – it is actually much more than that (it is endowed with a topology, and so on).

An affine scheme over a field k is a scheme $\operatorname{Spec}(A)$ where A has the structure of a k-algebra.

So why should a morphism be defined this way? For algebraic varieties, functions are defined on the target pulled back to the source. This leads us to scheme theory perspective switch #2: this is now the definition of a morphism; don't worry about functions given by formulas, partitions of unity, and so on.

So what happened to the points? Points of a (classical) variety can be recovered as the maximal ideals in the ring of functions. Roughly, this is Hilbert's Nullstellensatz, which is the primary vehicle used to move between the worlds of algebra and geometry:

Theorem 1.1.7. *Every maximal ideal of the ring $A = [x_1, \ldots, x_n]/\langle f_1, \ldots, f_k\rangle$ is $\langle x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n\rangle$ where $(a_1, \ldots, a_n) \in V(f_1, \ldots, f_k)$.*

Of course, this gives a bijection between maximal ideals of A and points of $V(f_1, \ldots, f_k)$. So perhaps you'd guess that the points of $\operatorname{Spec}(A)$ should be maximal ideals of A. You'd be pretty close – you gain a little more by looking at prime ideals of A instead.

A morphism of affine schemes should give a map on points. How can we construct the map? In other words, given a ring homomorphism $B \to A$, what objects on A pull back to the same kind of object on B? It is not hard to show

Proposition 1.1.8. *Given a map $f\colon B \to A$ between commutative rings with unity, for every prime ideal $\mathfrak{p} \subset A$, the set $f^{-1}(\mathfrak{p})$ is a prime ideal of B.*

This is not true generally if you replace "prime" by "maximal". This proposition might appear on our homeworks – but we'll get to administrative details on Thursday.

Definition 1.1.9. A point of an affine scheme Spec (A) is a prime ideal $\mathfrak{p}$ of A. △

Under some mild additional hypotheses, a map of affine schemes takes classical points (ie., maximal ideals) to classical points.

The point is that really $\operatorname{Spec}(A)$ should not be thought of as the set of prime ideals, but as endowed with all this extra structure that you can pull from A (that he will tell us in the future, but it will take some time). As an aside, the point set of $\operatorname{Spec}(A)$ has a natural topological structure, called the Zariski topology.

Summary. Why bother with schemes?

- Schemes have a generic (non-classical) point.
- Schemes can be defined over arbitrary rings (for example, $\operatorname{Spec}(\mathbb{Z})$ is a scheme, but not a variety for any field k).
- Schemes can have nilpotent elements. For example, $\operatorname{Spec}(k[x, y]/\langle y\rangle) \neq \operatorname{Spec}(k[x, y]/\langle y^2\rangle)$.

On Thursday we'll start talking about sheaves (not much more on varieties). We'll go "somewhat slowly".

2 Sheaves on spaces

2.2 Jan 24, 2019

There was a questionnaire to fill out. There is a website here. Last time was preamble, today we'll get a little technical. We'll define sheaves today.

Let $f\colon Y \to X$ be a continuous map between topological spaces, and let $V \subseteq U \subseteq X$ be open subsets. We define

$$\mathcal{F}(U) := \{s\colon U \to Y \text{ continuous maps such that } f \circ s = \mathrm{id}_U\}.$$

If you are familliar with vector bundles, these are sometimes called local sections over U. Note that restricting s to V, we get a map between sets

$$\begin{aligned}\rho^U_V\colon \mathcal{F}(U) &\to \mathcal{F}(V)\\ s &\mapsto s|_V\end{aligned}$$

By definition, if we have subsets $W \subseteq V \subseteq U$, we have $(s|_V)|_W = s|_W$, or equivalently, we have $\rho^U_U = \mathrm{id}$ and $\rho^U_W = \rho^V_W \circ \rho^U_V$.

Definition 2.2.1. A presheaf of sets (or of groups, rings, etc.) $\mathcal{F}$ assigns to every open subset U of X a set (or group, ring, etc.) $\mathcal{F}(U)$ and every inclusion of open subsets $V \subseteq U$ a map of sets (or groups, rings, etc.) $\rho^U_V\colon \mathcal{F}(U) \to \mathcal{F}(V)$, called "restriction maps", such that $\rho^U_U = \mathrm{id}$ and $\rho^U_W = \rho^V_W \circ \rho^U_V$ holds for all inclusions $W \subseteq V \subseteq U$. △

Definition 2.2.2. A presheaf $\mathcal{F}$ is a sheaf if it satisfies the sheaf condition/axiom: if $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of $\mathcal{U}$, and $s_i \in \mathcal{F}(U_i)$ for all $i \in I$, and $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, then there exists a *unique* $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$. △

Equivalently, this says that

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I}\mathcal{F}(U_i) \rightrightarrows \prod_{i,j}\mathcal{F}(U_i\cap U_j)$$

$$s_i \longmapsto s_i|_{U_i\cap U_j}$$

is an equalizer diagram (if the category has products).

Remark 2.2.3. Note that by an open covering of $\mathcal{U}$, we can take I to be any set, possibly empty, and each U_i is open (but possibly $U_i = \emptyset$). This has nontrivial technical consequences: for a sheaf $\mathcal{F}$, since $\emptyset = \cup_{i\in\emptyset}U_i$, then $\mathcal{F}(\emptyset) = \{*\}$, the final object in the category of sets. Consequently, if $U = V \sqcup W$, then the sheaf condition says

$$\mathcal{F}(U) = \underbrace{\mathcal{F}(V) \times_{\mathcal{F}(\emptyset)} \mathcal{F}(W)}_{\text{fiber product over } \mathcal{F}(\emptyset)} = \mathcal{F}(V) \times \mathcal{F}(W)$$

since we knew $\mathcal{F}(\emptyset)$ is the final object. So there are important consequences in the structure of sheaves. △

Example 2.2.4 (Constant sheaf)**.** Let X be a topological space and S a fixed set, and define $\mathcal{F}(U) = S$ for all opens $U \subseteq X$ (with the caveat that $\mathcal{F}(\emptyset) := \{*\}$), and for all $V \subseteq U \subseteq X$, define $\rho^U_V = \mathrm{id}_S$. This is not a sheaf in general, when X is not connected, say $X = V \sqcup W$, we must have $\mathcal{F}(X) = \mathcal{F}(V) \times \mathcal{F}(W)$. △

But there is such a sheaf such that $\mathcal{F}(U) = S$ whenever $\emptyset \neq U$ is connected. We call this S_X.

Give S the discrete topology. Then $\pi\colon X \times S \to X$ is continuous. For all $U \neq \emptyset$, define

$$S_X(U) := \{\text{locally constant sections of } \pi\colon X \times S \to X\}.$$

Remark 2.2.5. If X is a "reasonable" topological space (for instance, paracompact or something like this), the cohomology of $\mathbb{Z}_X$ is isomorphic to the singular cohomology $H^*_{\text{sing}}(X, \mathbb{Z})$. $\triangle$

Example 2.2.6. Let X be a topological space, and let $\mathcal{F}$ be given by real valued functions on U. This is a sheaf (though it is badly behaved, because the functions only have to be real valued). $\triangle$

Example 2.2.7. Let X be a differentiable manifold and let $\mathcal{F}$ be given by smooth functions on U. $\triangle$

In this sense sheaves axiomatize the gluing tricks that you see in manifold theory.

Let X be a topological space. We denote by $\mathbf{PSh}(X)$ to be the presheaves on X, $\mathbf{Sh}(X)$ to be sheaves on X, $\mathbf{PShAb}(X)$ to be presheaves of abelian groups on X, $\mathbf{Ab}(X)$ to be sheaves of abelian groups on X, and arguably most importantly $\mathbf{Mod}(\mathcal{O}_X)$ to be the sheaf of $\mathcal{O}_X$-modules on X. What's the $\mathcal{O}$ doing here?

Definition 2.2.8. A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a *sheaf* of rings (not a presheaf). $\triangle$

Definition 2.2.9. Let $(X, \mathcal{O}_X)$ be a ringed space. Then a sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$ is given by a sheaf of abelian groups $\mathcal{F}$ endowed with a map of sheaves

$$\mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$$

such that for all $U \subseteq X$ open, $\mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ makes $\mathcal{F}(U)$ into an $\mathcal{O}_X(U)$-module as a sheaf of sets. Here $(\mathcal{O}_X \times \mathcal{F})(U) := \mathcal{O}_X(U) \times \mathcal{F}(U)$. $\triangle$

A natural question one can ask is the following: given sheaves of $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, how do we define $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$? The naive thing to do is to map $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$, but in general, this is only a *presheaf*, that is to say only $\mathbf{PShMod}(\mathcal{O}_X)$ has a tensor product. This leads to interesting phenomena in algebraic geometry.

Definition 2.2.10 (Adjoint functors)**.** Let $\mathcal{C}, \mathcal{D}$ be categories with functors $U \colon \mathcal{C} \to \mathcal{D}$ and $V \colon \mathcal{D} \to \mathcal{C}$. We say U and V are adjoint if $\mathrm{Mor}_{\mathcal{C}}(X, VY) \cong \mathrm{Mor}_{\mathcal{D}}(UX, Y)$ for all $X \in \mathrm{Ob}(\mathcal{C}), Y \in \mathrm{Ob}(\mathcal{D})$, and this isomorphism is functorial in X and Y. More specifically, we say that U is a left adjoint of V, and we denote it $U \dashv V$. $\triangle$

We note that when the functor and object is clear, sometimes we drop parentheses (so we write VY instead of $V(Y)$).

Example 2.2.11. Consider the forgetful functor $U \colon \mathbf{Ab} \to \mathbf{Set}_S$ (so $M \mapsto M$), and $F \colon \mathbf{Set} \to \mathbf{Ab}_S$ mapping $S \mapsto \oplus_{s \in S} \mathbb{Z}_S$. We have $\mathrm{Mor}_{\mathbf{Set}_S}(S, UM) \cong \mathrm{Mor}_{\mathbf{Ab}}(F(S), M)$. $\triangle$

Example 2.2.12. Fix a ring homomorphism $R \to S$, and let N be an R-module and M an S-module. We have $\mathrm{Hom}_R(N, M_R) \cong \mathrm{Hom}_S(S \otimes_R N, M)$. This commutative algebra fact is just the fact that tensoring $\cdot \otimes_R S$ and restriction of scalars $(\cdot)_R$ are adjoints. $\triangle$

Example 2.2.13 (Sheafification)**.** We have $\mathbf{PSh}(X) \to \mathbf{Sh}(X)$ given by $\mathcal{F} \mapsto \mathcal{F}^\sharp$. This functor is adjoint to inclusion, that is,

$$\mathrm{Mor}_{\mathbf{PSh}(X)}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Mor}_{\mathbf{Sh}(X)}(\mathcal{F}^\sharp, \mathcal{G}).$$

$\triangle$

We'll describe what $\mathcal{F}^\sharp$ is later.

The sheafification $\mathcal{F}^\sharp$ of $\mathcal{F}$ comes with a map of presheaves $f_c \colon \mathcal{F} \to \mathcal{F}^\sharp$ such that for all morphisms of presheaves $\alpha \colon \mathcal{F} \to \iota\mathcal{G}$ there exists a unique factorization

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[dr] & \iota\mathcal{G} \\
& \iota\mathcal{F}^\sharp \arrow[u, dotted, "\exists!\alpha^\sharp"']
\end{tikzcd}$$

The idea of sheafification is that we force the sheaf condition to hold. Let $U \subseteq X$ be an open set and $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of U. Let $\mathcal{F}$ be a presehaf of sets on X. Then we have

$$\begin{aligned}\check{H}^0(\mathcal{U}, \mathcal{F}) &:= \ker\left(\prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)\right) \\ &= \{(s_i)_{i\in I} \colon s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}\ \forall i,j \in I\}\end{aligned}$$

Lemma 2.2.14. *There is a natural map*

$$\begin{aligned}\mathcal{F}(U) &\to \check{H}^0(\mathcal{U}, \mathcal{F}) \\ s &\mapsto (s|_{U_i})_{i\in I}\end{aligned}$$

If $\mathcal{F}$ is a sheaf, this is an isomorphism.

Given a presheaf $\mathcal{F}$, define $\mathcal{F}^+$ to be a presheaf with value

$$\mathcal{F}^+(\mathcal{U}) := \mathrm{colim}_U \check{H}^0(\mathcal{U}, \mathcal{F}).$$

Definition 2.2.15. A covering $\mathcal{U} = \cup_{i\in I} U_i$ is a refinement of $\mathcal{U}' = \cup_{i\in I} U'_i$ if there is an $\alpha \colon I \to I'$ such that $U_i \subseteq U'_{\alpha(i)}$ for all $i \in I$. △

Given such an α, we can define

$$\begin{aligned}\check{H}^0(\mathcal{U}', \mathcal{F}) &\to \check{H}^0(\mathcal{U}, \mathcal{F}) \\ (s_i)_{i\in I} &\mapsto (s_{\alpha(i)}|_{U_i})_{i\in I}\end{aligned}$$

This is well-defined and independent of choices f α, that is, since $U_i \subseteq U_{\alpha(i)}$ then $U_i \subseteq U_{\alpha(i)} \cap U_{\beta(i)}$, and since $U_j \subseteq U_{\beta(j)}$ then $s_{\alpha(i)}|_{U_i} = (s_{\alpha(i)}|_{U_{\alpha(i)}\cap U_{\beta(i)}})|_{U_i} = (s_{\beta(i)})|_{U_i}$.

Observation 2.2.16. Any two open coverings $\mathcal{U}_1$ and $\mathcal{U}_2$ of U have a common refinement, so the set of covers of U is a poset, and colim_U is a directed colimit! Thus,

$$\mathcal{F}(U) = \left(\coprod_{\mathcal{U}} \check{H}^0(\mathcal{U}, \mathcal{F})\right) / \sim .$$

Where $s_i \in \check{H}^0(U_i, \mathcal{F})$ for $i = 1, 2$ are equivalent if there is a common refinement where s_1 and s_2 are the same. △

So we can define restriction mappings in the following way: If $V \subseteq U \subseteq X$ are open subsets and

$$\mathcal{U} \colon U = \bigcup_{i\in I} U_i, \quad s = (s_i)_{i\in I} \in \check{H}^0(\mathcal{U}, \mathcal{F}),$$

then

$$\mathcal{V} \colon V = \bigcup_{i\in I} U_i \cap V, \quad s|_V := (s_i|_{U_i\cap V}) \in \check{H}^0(\mathcal{V}, \mathcal{F}),$$

since there exists a canonical map of presheaves $\theta \colon \mathcal{F} \to \mathcal{F}^+$ by regarding U as an open cover of itself,

$$\begin{aligned}\mathcal{F}(U) &\to \mathcal{F}^+(U) \\ s &\mapsto [s] \in \coprod \check{H}^0(\mathcal{U}, \mathcal{F})/\sim\end{aligned}$$

2.3 Jan 29, 2019

[Homeworks are due Thursday on Gradescope]
[For those interested in class field theory reading group, talk to Brian after class]

Recall that a presheaf $\mathcal{F}$ is a sheaf if for all open coverings $\{U_i\}_{i\in I}$ of U, given $s_i \in \mathcal{F}(U_i)$ for all $i \in I$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, there is a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i \in \mathcal{F}(U_i)$.

We were in the process of defining sheafification, with motivation coming from $\otimes$ for $\mathcal{O}_X$-modules. It is a left adjoint to $\iota\colon \mathbf{Sh}(X) \hookrightarrow \mathbf{PSh}(X)$. We had notation: for $\mathcal{F}$ a presheaf, we had the Čech cohomology

$$\check{H}^0(\mathcal{U}, \mathcal{F}) := \{(s_i)_{i\in I} \colon s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}\}$$

and

$$\mathcal{F}^+(U) := \operatorname{colim}_U \check{H}^0(\mathcal{U}, \mathcal{F}) = \coprod \check{H}^0(\mathcal{U}, \mathcal{F})/\sim$$

where the colimit is directed.

Definition 2.3.1. A presheaf $\mathcal{F}$ is separated if for any open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of U, the natural map

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i)$$

is injective. △

This is an important definition: the operation $\mathcal{F} \mapsto \mathcal{F}^+$ takes presheaves to separable presheaves, and it takes separable presheaves to sheaves. In light of this, sheafification will be $\mathcal{F} \mapsto (\mathcal{F}^+)^+ =: \mathcal{F}^\sharp$ [we'll make this precise in Theorem 2.3.5].

Remark 2.3.2. For many presheaves, $\mathcal{F}^+$ is already a sheaf. But not generally. △

Example 2.3.3. Let $X = \{x, y\}$ with the discrete topology. Define the presheaf $\mathcal{F}$ such that $\mathcal{F}(\emptyset) = \{0\}$, $\mathcal{F}(\{x\}) = \mathcal{F}(\{y\}) = \mathbb{Z}/2\mathbb{Z}$, and $\mathcal{F}(\{x, y\}) = (\mathbb{Z}/2\mathbb{Z})^3$, with maps $\mathcal{F}(\{x, y\}) \to \mathcal{F}(\{x\})$ is given by projection onto the first factor π_1 and $\mathcal{F}(\{x, y\}) \to \mathcal{F}(\{y\})$ is given by projection onto the second factor π_2. Since $\mathcal{F}(\{x, y\}) \to \mathcal{F}(\{x\}) \times \mathcal{F}(\{y\})$ is not injective, $\mathcal{F}$ is not separated. △

Exercise: Check that $\mathcal{F}^+$ is separated.

Example 2.3.4. Consider the constant presheaf of sets $\mathcal{F}(U) = S$ with $|S| > 1$, with restriction maps $\rho^U_V = \mathrm{id}_S$ for $V \subseteq U$ arbitrary. Then:

- $\mathcal{F}^+$ doesn't change anything except for $\mathcal{F}(\emptyset) = \emptyset$. This is just the usual constant presheaf, so not a sheaf.
- But $(\mathcal{F}^+)^+$ is the constant sheaf.

△

Theorem 2.3.5. *Let $\mathcal{F}$ be a presheaf. Then:*

1. *$\mathcal{F}^+$ is a separated presheaf.*
2. *If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf.*
3. *If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \simeq \mathcal{F}^+$*
4. *The construction $\mathcal{F} \mapsto (\mathcal{F} \xrightarrow{\theta} \mathcal{F}^+)$ is functorial in $\mathcal{F}$ and given a morphism of presheaves $\mathcal{F} \to \mathcal{G}$ there exists a factorization*

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\varphi"] \arrow[dr, "\theta"'] & & \mathcal{G} \\ & \mathcal{F}^+ \arrow[ur, dashed, "\exists!\varphi^+"'] & \end{tikzcd}$$

Proof. We'll start with part 2. Let $\mathcal{F}$ be separated. Pick $U \subseteq X$ open. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover. Pick $s_i \in \mathcal{F}^+(U_i)$, where $\mathcal{U}_i[= \{U_{ik}\}_{k\in I_i}]$ is the induced subcover of U_i. Then $s_{ik} \in \mathcal{F}(U_i \cap U_k) = \mathcal{F}(U_{ik})$, so given j, k we have $s_{ik}|_{U_{ik}\cap U_{ij}} = s_{ij}|_{U_{ik}\cap U_{ij}}$. We have $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \in \mathcal{F}^+(U_{ij})$.

Now $s_i|_{U_i\cap U_j}$ is given by $(s_{ik}|_{U_{ik}\cap U_j})_{k\in I}$. As elements of $\mathcal{F}^+(U_i \cap U_j)$ this is equal to $s_j|_{U_i\cap U_j}$ given by $(s_{jk}|_{U_{jk}\cap U_i})_{k\in I}$. We'll denote this agreement by $(*)$. Thus it remains to show that there exists a unique section $s \in \mathcal{F}^+(U)$ such that $s|_{U_i} = s_i \in \mathcal{F}^+(U_i)$.

Consider the refinement $\tilde{\mathcal{U}}$ of $\mathcal{U}$ where $\mathcal{U} = \{U_{ik}\}_{i,k\in I}$, and pick elements $s_{ik} \in \mathcal{F}(U_{ik})$. It follows by construction that $s_{ik}|_{U_{ik}\cap U_{jk'}} = s_{jk}|_{U_{ik}\cap U_{jk'}}$. It follows that (s_{ik}) defines a section in $\check{H}^0(\tilde{\mathcal{U}}; \mathcal{F})$. Since $\mathcal{F}$ is separated, we get uniqueness.

If $i = j$, we are done by our initial hypothesis. If $i \neq j$, look at the open covering $\tilde{\mathcal{U}}_{jk} = \{U_{ik} \cap U_{jk'}\}_{k,k'\in I}$ of $U_i \cap U_j$, which are the open covers corresponding to $s_i|_{U_i\cap U_j}$ and $s_j|_{U_i\cap U_j}$. Then, $(s_{ik}|_{U_{ik}\cap U_{jk'}})$ and $(s_{jk'}|_{U_{ik}\cap U_{jk'}})$ are in $\check{H}^0(\tilde{\mathcal{U}}_{ij}, \mathcal{F})$ and these define the same element under further refinements by $(*)$. We have

Lemma 2.3.6. *If $\mathcal{F}$ is separated then all maps from refinements are injective, ie. if $\mathcal{U}'$ is a refinement of $\mathcal{U}$ then*

$$\check{H}^0(\mathcal{U}; \mathcal{F}) \to \check{H}^0(\mathcal{U}', \mathcal{F}).$$

Proof. Consider $\mathcal{U}''$ given by the intersections of the two coverings, that is by open sets

$$\bigcup_{\substack{i\in I\\ j\in I'}} U_i \cap U_j.$$

This is a refinement of $\mathcal{U}$ and $\mathcal{U}'$ and $\mathcal{U}''$ are refinements of each other because $\mathcal{U}'$ refines $\mathcal{U}$. Then

$$\check{H}^0(\mathcal{U}', \mathcal{F}) = \check{H}^0(\mathcal{U}'', \mathcal{F}).$$

Given $(s_i), (t_i) \in \check{H}^0(\mathcal{U}, \mathcal{F})$ with the same image in $\check{H}^0(\mathcal{U}'', \mathcal{F})$, then $s_i, t_i \in \mathcal{F}(U_i)$ have the same image in $\prod_i \mathcal{F}(U_i \cap U_i')$, so that $s_i = t_i \in \mathcal{F}(U_i)$ because $\mathcal{F}$ is separated ◻

In other words, the lemma implies $(s_{ik}|_{U_{ik}\cap U_{jk'}}) = (s_{jk'}|_{U_{ik}\cap U_{jk'}}) \in \check{H}^0(\mathcal{U}_{ij}, \mathcal{F})$. This completes the proof of part 2 of the theorem.

Part 3 of the theorem is immediate from the definition of $\mathcal{F}+$ as a directed colimit, and the functoriality in part 4 is also immediate, but we have

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \mathcal{G} \arrow[d, "\sim"] \\ \mathcal{F}^+ \arrow[r, dashed, "\exists!"] & \mathcal{G}^+ \end{tikzcd}$$

So it remains to show part 1; if we have a raw presheaf $\mathcal{F}$, then $\mathcal{F}^+$ is separated. That is to say, given $s, s' \in \mathcal{F}^+(U)$ such that the restriction to an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of U agree, we need to show that $s = s' \in \mathcal{F}^+(U)$.

For each $\mathcal{F}^+(U_i)$ we have $s|_{U_i} = s'|_{U_i}$ there exists a refinement $\mathcal{U}_i = \{U_{ik}\}_{k\in I}$ of the cover on U_i such that $s|_{U_{ik}} = s'|_{U_{ik}} \in \mathcal{F}(U_{ik})$. Since $\tilde{\mathcal{U}} = \{U_{ik}\}_{i,k\in I}$ is an open cover of U refining $\mathcal{U}$, $s = s'$. ◻

[For a reference, there is: Bredon, Sheaf Theory.]

Definition 2.3.7. Given a presheaf $\mathcal{F}$, we say that $\mathcal{F}^\sharp := (\mathcal{F}^+)^+$ with the morphism $\mathcal{F} \xrightarrow{\theta} \mathcal{F}^+ \xrightarrow{\theta^+} (\mathcal{F}^+)^+$ is the sheafification of $\mathcal{F}$, ie. the unique sheaf such that

$$\mathrm{Mor}_{\mathbf{PSh}(X)}(\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{\mathbf{Sh}(X)}(\mathcal{F}^\sharp, \mathcal{G}).$$

△

Definition 2.3.8. Let $x \in X$ and let $\mathcal{F}$ be a presheaf on X. The stalk of $\mathcal{F}$ at x is

$$\mathcal{F}(x) := \mathrm{colim}_{U \ni x} \mathcal{F}(U) = \{(U, s) \colon s \in \mathcal{F}(U), U \ni x\}/\sim$$

where the colim runs over all neighborhoods U containing x. The equivalence relation says that $(U, s) \sim (U', s')$ if and only if there exists an open $V \subseteq U \cap U'$ with $x \in V$ such that $s|_V = s'|_V$. △

The main properties of the stalks are as follows:

- $\mathcal{F}^\sharp_x = \mathcal{F}_x = \mathcal{F}^+_x$.
- The maps $\mathcal{F} \to \mathcal{F}_x$ are functorial in $\mathcal{F}$.

We can define maps between presheaves and sheaves at the level of stalks.

Proposition 2.3.9. *Let $\varphi \colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then*

1. *φ is an isomorphism if and only if for all $x \in X$, the map $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism, if and only if for all $U \subseteq X$ open, $\varphi \colon \mathcal{F}(U) \xrightarrow{\sim} \mathcal{G}(U)$.*
2. *φ is a monomorphism if and only if for all $x \in X$, the map $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is injective, if and only if for all $U \subseteq X$ open, we have $\varphi_U \colon \mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$*
3. *φ is an epimorphism if and only if for all $x \in X$, the map $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is surjective, if and only if for all $U \subseteq X$ open and $s \in \mathcal{G}(U)$, there is $\mathcal{U} = \{U_i\}$ of U such that $s|_{U_i}$ lifts to $\mathcal{F}(U_i)$.*

We'll talk more about this next time.

2.4 Jan 31, 2019

[Homework 2 will be posted tonight.]

[The main reference of this class is Hartshorne Chapter II, heavily supplemented. We'll follow Eisenbud-Harris, Chapter I for the beginning.]

We'll make some remarks about sheafification. The point is that any presheaf can be sheafified. Again, this is important because schemes rely on sheaves in a fundamental way. We defined

$$\mathcal{F}^\sharp(U) = (\mathcal{F}^+)^+,$$

where

$$\mathcal{F}^+(U) := \operatorname{colim}_{\mathcal{U}} \check{H}^0(\mathcal{U}, \mathcal{F}) = \ker\left(\prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)\right).$$

It comes with a canonical map $\theta\colon \mathcal{F} \to \mathcal{F}^+$ corresponding to the trivial cover $\mathcal{U} = \{U\}$. The key fact to note is that sheafification is defined by a universal property: $\mathcal{F}^\sharp$ is the unique sheaf such that

$$\operatorname{Mor}_{\mathbf{PSh}(X)}(\mathcal{F}, \mathcal{G}) = \operatorname{Mor}_{\mathbf{Sh}(X)}(\mathcal{F}^\sharp, \mathcal{G}),$$

that is to say, if $\mathcal{G}$ is a sheaf, then

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr, "\theta^\sharp"'] & \mathcal{G} \\ & \mathcal{F}^\sharp \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

Proof (of the equivalence). We have

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[d] & \mathcal{F}^+ \arrow[r, "\theta^+"] \arrow[d] & \mathcal{F}^\sharp \arrow[d] \\ \mathcal{G} \arrow[r, "\cong"] & \mathcal{G}^+ \arrow[r, "\cong"] & \mathcal{G}^\sharp \end{tikzcd}$$

[Stacks 7.10.12 gives more details] □

Recall that if $x \in X$ and $\mathcal{F}$ is a presheaf, we defined stalks

$$\mathcal{F}_x := \operatorname{colim}_{U \ni x} \mathcal{F}(U) = \{(U, s)\colon s \in \mathcal{F}(U), U \text{ open}\}/\sim$$

where $(U, s) \sim (U', s')$ if there exists an open $W \subseteq U \cap U'$ with $x \in W$ such that $s|_W = s'|_W$.

Fact 2.4.1. *We have*

- $\mathcal{F}^\sharp_x = \mathcal{F}_x = \mathcal{F}^+_x$.
- *The maps* $\mathcal{F} \to \mathcal{F}_x$ *are functorial in* $\mathcal{F}$.

This gives an alternate construction of $\mathcal{F}^\sharp$: we have

$$\mathcal{F}^\sharp(U) := \{\text{functions } s\colon U \to \bigcup_{x\in U} \mathcal{F}_x\}$$

such that $s(x) \in \mathcal{F}_x$ for all $x \in U$, and such that for all $x \in U$, there is a neighborhood $V \ni x$ with $V \subseteq U$ and $t \in \mathcal{F}(V)$ such that for all $y \in V$, we have $t_y = s(y) \in \mathcal{F}_y$. This is called the "espace étalé construction" of the sheafification (as opposed to the "hypercover construction"). We remark that this construction only works when there is a notion of "point" (ie. a topology), so you can actually take stalks. On the other hand, the hypercover construction works in more generality; for example you can think about sheaves on sites.

Lemma 2.4.2. *Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then*

(a) We have that φ is an isomorphism if and only if for all $x \in X$, the induced map $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism, if and only if for all $U \subseteq X$ open, $\varphi\colon \mathcal{F}(U) \xrightarrow{\sim} \mathcal{G}(U)$.

(b) We have that φ a monomorphism if and only if for all $x \in X$, the induced map $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is injective, if and only if for all $U \subseteq X$ open, $\mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$

(c) We have that φ is an epimorphism if and only if for all $x \in X$, we have $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is surjective, if and only if for all $U \subseteq X$ open and $s \in \mathcal{G}(U)$, there exists an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of U such that $s|_{U_i}$ lifts to $\mathcal{F}(U_i)$ for all $i \in I$.

We note that item (a) does not hold for presheaves – luckily, we can sheafify.

[Apr 25, 2019: As Kabir and Jake noted, it is crucial that these φ_x arise as the induced map of a morphism φ of sheaves. Otherwise, all invertible sheaves would be $\mathscr{O}_X$!]

Proof. We note that the stalk conditions and the open set conditions are readily seen to be equivalent. We'll show the equivalence of these conditions to the conditions on φ.

For part (a), the backwards direction is easy. Fix an isomorphism φ and pick $U \subseteq X$ open. We want to show that $\varphi\colon \mathcal{F}(U) \xrightarrow{\sim} \mathcal{G}(U)$. To show injectivity, let $s, s' \in \mathcal{F}(U)$ such that $\varphi(s) = \varphi(s')$. Their images in $\mathcal{G}_x$ are the same for all $x \in U$. Since φ_x is injective, we have $s_x = s'_x \in \mathcal{F}_x$ for all $x \in U$. By definition of stalks, there exists a neighborhood U_x of x such that $s|_{U_x} = s'|_{U_x}$. But $\mathcal{U} = \{\cup_{x \in U} U_x\}$ is an open cover of U, and hence $s = s'$ by the sheaf property.

To show surjection, we pick $t \in \mathcal{G}(U)$. By our hypothesis, for all $x \in U$, there exists a neighborhood $\tilde{U}_x$ containing x, and an $s \in \mathcal{F}(\tilde{U}_x)$ such that $\varphi(s_x) = t_x \in \mathcal{G}_x$. Hence by definition of stalks, there exists an open neighborhood $U_x \ni x$ such that $U_x \subseteq \tilde{U}_x$ where $\varphi(s)|_{U_x} = t|_{U_x}$. If s' is another such section in $\mathcal{F}(U'_x)$, then we know that $\varphi(s'x) = \varphi(s')x = \varphi(s)x = \varphi(sx)$ for all $x \in U_x \cap U'_x$. By the injectivity we just proved, we must have $s' = s$ on $U_x \cap U'_x$. By the sheaf property, these local sections $\{(U_x, s)\}$ glue to a section on U.

The moral of these proofs is that if you want to talk about maps between sheaves you should think about them in terms of the stalks. □

Remark 2.4.3. In $\mathbf{Sh}(X)$, there exist fiber products and pushouts. In other words, $\mathcal{F} \to G$ is a monomorphism if and only if we have an isomorphism of sheaves $\mathcal{F} \xrightarrow{\cong} \mathcal{F} \times_{\mathcal{G}} \mathcal{F}$, if and only if $\mathcal{F}_x \cong \mathcal{F}_x \times_{\mathcal{G}_x} \mathcal{F}_x$ (by (a) in the Lemma above), if and only if $\mathcal{F}_x \to \mathcal{G}_x$ is injective. △

Remark 2.4.4. In general, since taking stalks is a filtered colimit, it commutes with colimits and finite limits (in particular, finite products). △

Recall that $\mathbf{PShAb}(X)$, $\mathbf{Ab}(X)$, $\mathbf{Mod}(\mathcal{O}_X)$ are abelian categories. Thus, given $\varphi\colon \mathcal{F} \to \mathcal{G}$ in these categories, $\ker \varphi\colon U \mapsto \ker(\mathcal{F}(U) \to \mathcal{G}(U))$ remains in each category. We also have $\mathrm{PShCoker}\varphi\colon U \mapsto \mathrm{coker}(\mathcal{F}(U) \to \mathcal{G}(U))$ is in $\mathbf{PShAb}(X)$, and

$$\mathrm{coker}(\varphi) := (\mathrm{PShCoker}(\varphi))^\sharp$$

is in $\mathbf{Ab}(X), \mathbf{Mod}(\mathcal{O}_X)$. Moreover, taking kernels and cokernels commutes with taking stalks, and given a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

it is in $\mathbf{Ab}(X)$ or $\mathbf{Mod}(\mathcal{O}_X)$ if and only if for all $x \in X$,

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x \longrightarrow 0$$

is a short exact sequence.

Definition 2.4.5 (Skyscraper Sheaf)**.** Given $x \in X$ and S a set, the skyscraper sheaf $\iota_{x*}(S)$ is the (suggestively notated) sheaf of sets

$$\iota_{x*}(S)(U) := \begin{cases} S & x \in U \\ \{*\} & x \notin U \end{cases}$$

$\triangle$

An easy property of the skyscraper sheaf is that the stalks, for $y \in X$,

$$(\iota_{x*}(S))_u = \begin{cases} S & y \in \overline{\{x\}} \\ \{*\} & y \notin \overline{\{x\}} \end{cases}$$

As a cautionary note, the closure of a point is not always a point.

Remark 2.4.6. We have:

- If S is an abeilan group (or ring, monoid, etc.), then it is not hard to see that $\iota_{x*}(S)$ is also a sheaf of abelian groups (or rings, monoids, etc.).
- If $(X, \mathcal{O}_X)$ is a ringed space, if $x \in X$ and S is an $\mathcal{O}_{X,x}$-module (where $\mathcal{O}_{X,x}$ is the completion of $\mathcal{O}_X$ with respect to x), then $\iota_{x*}(S)$ is a $\mathcal{O}_X$-module.
- There are also nice adjointness proeprties: we have $\mathrm{Mor}_{\mathbf{PSh}(X)}(\mathcal{F}, \iota_{x*}(S)) \cong \mathrm{Mor}_{\mathbf{Sets}}(\mathcal{F}_x, S)$ (and similarly for $\mathbf{Sh}(X)$, $\mathbf{Ab}(X)$, $\mathbf{Mod}(\mathcal{O}_X)$). This is why $\mathcal{F}_x^\sharp = \mathcal{F}_x$ for any presheaf $\mathcal{F}$.

$\triangle$

Proof (of $\mathcal{F}_x^\sharp = \mathcal{F}_x$). We have $\mathrm{Mor}_{\mathbf{PSh}(X)}(\mathcal{F}, \iota_{x*}(S)) = \mathrm{Mor}_{\mathbf{Sets}}(\mathcal{F}_x, S)$, while on the other hand

$$\mathrm{Mor}_{\mathbf{PSh}(X)}(\mathcal{F}, \iota_{x*}(S)) = \mathrm{Mor}_{\mathbf{Sh}(X)}(\mathcal{F}^\sharp, \iota_{x*}(S)) = \mathrm{Mor}_{\mathbf{Psh}(X)}(\mathcal{F}^\sharp, \iota_{x*}(S)) = \mathrm{Mor}_{\mathbf{Sets}}(\mathcal{F}_x^\sharp, S).$$

$\square$

Since $\iota_x \colon \{x\} \hookrightarrow X$ is continuous, $\iota_{x*}(S)$ is the "pushforward" of S_x adjoint to the "pullback" which is taking stalks.

Let's talk about tensor products. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}, \mathcal{G} \in \mathbf{Mod}(\mathcal{O}_X)$. We have a presheaf of $\mathcal{O}_X$-modules

$$\mathcal{F} \otimes_{\mathbf{PSh}(\mathcal{O}_X)} \mathcal{G}(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

Thus,

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} := (\mathcal{F} \otimes_{\mathbf{PSh}(\mathcal{O}_X)} \mathcal{G})^\sharp$$

lies in $\mathbf{Mod}(\mathcal{O}_X)$. An immediate consequence is that

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x.$$

Commutative algebra properties imply consequences for sheaves. For example, fix a ring homomorphism $A \to B$ and fix M an A-module and N a B-module. Then

$$\mathrm{Hom}_B(M \otimes_A B, N) = \mathrm{Hom}_A(M, N_A).$$

This means that if X is a topological space and $\mathcal{O}_1 \to \mathcal{O}_2$ is a map of sheaves of rings, and $\mathcal{F}$ is an $\mathcal{O}_1$-module, and $\mathcal{G}$ is an $\mathcal{O}_2$-module, then

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{F}, \mathcal{G}|_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{F} \otimes_{\mathcal{O}_1} \mathcal{O}_2, \mathcal{G}).$$

Example 2.4.7. Let $X = \{p, q\}$ be endowed with the discrete topology, and consider the constant sheaf $\mathbb{C}_X = \mathcal{O}_X$. Let $U = \{p\}$ and $V = \{q\}$. What is an $\mathcal{O}_X$-module on X?

Thus $\mathcal{O}_X(U) = \mathbb{C} = \mathcal{O}_X(V)$. Thus $\mathcal{F}(U) = K_1$ is a complex vector space (a module over $\mathbb{C}$). Similarly $\mathcal{F}(V) = K_2$ is a complex vector space. Since $\mathcal{F}$ is a sheaf we have $\mathcal{F}(\emptyset) = \{0\}$. We also have $\mathcal{F}(X) = K_1 \oplus K_2$ as a $(\mathbb{C} \oplus \mathbb{C})$-module (this is more structure than thinking of $K_1 \oplus K_2$ as a complex vector space).

Let $\mathcal{G}$ be an $\mathcal{O}_X$-module; $\mathcal{G}(U) = L_1, \mathcal{G}(V) = L_2$. We have

$$(\mathcal{F} \otimes_{\mathbf{PSh}(\mathcal{O}_X)} \mathcal{G})(U) \times (\mathcal{F} \otimes_{\mathbf{PSh}(\mathcal{O}_X)} \mathcal{G})(V) = (K_1 \otimes_{\mathbb{C}} L_1) \times (K_2 \otimes_{\mathbb{C}} L_2).$$

On the other hand

$$(\mathcal{F} \otimes_{\mathbf{PSh}(\mathcal{O}_X)} \mathcal{G})(X) = (K_1 \oplus K_2) \otimes_{\mathbb{C} \oplus \mathbb{C}} (L_1 \oplus L_2) = (K_1 \otimes_{\mathbb{C}} L_1) \oplus (K_2 \otimes_{\mathbb{C}} L_2).$$

We check for the sheaf condition:

$$(\mathcal{F} \otimes_{\mathbf{PSh}(\mathcal{O}_X)} \mathcal{G})(X) \to (\mathcal{F} \otimes_{\mathbf{PSh}(\mathcal{O}_X)} \mathcal{G})(U) \times (\mathcal{F} \otimes_{\mathbf{PSh}(\mathcal{O}_X)} \mathcal{G})(V) \to 0$$

This is in fact a sheaf. △

2.5 Feb 5, 2019

Last time we defined skyscraper sheaves: fix a set S and an $x \in X$; the skyscraper sheaf suggestively denoted

$$\iota_{x*}(S)(U) = \begin{cases} S & \text{if } x \in U \\ \{*\} & \text{if } x \notin U \end{cases}$$

The key point is that it is the pushforward of $S_{\{x\}}$ on $\{x\}$ that is adjoint to "ι^{-1}" (taking stalks).

We also talked about tensor products. If $(X, \mathcal{O}_X)$ is a ringed space, and $\mathcal{F}, \mathcal{G}$ are sheaves of $\mathcal{O}_X$-modules, we have a presheaf

$$(\mathcal{F} \otimes_{\mathbf{PSh}\mathcal{O}_X} \mathcal{G})(U) := \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

and so we can define

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) := (\mathcal{F} \otimes_{\mathbf{PSh}\mathcal{O}_X} \mathcal{G})^\sharp$$

Let's give an example where $\mathcal{F} \otimes_{\mathbf{PSh}\mathcal{O}_X} \mathcal{G}$ is not a sheaf. Let $X = \mathbb{R}$ and $\mathcal{O}_X = \mathbb{Z}_X$. Define $\mathcal{F} = \iota_{0*}(\mathbb{Z}) \oplus \iota_{1*}(\mathbb{Z})$. Note that $\mathcal{O}_{X,0} = \mathbb{Z}$, and $\mathcal{O}_{X,1} = \mathbb{Z}$. We see that

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F} = \mathcal{F}$$

by looking at stalks. In particular,

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}(X) = \mathbb{Z} \oplus \mathbb{Z}.$$

On the other hand,

$$\mathcal{F} \otimes_{\mathbf{PSh}\mathcal{O}_X} \mathcal{F}(X) \cong (\mathbb{Z} \oplus \mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z}) = \mathbb{Z}^4.$$

Let's talk about the functoriality of sheaves.

There is a philosophy called the 6-functor formalism: if you want to develop a sheaf theory on a new kind of space, everything can be deduced from the six functors $f_*, f^*, f_!, f^!, \otimes^{\mathbb{L}}, \mathrm{RHom}$. Hence you should figure out what these should be and the homological algebra should give everything else.

Let $f\colon X \to Y$ be a continuous map. This induces $f_*\colon \mathbf{PSh}(X) \to \mathbf{PSh}(Y)$ given by

$$(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V))$$

for V ope in Y, with obvious restriction maps from $\mathcal{F}$. If $\mathcal{F}$ is a sheaf, then $f_*\mathcal{F}$ is a sheaf. Proof is in homework 2.

We want to define the adjoint of f_*, which we usually denote f_p for reasons that will be clear soon:

$$\begin{tikzcd} \mathbf{PSh}(X) \arrow[r, "f_*"'] & \mathbf{PSh}(Y) \arrow[l, bend right, "f_p"'] \end{tikzcd}$$

We note that the adjoint of ι_{x*} is like taking stalks. So we also need a colimit. Here's a naive idea: suppose we analogously try to define

$$f^*\mathcal{G}(U) := \mathcal{G}(f(U))$$

for U open in X. Unfortunately $f(U)$ is not always open, so we need to "approximate by open sets". In fact, this is precisely what we will do.

Definition 2.5.1. Given $\mathcal{G}$ in $\mathbf{PSh}(Y)$, define $f_p(\mathcal{G})$ in $\mathbf{PSh}(X)$

$$f_p(\mathcal{G})(U) := \operatorname{colim}_{\substack{f(U) \subseteq V \\ V \text{ open}}} \mathcal{G}(V)$$

with restriction maps, for $U_1 \subseteq U_2 \subseteq X$ open

$$\begin{tikzcd}
f_p(\mathcal{G})(U_2) \arrow[r, dashed] \arrow[d, leftrightarrow, "="'] & f_p(\mathcal{G})(U_1) \arrow[d, leftrightarrow, "="] \\
\operatorname{colim}_{V \supseteq f(U_2)} \mathcal{G}(V) \arrow[r] & \operatorname{colim}_{V \supseteq f(U_1)} \mathcal{G}(V)
\end{tikzcd}$$

where the map in the bottom row is given by $\mathrm{id}_{\mathcal{G}(V)}$, because if $V \supseteq f(U_2)$ then $V \supseteq f(U_1)$. △

As an immediate corollary we have $(f_p\mathcal{G})_x = \mathcal{G}_{f(x)}$.

This is not generally a sheaf. For example, let $Y = \{u\}$ and $\mathcal{G}$ any sheaf in Y. Then $f_p\mathcal{G}$ is a constant presheaf.

Lemma 2.5.2. *We have* $\mathrm{Mor}_{\mathbf{PSh}(X)}(f_p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathbf{PSh}(X)}(\mathcal{G}, f_*\mathcal{F})$.

Proof. A map from a colimit is a compatible collection of maps on objects in the colimit, that is,

$$\varphi\colon f_p\mathcal{G}(U) = \operatorname{colim}_{V \supseteq f(U)} \mathcal{G}(V) \to \mathcal{F}(U)$$

is given by

$$\varphi_{U,V} : \mathcal{G} \to \mathcal{F}(U)$$

for all diagrams

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, "f"'] & X \arrow[d, "f"] \\
V \arrow[r, hook] & Y
\end{tikzcd}$$

compatible with restrictions, that is, whenever

$$\begin{tikzcd}
U' \arrow[r, hook] \arrow[d, "f"'] & U \arrow[r, hook] \arrow[d, "f"'] & X \arrow[d, "f"] \\
V' \arrow[r, hook] & V \arrow[r, hook] & Y
\end{tikzcd}$$

then the square

$$\begin{tikzcd}
\mathcal{F}(U') & \mathcal{F}(U) \arrow[l] \\
\mathcal{G}(V') \arrow[u, "\varphi_{U',V'}"] & \mathcal{G}(V) \arrow[l] \arrow[u, "\varphi_{U,V}"']
\end{tikzcd}$$

commutes. On the right hand side, a map $\psi\colon \mathcal{G} \to f_*\mathcal{F}$ is given by a collection of maps

$$\psi_V : \mathcal{G}(V) \to f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

and compatible with restrictions. For the forward direction, it is enough to define $\psi_V := \varphi_{f^{-1}(V),V}$ and check straightforward details. The backwards direction is trickier. We define $\varphi_{U,V} := \rho_U^{f^{-1}(V)} \circ \psi_V$. Let's check that we get maps LHS to RHS to LHS, that is, $\varphi \mapsto \psi \mapsto \varphi'$:

$$\varphi'_{U,V} = \rho_U^{f^{-1}(V)} \circ \psi_V = \rho_U^{f^{-1}(V)} \circ \varphi_{f^{-1}(V,V)} = \varphi_{U,V},$$

and that we get maps RHS to LHS to RHS, that is, $\psi \mapsto \varphi \mapsto \psi'$:

$$\psi'_V = \varphi_{f^{-1}(V),V} = \rho^{f^{-1}(V)}_{f^{-1}(V)} \circ \psi_V = \psi_V.$$

□

We remark that any $\varphi \in \mathrm{Mor}_{\mathbf{PSh}(X)}(f_p\mathcal{G}, \mathcal{F})$ or $\psi \in \mathrm{Mor}_{\mathbf{PSh}(X)}(\mathcal{G}, f_*\mathcal{F})$ gives a map $\mathcal{G}_{f(x)} \to \mathcal{F}_x$.

Definition 2.5.3. If $\mathcal{G}$ is a sheaf on Y, define the inverse image sheaf

$$f^{-1}(\mathcal{G}) := (f_p(\mathcal{G}))^\sharp.$$

△

Remark 2.5.4. In topology this is f^*, but we'll need the notation f^* later for something a little more fundamental. △

Proposition 2.5.5. *We have* $\mathrm{Mor}_{\mathbf{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{G}) = \mathrm{Mor}_{\mathbf{Sh}(Y)}(\mathcal{G}, f_*\mathcal{F})$.

Proof. We have

$$\begin{aligned} Mor_{\mathbf{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{G}) &= \mathrm{Mor}_{\mathbf{PSh}(X)}(f_p\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{\mathbf{PSh}(Y)}(\mathcal{G}, f_*\mathcal{F}) \\ &= \mathrm{Mor}_{\mathbf{Sh}(Y)}(\mathcal{G}, f_*\mathcal{F}) \end{aligned}$$

because $f_*\mathcal{F}$ is a sheaf. □

Corollary 2.5.6. *We have a canonical isomorphism* $(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}$.

Proof. Let $\iota_{x*}S$ be a skyscraper in $\mathbf{Sh}(X)$. Then

$$\begin{aligned} \mathrm{Mor}_{\mathbf{Sh}(X)}(f^{-1}\mathcal{G}, \iota_{x*}S) &= \mathrm{Mor}_{\mathbf{Sh}(Y)}(\mathcal{G}, (f \circ \iota_x)_*S) \\ &= \mathrm{Mor}_{\mathbf{Sh}(Y)}(\mathcal{G}, \iota_{f(x)*}S) \\ &= \mathrm{Mor}_{\mathbf{Sets}}(\mathcal{G}_{f(x)}, S) \end{aligned}$$

implies $(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}$. □

Let's talk about morphisms of ringed spaces.

Our goal is as follows: Given a ring homomorphism $A \to B$, we want to define morphisms between affine schemes (examples of ringed spaces)

$$\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A) \quad \text{and} \quad \mathrm{Mor}_{\mathbf{Rings}}(A, B) = \mathrm{Mor}_{\mathbf{AffSch}}(\mathrm{Spec}\, B, \mathrm{Spec}\, A).$$

We'll give some motivation from smooth manifolds. Let $\psi\colon M \to N$ be a smooth map between smooth manifolds. We can think of it as a map between ringed spaces $(M, C^\infty_M) \to (N, C^\infty_N)$. Suppose $U \subseteq M$, $V \subseteq N$ are open such that if $h \in C^\infty(N)(V)$ then $h \circ \psi \in C^\infty(M)(U)$, that is, we have the commutative diagram

$$\begin{tikzcd} & U \arrow[r, hook] \arrow[d, "\psi"] \arrow[dl, dashed] & M \arrow[d, "\psi"] \\ \mathbb{R} & V \arrow[l, "h"'] \arrow[r, hook] & N \end{tikzcd}$$

In other words, this gives a map

$$\begin{tikzcd} \psi_p C^\infty_N \arrow[r] \arrow[dr, dashed] & C^\infty_M \\ & \psi^{-1} C^\infty_N \arrow[u] \end{tikzcd}$$

or equivalently a map $C^\infty_N \to \psi_* C^\infty_M$.

Definition 2.5.7. A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ where $f\colon X \to Y$ is continuous and $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ (equivalently, $f^\sharp\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$), that is to say, a collection of continuous maps

$$f^\sharp_{U,V}\colon \mathcal{O}_Y(V) \to \mathcal{O}_X(U)$$

for all diagrams

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "f"] & X \arrow[d, "f"] \\ V \arrow[r, hook] & Y \end{tikzcd}$$

△

Given such a morphism $(f, f^\sharp)$, we get:

(i) $f_*\colon \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_Y)$ given by $\mathcal{F} \mapsto f_*\mathcal{F}$, where $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module regarded as an $\mathcal{O}_Y$-module via $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$.

(ii) $f^*\colon \mathbf{Mod}(\mathcal{O}_Y) \to \mathbf{Mod}(\mathcal{O}_X)$ given by $\mathcal{G} \mapsto f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X =: f^*\mathcal{G}$.

Note that $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is the canonical one adjoint to $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$. Everything we did in this course so far led up to $f^*\mathcal{G}$ above.

Theorem 2.5.8. *We have adjointness:* $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F})$.

Proof. The proof is as [“]trivial[”] as always

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) &= \mathrm{Hom}_{f^{-1}\mathcal{O}_Y}(f^{-1}\mathcal{G}, (\mathcal{F})_{f^{-1}\mathcal{O}_Y}) \\ &= \mathrm{Hom}\mathcal{O}_X(f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X, \mathcal{F}) \\ &= \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}). \end{aligned}$$

□

Corollary 2.5.9. *We have* $f^*\mathcal{O}_Y = \mathcal{O}_X$ *and* $(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \otimes O_{X,x}$.

Let's talk about schemes.

Definition 2.5.10. A locally ringed space $(X, \mathcal{O}_X)$ is a ringed space such that all stalks $\mathcal{O}_{X,x}$ are local rings. A morphism of locally ringed spaces is a morphism $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces such that for all $x \in X$, the map $f^\sharp_x\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism of local rings (in particular maps maximal ideals to lie in the maximal ideal). △

Lemma 2.5.11. *If* X, Y *are ringed saces and* $f\colon X \to Y$ *is an isomorphism (in the category of ringed spaecs), then* f *is also an isomorphism in the category of locally ringed spaces.*

We want to work in this generality because given $(X, \mathcal{O}_X)$ locally ringed spaces and $j\colon U \hookrightarrow X$ inclusion of some open set, then $(U, j^{-1}\mathcal{O}_X =: \mathcal{O}_X|_U)$ is also a locally ringed space. For example:

Example 2.5.12. Let R be a commutative ring. Then $(\operatorname{Spec} R, R)$ is a locally ringed space. These are affine schemes. △

3 Schemes

3.6 Feb 7, 2019

Last time, we defined pushforwards, which were easy, and pullbacks, which were more complicated. In particular, if $f\colon X \to Y$ we can define the pushforward $f_*\colon \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$; on the other hand we needed to define f_p and $f^{-1} := (f_p)^\sharp$. This defined $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$, which became the pullback $f^*\colon \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$.

Let's talk about schemes as ringed spaces (and locally ringed spaces). The motivation is that we want

$$\mathrm{Mor}_{\mathbf{RingedSpaces}}(\operatorname{Spec} A, \operatorname{Spec} B) = \mathrm{Mor}_{\mathbf{Rings}}(B, A).$$

However, there is an issue: this is not literally true, so we need to add some extra algebraic data. For example:

Example 3.6.1. Let R be a discrete valuation ring. Then $\operatorname{Spec} R = \{\langle 0\rangle, \mathfrak{m}\}$. For a DVR, $\mathfrak{m}$ is usually called the closed point, whereas $\langle 0\rangle$ is called the open point. Suppose that K is the faction field of R, so that $\operatorname{Spec} K = \{\langle 0\rangle\}$ (which is also called the open point, as is the case in general with domains). We have a ring homomorphism $R \hookrightarrow K$ sending $r \mapsto r/1$.

But there are two maps of ringed spaces $\operatorname{Spec} K \to \operatorname{Spec} R$: the first maps the open point to the closed point, giving a map $R = \mathcal{O}_{\operatorname{Spec} R, \mathfrak{m}} \to \mathcal{O}_{\operatorname{Spec} K, \langle 0\rangle}$, and the second maps the open point to the open point, which induces a map $K = \mathcal{O}_{\operatorname{Spec} R, \langle 0\rangle} \mapsto \mathcal{O}_{\operatorname{Spec} K, \langle 0\rangle} = K$. △

Definition 3.6.2. We'll define locally ringed spaces:

(a) A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that all stalks $\mathcal{O}_{X,x}$ are local rings.

(b) A morphism of locally ringed spaces is a morphism

$$(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

of ringed spaces, suh that for all $x \in X$ we have that $f^\sharp_X\colon \mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism of rings.

△

Let $x \in X$. We write $K(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$.

We view functions $f \in \mathcal{O}_X$ and evaluate them, then pass to the residue field, that is,

$$\text{“}f(x) := \text{quotient of } f \text{ in } K(x)\text{”}.$$

Recall that an affine scheme is $(\operatorname{Spec} R, R)$ for R a ring. Note that $\operatorname{Spec} R$ is the set of prime ideals in R, whereas R should be thought of as functions on $\operatorname{Spec} R$. To make this a locally ringed space, we need to say what the topology on $\operatorname{Spec} R = X$ is.

Given an ideal $\mathfrak{a} \subset R$, we have $V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec} R \colon \mathfrak{p} \supseteq \mathfrak{a}\}$, called the algebraic subsets. These will be our closed sets of X. The corresponding topology is the called the Zariski topology. This is a bad topology to work with (non-Hausdorff etc.), but it is the right topology to work with when trying to glue things together.

An important role is played by distinguished open sets: for $f \in R$, define $D(f) = X \backslash V(\langle f\rangle)$. An important fact is that the $D(f)$, for $f \in R$, form a basis for the topology on $\operatorname{Spec} R$.

If $V(\mathfrak{a})$ is a closed set and $\mathfrak{p} \notin V(\mathfrak{a})$, then $\mathfrak{p} \not\supseteq \mathfrak{a}$, so there is $f \notin \mathfrak{p}$. Then $\mathfrak{p} \in D(f)$ and $D(f) \cap V(\mathfrak{a}) = \emptyset$.

Lemma 3.6.3. *Let R be a ring and let M be an R-module. If $f, g \in R$ are such that $D(g) \subseteq D(f)$ (that is, $V(\langle f\rangle) \subseteq V(\langle g\rangle)$), then*

- *f is invertible in R_g*
- *$g^e = af$ for some $e \geq 1, a \in R$*
- *There exists a canonical map $R_f \to R_g$*
- *There exists a canonical R_f-module homomorphism $M_f \to M_g$.*

Also, any open covering of

$$D(f) = \bigcup_{i=1}^{n} D(g_i).$$

If $g_1, \ldots, g_n \in R$ and $D(f) \subseteq \cup_i D(g_i)$, then $g_1, \ldots, g_n$ generate the unital ideal in R_f.

Now let $\mathcal{B}$ be the collection of distinguished opens of $\operatorname{Spec} R = X$. This collection is sometimes called the standard opens. Recall that by HW 2, to define a sheaf on X it is enough to define it on $\mathcal{B}$.

Given an R-module M, we have the "tilde construction". We define a presheaf $\widetilde{M}$ on $\mathcal{B}$ via $\widetilde{M}(D(f)) := M_f$. This is well defined: consider $D(f) = D(g)$. Then f is invertible in R_g, and hence $M_{gf} = (M_g)_f = M_g$. Similarly we have $M_{gf} = M_f$.

If $D(g) \subseteq D(f)$, then $\rho^{D(f)}_{D(g)} \colon \widetilde{M}(D(f)) = M_f \to M_g = \widetilde{M}(D(g))$ is just the canonical R_f-module homomorphism guaranteed by Lemma 3.6.3.

Note that the sheaf axiom here says that we have a standard covering of

$$D(f) = \bigcup_{i=1}^{n} D(g_i),$$

where here we are using implicitly the quasi-compactness of $\operatorname{Spec} R$, then we have the sequence

$$0 \longrightarrow \widetilde{M}(D(f)) \longrightarrow \bigoplus_i \widetilde{M}(D(g_i)) \longrightarrow \bigoplus_{i,j} \widetilde{M}(D(g_i g_j))$$

and get the exactness by the gluing lemma on the sheafification and $D(g) \subseteq D(f)$ implies f is a unit in R_g. This means that $D(g) = D(gf)$ and $M_g = (M_f)_g$ and $M_{g_i g_j} = (M_f)_{g_i g_j}$.

Fact 3.6.4. *There is an equivalence of categories $\mathbf{Sh}_{\mathcal{B}}(\operatorname{Spec} R) = \mathbf{Sh}(\operatorname{Spec} R)$, where we usually take $\mathcal{B} = \{D(f)\}_{f \in R}$, and $\mathbf{Sh}_{\mathcal{B}}$ denotes the category of $\mathcal{B}$-sheaves.*

Proof. Is in the homework. But one remark: say $U \subseteq X$ is an open set. Let $\Gamma(U, \widetilde{M})$ denote the set of elements $\{s_p\} \in \prod_{p \in U} M_p$ for which there exists a covering of U by $D(f_\alpha)$s together with elements $s_\alpha \in M_{f_\alpha}$ such that s_β equals s_α under the restriction $M_{f_\alpha} \to M_\beta$. We have the natural restriction maps $V \subseteq U$ by coordinate projection $\prod_{p \in U} M_p \to \prod_{p \in V} M_p$. One can check that $\Gamma(-, \widetilde{M})$ is a sheaf. By the gluing lemma, we have $\Gamma(D(f), \widetilde{M}) = M_f$. This gives us the equivalence of categories needed. □

Remark 3.6.5. Hopefully this clarifies the $\widetilde{M}$ construction in Hartshorne II.2. △

Summary. There exists a *unique* sheaf of rings $\mathcal{O}_{\operatorname{Spec} R}$ such that $\mathcal{O}_{\operatorname{Spec} R}(D(f)) = \widetilde{R}(D(f)) = R_f$. Moreover, for any R-module M, there exists a unique sheaf of $\mathcal{O}_{\operatorname{Spec} R}$-modules $\mathcal{F} = \widetilde{M}$ such that

$$\mathcal{F}(D(f)) = \widetilde{M}(D(f)) = M_f$$

as a $\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R_f$-module. In particular, $\Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) = R$.

Definition 3.6.6. An affine scheme is a locally ringed space isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R. △

Definition 3.6.7. A scheme is a locally ringed space such that every point has an open neighborhood which is an affine scheme. △

Remark 3.6.8. Let $\mathcal{F} = \widetilde{M}$.

- Then a point $x \in \operatorname{Spec} R$ corresponds to $\mathfrak{p} \subseteq R$. We have
$$\mathcal{F}_x = \operatorname{colim}_{D(f) \ni x} \mathcal{F}(D(f)) = \operatorname{colim}_{f \in R, f \in \mathfrak{p}} M_f = M_{\mathfrak{p}}$$
(since we know $f, g \in R \backslash \mathfrak{p}$ implies $fg \in R \backslash \mathfrak{p}$ and $D(fg) \subseteq D(f) \cap D(g)$).

- The functor $\mathcal{F} \to \mathcal{F}_x$ (or $\widetilde{M} \to M_{\mathfrak{p}}$) is exact.

- If $\varphi \colon \widetilde{M} \to \widetilde{N}$ is a $\mathcal{O}_{\operatorname{Spec} R}$-module map, then we get the induced map on global sections, that is, we get $\varphi \colon M \to N$ back.

△

Example 3.6.9 (Why we define $\widetilde{M}$ on just $D(f)$)**.** Consider $X = \operatorname{Spec} k[x, y]$, and let $U = X \backslash \{0\}$, where $\{0\}$ here corresponds to the maximal ideal $\langle x, y \rangle \subseteq k[x, y]$. We claim that $\mathcal{O}_X(U) = k[x, y]$, This is because we have
$$0 \longrightarrow \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(D(x)) \oplus \mathcal{O}_X(D(y)) \longrightarrow \mathcal{O}_X(D(xy))$$
which implies
$$\mathcal{O}_X(U) = \ker(k[x, y, 1/x] \oplus k[x, y, 1/y] \to k[x, y, 1/xy]) = k[x, y]$$
This is reminiscent of Hartog's theorem in complex analysis. △

3.7 Feb 12, 2019

Last time we talked about the $\widetilde{\ }$ construction for affine schemes. In particular, let $X = \operatorname{Spec} R$ and let M be an R-module. Then $\widetilde{M}(D(f)) = M_f$, which gives a unique sheaf of $\mathcal{O}_{\operatorname{Spec} R}$ modules. In the special case that $M = R$, then we get this sheaf of rings $\mathcal{O}_{\operatorname{Spec} R}$. We also defined affine schemes, which are locally ringed spaces isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$. A scheme is a locally ringed space such that every point has a local neighborhood that is an affine scheme. Then we defined

$$\Gamma(U, \widetilde{M}) := \left\{ \{s_p\} \in \prod_{p \in U} M_p \colon \exists \text{ covering } \mathcal{U} = \bigcup_\alpha D(f_\alpha) \text{ and } s_\alpha \in M_{f_\alpha} \text{ such that } s_p = s_\alpha \text{ under } M_{f_\alpha} \to M_p \right\}.$$

Our main goal was to establish

$$\operatorname{Mor}_{\mathbf{Rings}}(A, B) = \operatorname{Mor}_{\mathbf{RingedSpaces}}(\operatorname{Spec} B, \operatorname{Spec} A)$$

but we saw that this was not quite enough (ie. there were not enough morphisms on one side). Today we'll see that the corrected statement:

$$\operatorname{Mor}_{\mathbf{Rings}}(A, B) = \operatorname{Mor}_{\mathbf{LocRingedSpaces}}(\operatorname{Spec} B, \operatorname{Spec} A).$$

Example 3.7.1. Consider $(\operatorname{Spec} \mathbb{Z}, \mathcal{O}_{\operatorname{Spec} \mathbb{Z}})$. We have prime ideals $\langle 0 \rangle$ and $\langle p \rangle$ for all primes p, where the $\langle p \rangle$ are classical points, and the $\langle 0 \rangle$ is stitched everywhere. We call $\langle 0 \rangle$ an open point.

Functions on $\operatorname{Spec} \mathbb{Z}$ are elements $n \in \mathbb{Z}$. For example, if $n = 12$, then the function 12 evaluated at the point $\langle 2 \rangle$ is an element of the quotient field $\mathbb{Z}/2\mathbb{Z}$; we have $12(2) = 0 \in \mathbb{Z}/2\mathbb{Z}$. Similarly we have $12(3) = 0 \in \mathbb{Z}/3\mathbb{Z}$, and $12(5) = 2 \in \mathbb{Z}/5\mathbb{Z}$, $12(7) = 5 \in \mathbb{Z}/7\mathbb{Z}$, $12(11) = 11 \in \mathbb{Z}/11\mathbb{Z}$, and so on.

Note that the standard opens in our space are $D(n)$, and $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}(D(n)) = \mathbb{Z}[1/n]$. △

Theorem 3.7.2. *Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be locally ringed paces and assume Y is affine, say, isomorphic to* $\operatorname{Spec} R$. *Then*

$$\operatorname{Mor}_{\mathbf{LocRinSp}}(X, Y) = \operatorname{Hom}(R, \Gamma(X, \mathcal{O}_X))$$

and this equality is functorial in X.

Remark 3.7.3. By Yoneda's lemma, such maps for all X uniquely determine Y as a locally ringed space (because $R = \Gamma(Y, \mathcal{O}_Y)$). △

Proof of Theorem 3.7.2. Given $(\psi, \psi^\sharp) \in \operatorname{Mor}_{\mathbf{LocRinSp}}(X, Y)$ we get a ring homomorphism from $\psi^\sharp$

$$\alpha \colon R = \Gamma(Y, \mathcal{O}_Y) \to \Gamma(Y, \psi_* \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X).$$

We want to express ψ as a map of sets in terms of α: given $x \in X$, how do we define $\psi(x) \in Y$? We have the diagram

$$\begin{array}{ccc} \Gamma(X, \mathcal{O}_X) & \longrightarrow & \mathcal{O}_{X,x} \\ \uparrow & & \uparrow \\ R = \Gamma(Y, \mathcal{O}_Y) & \longrightarrow & \mathcal{O}_{Y, \psi(x)} \end{array}$$

which commutes because it is a map between locally ringed spaces.

Well, if $\psi(x) = \mathfrak{p} \in \operatorname{Spec} R$, then we get

$$\begin{array}{ccc} & & \mathcal{O}_{X,x} \\ & & \uparrow \\ R & \longrightarrow & R_\mathfrak{p} \end{array}$$

and applying Spec gives

$$\begin{tikzcd} & \operatorname{Spec} \mathcal{O}_{X,x} \arrow[d] \\ \operatorname{Spec} R & \operatorname{Spec} R_{\mathfrak{p}} \arrow[l] \end{tikzcd}$$

Since $R_{\mathfrak{p}} \to \mathcal{O}X, x$ is a local homomorphism of local rings, the unique closed point of $\operatorname{Spec} \mathcal{O}_{X,x}$ must be mapped to the closed point of $\operatorname{Spec} R_{\mathfrak{p}}$ (note that this is one reason why we need locally ringed spaces). Thus, this closed point is mapped to $\mathfrak{p}$ under $\operatorname{Spec} R_{\mathfrak{p}} \to \operatorname{Spec} R$.

Hence $\psi(x) \in Y$ corresponds to $\mathfrak{p} \subseteq R$ which is the kernel of the composite of the maps

$$R \xrightarrow{\alpha} \Gamma(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{X,x} \longrightarrow K(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$$

□

Lemma 3.7.4. *Given any locally ringed space $(X, \mathcal{O}_X)$ and any global section $f \in \Gamma(X, \mathcal{O}_X)$, we want to show that the set $D(f) := \{x \in X : f \notin \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}\}$ is open in X and furthermore that $f \in \Gamma(D(f), \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ is called the sheaf of units in $\mathcal{O}$.*

Proof. For all $x \in D(f)$ we have $f \notin \mathfrak{m}_x$. This means that f is a unit in $\mathcal{O}_{X,x}$, and hence there is $g \in \mathcal{O}_{X,x}$ such that $fg = 1 \in \mathcal{O}_{X,x}$. By the definition of a stalk we know there exists $U \ni x$ sch that $fg = 1$ on U. This means that $f \in \Gamma(U, \mathcal{O}_X^*)$. Since this works for all $x \in D(f)$, we see that $D(f)$ is open. Furthermore, we get $f \in \Gamma(D(f), \mathcal{O}_X^*)$. □

From a ring homomorphism $\alpha\colon R \to \Gamma(X, \mathcal{O}_X)$ we want to construct a morphism of locally ringed spaces $(\psi, \psi^\sharp)$. Set

$$\psi(x) := \ker\left(R \xrightarrow{\alpha} \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x} \to K(x)\right).$$

It turns out that ψ is continuous (because we have $\psi^{-1}(D(f)) = D(\alpha(f))$, which is open by our lemma). For $\psi^\sharp\colon \mathcal{O}_Y \to \psi_*\mathcal{O}_X$, it's enough to show on standard opens that

$$\begin{tikzcd} R_f \arrow[r, dashed] & \Gamma(D(\alpha(f)), \mathcal{O}_X) \\ R \arrow[u] \arrow[r, "\alpha"] & \Gamma(X, \mathcal{O}_X) \arrow[u] \end{tikzcd}$$

and since $R_f = \Gamma(D(f), \mathcal{O}_Y)$ and $\Gamma(D(\alpha(f)), \mathcal{O}_X) = \Gamma(D(f), \psi_*\mathcal{O}_X)$, then we get an induced map

$$\Gamma(D(f), \mathcal{O}_Y) \to \Gamma(D(f), \psi_*\mathcal{O}_X).$$

By the lemma, $\alpha(f)$ is a unit in $\Gamma(D(\alpha(f)), \mathcal{O}_X)$. Using the universal property of localization, we see that α lifts to a map $R_f \to \Gamma(D(\alpha(f)), \mathcal{O}_X)$. Since we've defined $\psi^\sharp$ on distinguished open sets, we are done. To end, you need to check that this functor is inverse to $\Gamma(Y, -)$.

Summary. There is an antiequivalence of categories

$$\begin{tikzcd} \mathbf{AffSch} \arrow[r, bend left, "\Gamma(\cdot)"] & \mathbf{Rings} \arrow[l, bend left, "\operatorname{Spec}"] \end{tikzcd}$$

and our theorem says that Spec (as a functor) is fully faithful.

Corollary 3.7.5. *If Y is an affine scheme and $f \in \Gamma(Y, \mathcal{O}_Y)$, then $(D(f), \mathcal{O}_Y|_{D(f)}) \cong (\operatorname{Spec} R_f, \mathcal{O}_{\operatorname{Spec} R_f})$. Thus, any scheme has a topology whose basis is given by affine opens.*

The details are left to the homework. The moral is that whenever there is something you want to understand locally, you can reduce to the case where we have an affine open and then reason from that point (and you can even assume it's of this nice form).

Immersions of locally ringed spaces. Let $(X, \mathcal{O}_X)$ be a locally ringed space, with $U \subseteq X$ open. Then $(U, \mathcal{O}_X|_U)$ is an open subspace of locally ringed spaces.

Definition 3.7.6. An open immersion of locally ringed spaces is a morphism $j\colon V \to Y$ such that j is a homeomorphism onto an open subset of Y and $j^\sharp\colon j^{-1}\mathcal{O}_Y \to \mathcal{O}_V$ is an isomorphism. △

Lemma 3.7.7. *Let $X \xrightarrow{f} Y$ be a morphism of locally ringed spaces. Let $U \subseteq X$ and $V \subseteq Y$ be opens such that $f(U) \subseteq V$. Then the following diagram commutes (in the category of locally ringed spaces):*

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "f|_U"'] & X \arrow[d, "f"] \\ V \arrow[r, hook] & Y \end{tikzcd}$$

Proof. This follows from the previous theorem and the homework problem on open subsets. □

We note that closed immersions are trickier to define. The definition in Hartshorne for closed immersions between schemes doesn't work in general for locally ringed spaces.

Example 3.7.8. Let $X = \mathbb{R}$ and $\mathcal{O}_X$ be the constant sheaf $\mathbb{Z}/2\mathbb{Z}$. Let $Z = \{0\}$ and $\mathcal{O}_Z = \mathbb{Z}/2\mathbb{Z}$. We have $\iota\colon Z \to X$ and $\iota^\sharp$ being the natural map. Should this be a closed immersion?

The answer is "yes?". This agrees with the usual definition of closed immersion for schemes. But this is not good because we want closed sets that are cut out by ideals of regular functions. △

This leads to the following definition:

Definition 3.7.9. Let $\iota\colon Z \hookrightarrow X$ be a morphism of locally ringed spaces. We say ι is a closed immersion if:

1. ι is a homeomorphism of Z onto a closed subset of X,
2. Our map $\iota^\sharp\colon \mathcal{O}_X \to \iota_*\mathcal{O}_X$ is surjective with kernel J, and
3. As an $\mathcal{O}_X$-module, J is locally generated by sections, that is to say, for all $x \in X$ there is an open $U \subseteq X$ such that $x \in U$, and sections $s_i \in \widetilde{J}(U)$, for $i \in I, \cup U_i = U$, such that

$$\bigoplus_{i \in I} \mathcal{O}_X|_{U_i} \to J|_U$$
$$(f_i)_{i\in I} \mapsto \sum f_i s_i$$

is surjective.

△

Example 3.7.10 (Example 3.7.8 revisited)**.** Note that $J = \ker(\mathcal{O}_X \to \iota_*\mathcal{O}_Z)$ is not locally generated by sections. For all $U \ni x$ such that U is connected, we have

$$\mathbb{Z}/2\mathbb{Z} = \mathcal{O}_X(U) \xrightarrow{\cong} \iota_*\mathcal{O}_Z(U) = \mathbb{Z}/2\mathbb{Z}$$

and hence $J(U)$ is 0. If there existed a U such that it were generated by (constant) sections, we would have $J(U) \neq 0$ becuase $I_y \cong \mathbb{Z}/2\mathbb{Z} \neq 0$ for all $y \in U$ and $Y \neq 0$. Hence, don't consider $(\{0\}, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow (\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$ as a closed immersion. △

Summary. If $\iota\colon Z \hookrightarrow X$ is a closed immersion, then for all $z \in Z$ there is an open $U \subseteq X$ with $\iota(z) \in U$ and $f_i \in \mathcal{O}_X(U)$ such that $\iota(Z) \cap U$ is cut out by vanishing sets of f_j's, that is,

$$\bigcap_j \{x \in U\colon f_j = 0 \text{ in } K(x)\}$$

3.8 Feb 14, 2019

[Use the regrading feature on Gradescope if you feel like you need to (to create some dialogue, also as a semi-replacement for the lack of OH, ...)]

Last time, we had the following

Theorem 3.8.1. *Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be locally ringed spaces, and let $Y = \operatorname{Spec} R$. Then*

$$\operatorname{Mor}_{\mathbf{LocRinSp}}(X, Y) = \operatorname{Hom}_{\mathbf{Rings}}(R, \Gamma(X, \mathcal{O}_X))$$

and this is functorial in X.

and the following

Corollary 3.8.2. *There is an anti-equivalence of categories*

$$\begin{tikzcd} \mathbf{AffSch} \arrow[r, bend left, "\Gamma(\cdot)"] & \mathbf{Rings} \arrow[l, bend left, "\operatorname{Spec}"] \end{tikzcd}$$

and Spec *is fully faithful (as a functor).*

We talked about open vs closed immersions of locally ringed spaces. As a warning, closed immersions for locally ringed spaces are not what we want for schemes (i.e. we need this surjectivity on sheaf of rings and locally generated by sections condition).

Fact 3.8.3. *If $X = \operatorname{Spec} R$ is an affine scheme, then any closed immersion $\iota\colon Z \hookrightarrow X$ is of the form $\operatorname{Spec} R/I \xrightarrow{\varphi} \operatorname{Spec} R$ for a unique ideal $\subseteq R$. Furthermore, $\ker(\mathcal{O}_{\operatorname{Spec} R} \to \varphi_* \mathcal{O}_{\operatorname{Spec} R/I}) = \widetilde{I} \subseteq \widetilde{R} = \mathcal{O}_{\operatorname{Spec} R}$.*

We note that replacing I with $\sqrt{I}$ leaves the maximal ideals invariant (ie. it will be the same as a variety, but different as a scheme).

We saw this nice result in the homework:

Lemma 3.8.4. *If X is a scheme and $U \subseteq X$ is open, then U is a scheme.*

Equivalently:

Lemma 3.8.5. *Let $j\colon X \to Y$ is an open immersion and Y is a scheme, then X is a scheme.*

Warning. If Y is affine, this X may not be affine in general.

Example 3.8.6. Consider $\mathbb{A}^2_k = \operatorname{Spec} k[x, y] \supseteq U = D(x) \cup D(y)$. So we are looking at the open set of $\mathbb{A}^2_k$ with the origin removed [picture coming later, or maybe never lol]. This is not affine (you can check this using, for example, the correspondence between affine schemes and ring homomorphisms). On the other hand, open subsets of affines are called quasi-affine (these are related, but not quite the same). △

Using Fact 3.8.3, we get

Lemma 3.8.7. *If $i\colon X \to Y$ is a closed immersion and Y is a scheme, then X is a scheme.*

This is because the fact implies $i^{-1}(\operatorname{Spec} R) = \operatorname{Spec} R/I$ for a unique ideal $I \subseteq R$. **Warning.** The convention in Hartshorne is that he always requires X to be a scheme.

Definition 3.8.8. A morphism $f\colon X \to Y$ is said to be a <u>locally closed immersion</u> (often just called an <u>immersion</u>) f such that $f = j \circ i$ where i is a closed immersion and j is an open immersion. △

This should remind you of the notion of being locally closed in topology.

Lemma 3.8.9. *An immersion is closed if and only if its image is closed.*

Here's another important fact that we will use:

Lemma 3.8.10. *Let X be a scheme. Then any irreducible closed subset has a unique generic point.*

This property is called "being sober".

Proof. Let $Z \subseteq X$ be closed and irreducible. Pick an open affine $U = \operatorname{Spec} R \subseteq X$ such that $U \cap Z \neq \emptyset$. Then $U \cap Z$ is irreducible, closed, and by Fact 3.8.3, this corresponds to a unique (radical) ideal $\mathfrak{p}$. Since this is irreducible and $\mathfrak{p}$ is prime, $\mathfrak{p} \in \operatorname{Spec} R \subseteq X$ has the property that $\overline{\{\mathfrak{p}\}} = Z$ because $\{\mathfrak{p}\} \cap Z$ contains an open subset $U \cap Z \subseteq Z$ and Z is irreducible. Uniqueness follows because any generic point of Z is in U. $\square$

Lemma 3.8.11. *The open affines of a scheme X form a basis for its topology.*

Warning. If U, V are affine schemes then $U \cap V$ is not necessarily affine. (This is true for varieties, but not schemes.)

But nonetheless, we have

Lemma 3.8.12. *Let X be a scheme and let $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$ be affine schemes in X. Pick $x \in U \cap V$. Then there exists a $W \subseteq U \cap V$ such that W distinguished open in both U and V.*

Proof. Choose $f \in A$, we have $D(f) \subseteq U \cap V$, and also have $g \in B$ such that $D(g) \subseteq D(f) \subseteq U \cap V$. Then $g \in B = \Gamma(V, \mathcal{O}_X)$ restricts to an element. We have

$$\frac{a}{f^n} \in A_f = \Gamma(D(f), \mathcal{O}_X),$$

and hence $D(g) = D(af)$ is also a standard open in $\operatorname{Spec} A = U$) $\square$

Definition 3.8.13. A scheme $X = (X, \mathcal{O}_X)$ is said to be reduced if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced (i.e., has no nilpotents). $\triangle$

Lemma 3.8.14. *The scheme X is reduced if and only if for all $U \subseteq X$ open, the ring $\mathcal{O}_X(U)$ is reduced.*

Proof. The backwards direction is just commutative algebra: the colimit of reduced rings is reduced. For the forwards direction, we note that for all $U \subseteq X$ open, pick $f \in \mathcal{O}_X(U)$ such that $f^n = 0$. This implies that $f = 0 \in \mathcal{O}_{X,x}$ for all $x \in U$. Since $\mathcal{O}_{X,x}$ is reduced, then $f = 0$ in $\mathcal{O}_{X,x}$. Using that $\mathcal{O}_{X,x}$ is a sheaf, we get $f = 0$ on $\mathcal{O}_X(U)$. $\square$

Corollary 3.8.15. *We have $X = \operatorname{Spec} R$ is reduced if and only if R is reduced.*

Corollary 3.8.16. *Let X be a scheme. The following are equivalent:*

(i) X is reduced

(ii) there exists an open cover $X = \cup_i U_i$ with $\Gamma(U_i, \mathcal{O}_X)$ is reduced for all i

(iii) For all U affine and open, we have $\Gamma(U, \mathcal{O}_X)$ reduced, and

(iv) For all U open, $\Gamma(U, \mathcal{O}_X)$ is reduced.

Definition 3.8.17. Let X be a scheme and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say $\mathcal{F}$ is quasi-coherent if for all $\operatorname{Spec} R = U \subseteq X$ affine open, we have

$$\mathcal{F}|_U = \widetilde{M} \quad \text{for some } R\text{-module } M$$

$\triangle$

Lemma 3.8.18. *It's enough to check this for a single affine open cover of X.*

Proof. Homework. $\square$

Suppose that X is a scheme and $\mathcal{I}$ is a quasicoherent sheaf of ideals. Then for all affine opens $U = \operatorname{Spec} R$, we have

$$\mathcal{I}|_U = \widetilde{I} \quad \text{for some ideal } I \subseteq R.$$

By looking at the short exact sequences of sheaves given by

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{I} \longrightarrow 0$$

on X, and

$$0 \longrightarrow \widetilde{I} \longrightarrow \widetilde{R} \longrightarrow \widetilde{R/I} \longrightarrow 0$$

on U.

Fact 3.8.19. *There is a unique closed subscheme $\iota\colon Z \hookrightarrow X$ such that $\mathcal{I} = \ker(\mathcal{O}_X \to \iota_* \mathcal{O}_Z)$ such that on each affine open $U = \operatorname{Spec} R$, we have $Z \cap U = \operatorname{Spec} R/I$.*

This gives us

Fact 3.8.20. *For a scheme X, there is an inclusion-reversing bijection*

$$\begin{aligned} \{\text{closed subschemes of } X\} &\leftrightarrow \{\text{quasicoherent sheaves of ideals of } \mathcal{O}_X\} \\ (\iota\colon Z \hookrightarrow X) &\mapsto \ker(\mathcal{O}_X \to \iota_* \mathcal{O}_Z) \end{aligned}$$

Lemma 3.8.21. *Let X be a scheme and $T \subseteq X$ a closed subset. Then there exists a unique closed subscheme $Z \subseteq X$ such that*

(a) $Z = T$ as sets

(b) Z is reduced.

Proof. To construct Z, it suffices to construct a quasi-coherent sheaf of (radical) ideals (by the fact above). Indeed, given $T \subseteq X$ closed, define

$$\mathcal{I}(U) := \{f \in \mathcal{O}_X(U)\colon f(t) = 0 \pmod{\mathfrak{m}_t} \text{ for all } t \in U \cap T$$

where $\mathfrak{m}_t$ is the maximal ideal of $\mathcal{O}_{X,t}$. Note that $\mathcal{I}$ is a subsheaf of $\mathcal{O}_X$. It remains to show that it is quasi-coherent. So pick $U = \operatorname{Spec} R \subseteq X$ affine and open. Since T is closed in X, we know $T \cap U = V(I)$ for some (radical) ideal $I \subseteq R$. Such an I is unique, since

$$I = \bigcap_{\mathfrak{p} \in T \cap U} \mathfrak{p} = \Gamma(U, \mathcal{I}) \subseteq R.$$

We need to show $\mathcal{I}|_U = \widetilde{I}$. Recall that it suffices to check this on standard opens: for all $f \in R$, we have

$$\begin{aligned} \Gamma(D(f), \mathcal{I}|_U) &= \Gamma(D(f), \mathcal{I}) \\ &= \left\{ h \in \mathcal{O}_X(D(f))\colon h(t) \equiv 0 \pmod{\mathfrak{m}_t} \text{ for all } D(f) \cap T \right\} \\ &= \bigcap_{\mathfrak{p} \in D(f) \cap V(I)} \mathfrak{p} \\ &= I_f = \widetilde{I}(D(f)). \end{aligned}$$

Then we can take Z to be the closed subscheme associated with $\widetilde{I}$. ∎

3.9 Feb 19, 2019

[This week, OH will be moved from Friday to Wednesday at 2:30-4?]

[Look out for emails! There will be a nuts and bolts survey, and solutions for homework.]

Last time we saw many things. We saw that reduced schemes are those schemes whose local rings have no nilpotents. We saw that closed subschemes of X correspond to quasicoherent sheaves of ideals of $\mathcal{O}_X$. Also, we saw that given X a scheme and $T \subseteq X$ any closed subset, then there exists a unique closed subscheme structure on T where $T = Z$ is reduced (there are infinitely many closed subscheme structures you can endow T with, but they're not reduced). This is usually called the (reduced) induced closed scheme structure on T.

Definition 3.9.1. A scheme X is integral if for all $U \subseteq X$ open, the ring $\mathcal{O}_X(U)$ is an integral domain. $\triangle$

Lemma 3.9.2. *The scheme X is integral if and only if X is irreducible and reduced.*

Remark 3.9.3. Varieties over a field k, when viewed as schemes, are usually taken to be integral. But this choice means that varieties are irreducible, which is not what people always mean when they say "variety". $\triangle$

Let's talk about characterizations of closed immersions.

Lemma 3.9.4. *Let $\iota\colon Z \hookrightarrow X$ be a morphism of schemes. The following are equivalent:*

1. *ι is a closed immersion (as defined before).*
2. *For all $U = \operatorname{Spec} R \subseteq X$ open affine, we have*

$$\iota^{-1}(U)(= \iota|_{\iota^{-1}(U)})\colon \operatorname{Spec} R/I \to \operatorname{Spec} R$$

   corresponding to some ideal $I \subseteq R$.

3. *There exists an open affine covering $X = \cup_j U_j$ with $U_j = \operatorname{Spec} R_j$ such that $\iota^{-1}(U_j) = \operatorname{Spec} R_j/I_j$.*
4. *(Hartshorne)*
   - *(a) ι is a homeomorphism onto a closed subset of X*
   - *(b) The induced map $\iota^\sharp\colon \mathcal{O}_X \to \iota_*\mathcal{O}_Z$ is surjective.*
5. *Both (a) and (b) hold, and furthermore that $\ker \iota^\sharp\colon \mathcal{O}_X \to \iota_*\mathcal{O}_Z$ is a quasicoherent sheaf of ideals.*
6. *Both (a) and (b) hold, and furthermore that the subsheaf $\ker \iota^\sharp$ of $\mathcal{O}_X$ is a sheaf of ideals, locally generated by sections.*

Remark 3.9.5. The assumption that ι is a morphism of schemes is key for these equivalences. $\triangle$

Given a quasicoherent sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$, there exists a closed immersion $\iota\colon Z \to X$ such that $\mathcal{I} = \ker(\iota^\sharp)$.

Warning. Given an immersion $\iota\colon Z \hookrightarrow X$, recall that ι must factor as a "closed, then open", but there does not generally exist a factoring

$$\begin{tikzcd} Z \arrow[r, hook, "\iota"] \arrow[dr, hook, "\text{open}"'] & X \\ & \bar{Z} \arrow[u, hook, "\text{closed}"'] \end{tikzcd}$$

This is okay for varieties, but not for schemes (key property is reducedness).

Example 3.9.6. This example is common (somewhat like $\sin(1/x)$). Let $X = \operatorname{Spec}\mathbb{C}[x_1, x_2, \dots]$, which we'll also call $\mathbb{A}^\infty_{\mathbb{C}}$ for now. Let $U = \cup_{i=1}^\infty D(x_i)$. Let $Z \hookrightarrow U$ be defined on $D(x_i) = \operatorname{Spec}\mathbb{C}[x_1, x_2, \dots, x_{i-1}, \bar{x_i}, \dots]$ by the ideal $I_i := \langle x_1^i, x_2^i, \dots, x_{i=}^i, x_i - 1, x_{i+1}, x_{i+2}, \dots\rangle$. This ideal corresponds to a closed point $(0, \dots, 0, 1, 0, \dots)$ with a nonreduced structure.

On $D(x_i x_j)$, we have $I_i|_{D(x_i x_j)} = \operatorname{Spec}\mathbb{C}[x_1, x_2, \dots, 1/(x_i x_j)] = I_j|_{D(x_i x_j)}$. Thus the I_i's glue to give a closed subscheme in U. However, there is no closed subscehem structure on $\bar{Z}$ in X that restricts to this scheme structure of Z in U because for all $f \in \mathbb{C}[x_1, x_2, \dots]$ with $f|_{D(x_i)} \in I_i$ implies that $\deg f \geq i$, and hence $f = 0$. △

Let's talk about how to construct schemes. Say that for a given index set I, for all $i \in I$, we have schemes $(X_i, \mathcal{O}_i)$, and for all i, j, there exist open subschemes $U_{ij} \subseteq X_i$ and $U_{ij} \subseteq X_j$ such that there exists an isomorphism $\varphi_{ij}\colon U_{ij} \to U_{ji}$, such that if $i = j$ then $U_{ii} = X_i$ and $\varphi_{ii} = \mathrm{id}_{X_i}$, and for all I, j, k we have

1. $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$
2. The maps

$$\begin{tikzcd} U_{ij} \cap U_{ik} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ji}"'] & & U_{ki} \cap U_{jk} \arrow[dl, "\varphi_{kj}"] \\ & U_{ji} \cap U_{jk} & \end{tikzcd}$$

commute (this is sometimes called the cocycle condition). Suppose all of this is true; then we call $((X_i, \mathcal{O}_i); U_{ij}, \varphi_{ij})$ the <u>gluing data</u>.

Proposition 3.9.7. *Given a gluing datum, there eixsts a unique scheme X with open subschemes $U_i \subseteq X$ and isomorphisms $\varphi_i\colon X_i \to U_i$ such that*

1. $\varphi_i(U_{ij}) = U_i \cap U_j$
2. $\varphi_{ij} = \varphi_j^{-1}|_{U_i \cap U_j} \circ \varphi_i|_{U_{ij}}$

Also $\operatorname{Mor}_{\mathbf{Sch}}(XY) = \{(f_i)_{i \in I}$ such that $f_i\colon X_i \to Y$ and $f_j \circ \varphi_{ij} = f_i|_{U_{ij}}\}$

Example 3.9.8. This is another common counterexample: $\mathbb{A}^1$ with a doubled point. Let k be an algebraically closed field. Let $X_1 = \operatorname{Spec} k[x]$ and $X_2 = \operatorname{Spec} k[y]$, and pick $O_i \in X_i$. We have open sets $X_1 \supseteq U = D(x) = \operatorname{Spec} k[x, 1/x]$ and analogously $X_2 \supseteq V = D(y) = \operatorname{Spec} k[y, 1/y]$. Let $\varphi\colon U \to V$ be the isomorphism corresponding to the ring homomorphism $k[y, 1/y] \to k[x, 1/x]$ given by $y \mapsto x$ (if you use $y \mapsto 1/x$ and glue, you can get $\mathbb{P}^1$).

Let $X = X_1 \cup_{U=V} X_2$. What is $\Gamma(X, \mathcal{O}_X)$? We have

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(X_1, \mathcal{O}_{X_1}) \times \Gamma(X_2, \mathcal{O}_{X_2}) \xrightarrow{\varphi} \Gamma(U, \mathcal{O}_X)$$

with $\Gamma(X_1, \mathcal{O}_{X_1}) \times \Gamma(X_2, \mathcal{O}_{X_2}) = k[x] \times k[y]$ and $\Gamma(U, \mathcal{O}_X) = k[x, 1/x]$, and the map $\varphi\colon (f(x), g(y)) \mapsto f(x) - g(x)$. Thus $\Gamma(X, \mathcal{O}_X) \cong k[x]$.

Note that given $f \in \Gamma(X, \mathcal{O}_X)$ we must have $f(O_1) = f(O_2) \in k$, and so X is not affine, for example because affine schemes are always T_0 (that is, for all $x, y \in X$, at least one has a neighborhood not containing the other), and alternatively we can look at maps, say, out of X into affine schemes. △

Let's talk about relative schemes and fiber products. Relative schemes are schemes with a structure map $X \to S$ for S a scheme. Sometimes these are called "S-schemes".

Definition 3.9.9. Given two morphisms of schemes $f\colon X \to S$ and $g\colon Y \to S$, its fiber product is a scheme $X \times_S Y$ such that there are projection maps π_1 and π_2 and

$$\begin{tikzcd}
X \times_S Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & S
\end{tikzcd}$$

commutes. Its universal property is that given a scheme T and morphisms $a\colon T \to X$ and $b\colon T \to Y$, we have a commutative diagram

$$\begin{tikzcd}
T \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "a"'] \arrow[dr, dashed, "\exists!"] & & \\
& X \times_S Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & S
\end{tikzcd}$$

$\triangle$

Theorem 3.9.10. *Fiber products exist.*

The proof is painful, especially if you want to do it in a lot of generality.

Idea of proof. Some books give an explicit construction, but it's not so informative to go through this. Assuming $X \times_S Y$ exists,

- If $U \subseteq X$, $V \subseteq Y$, and $W \subseteq S$ are opens such that $f(U), g(V) \subseteq W$, then $U \times_W V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$ is an open subset of $X \times_S Y$. This is abstract nonsense.
- If $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $S = \operatorname{Spec} R$, then $X \times_S Y = \operatorname{Spec}(A \otimes_R B)$ because

$$\begin{aligned}
\operatorname{Mor}_{\mathbf{Sch}}(T, \operatorname{Spec}(A \otimes_R B)) &= \operatorname{Hom}(A \otimes_R B, \mathcal{O}_T(T)) \\
&= \operatorname{Hom}(A, \mathcal{O}_T(T)) \times_{\operatorname{Hom}(R, \mathcal{O}_T(T))} \operatorname{Hom}(B, \mathcal{O}_T(T)) \\
&= \operatorname{Mor}(T, \operatorname{Spec} A) \times_{\operatorname{Mor}(T, \operatorname{Spec} R)} \operatorname{Mor}(T, \operatorname{Spec} B)
\end{aligned}$$

- In general, take an affine open covering and glue.

$\square$

Example 3.9.11. If R is a ring, we have $\mathbb{A}^n_R = \operatorname{Spec} R[x_1, \ldots, x_n] \to \operatorname{Spec} R$. We have

$$\mathbb{A}^n_R \times_{\operatorname{Spec} R} \mathbb{A}^m = \mathbb{A}^{n+m}_R$$

because $R[x_1, \ldots, x_n] \otimes_R R[y_1, \ldots, y_m] \cong R[x_1, \ldots, x_n, y_1, \ldots, y_m]$. On points, $X \times_S Y$ is not necessarily $|X| \times_{|S|} |Y|$. Namely, consider $\mathbb{A}^2_{\mathbb{C}}$ versus $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}}$ on points. On one hand, we have $\mathbb{A}^1_{\mathbb{C}} = \{\langle x - \alpha \rangle \colon \alpha \in \mathbb{C}\} \cup \{\langle 0 \rangle\}$ and so on the product $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}}$ we have points

$$\{\langle x - \alpha, y - \beta \rangle, \langle x - \alpha \rangle, \langle y - \beta \rangle, \langle 0 \rangle \colon \alpha, \beta \in \mathbb{C}\}$$

and yet, for example, $\langle x^2 - y^3 \rangle \in \mathbb{A}^2_{\mathbb{C}}$, but is not in $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}}$. They do match on closed points, that is, maximal ideals, but not generally. $\triangle$

3.10 Feb 21, 2019

[disclaimer: many typos, i was not really awake]

Last time we were talking about fiber products: given X, Y, S schemes, we had the commutative diagram

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

and showed that fiber products exist for S-schemes.

Definition 3.10.1. Let K be a field and S a scheme. A K-point of S is a morphism $\operatorname{Spec} K \to S$. They can be viewed as sections

$$\begin{tikzcd} S \arrow[d] \\ \operatorname{Spec} K \arrow[u, dashed, bend right, "K\text{-points}"'] \end{tikzcd}$$

$\triangle$

We note that $\mathrm{Mor}_{\mathbf{Sch}}(K, S) = \{(s \in S, K(s) \hookrightarrow K)\}$. In particular, for all s, there is a canonical morphism $s = (\operatorname{Spec} K(s) \to S)$.

Last time we saw that $\mathbb{A}^2_{\mathbb{C}} \neq \mathbb{A}^1_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}}$; for example they don't even have the same points (eg. $\langle x^2 - y^3 \rangle$). But what is true is that $\mathbb{A}^2_{\mathbb{C}} = \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}}$ (on the algebraic side, we need $\mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y] \cong \mathbb{C}[x, y]$).

Given a morphism of schemes $f \colon X \to S$ and a point $s \in S$, the fiber of X at s, denoted (X_s), is

$$\begin{tikzcd} X_s = \operatorname{Spec} K(s) \arrow[r] \arrow[d] & x_f \arrow[d] \\ \operatorname{Spec} K(s) \arrow[r] & S \end{tikzcd}$$

Note that X_s naturally lie over $\operatorname{Spec} K$. For example, we have

$$\begin{tikzcd} X \arrow[d] & X_0 \arrow[d, dotted, hook] & X_2 \arrow[d, dotted, hook] & X_3 \arrow[d, dotted, hook] & \dots \\ \operatorname{Spec} \mathbb{Z} & \mathbb{Q} & \mathbb{F}_2 & \mathbb{F}_3 & \dots \end{tikzcd}$$

Another example is that of $\mathbb{A}^2_{\mathbb{C}} = X \to \mathbb{A}^1_{\mathbb{C}}$ sending $(x, y) \mapsto (x)$; in this case K-points are $\mathbb{C}[x] \to \mathbb{C}[x, y]$.

Over closed points, $\langle x - \alpha \rangle \subseteq \mathbb{C}[x]$. Then

$$\mathbb{C}[x, y] \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/\langle x - \alpha \rangle \cong \mathbb{C}[x, y]/\langle x - \alpha \rangle.$$

For a closed $s \in \operatorname{Spec} \mathbb{C}[x] \cong \mathbb{C}[y]$, we have $X_s \cong \mathbb{A}^1_{\mathbb{C}}$ (these are called "closed fibers", or in number theory sometimes "special fibers"). Over a generic point $\eta = \langle 0 \rangle$,

$$\mathbb{C}[x, y] \otimes_{\mathbb{C}[x]} = \mathbb{C}(x)[y]$$

which implies $X_\eta \cong \mathbb{A}^1_{\mathbb{C}(x)}$ (this is the "generic fiber").

Example 3.10.2. Let $X = \operatorname{Spec} \mathbb{Z}[x]/\langle x^2 - 50 \rangle$ map into $\operatorname{Spec} \mathbb{Z}$.

For any point $s \neq \langle 2\rangle, \langle 5\rangle$, that is a closed point (take, for example, $s = \langle 59\rangle$), then

$$X_S = \operatorname{Spec}(\mathbb{Z}[x]/\langle x^2 - 50, 59\rangle) \cong \operatorname{Spec}(\mathbb{F}_5[x]/\langle x^2 - 50\rangle).$$

Since 50 is not a square in $\mathbb{F}_{59}$, this should be one point. However if 50 is a square mod p, then for $p \neq 2, 5$, X_S consists of two points.

If $s = \langle 2\rangle$ or $\langle 5\rangle$, then

$$X_S \cong \operatorname{Spec} \mathbb{F}_p[x]/\langle x^2\rangle,$$

and we get a single non-reduced point. △

Example 3.10.3. (Families of curves.) Consider a map $\operatorname{Spec} \mathbb{C}[x, y, t]/\langle ty - x^2\rangle \to \operatorname{Spec} \mathbb{C}[t]$. This is sometimes called a 1-parameter family of curves (think of this as parabolas "expanding any contracting with respect to t"), so for $t = 1$ we have [picture coming soon i promise] and for $t = 4$ we have [picture coming soon i promise], but for $t = 0$ we have $\operatorname{Spec} \mathbb{C}[x, y]/\langle x^2\rangle$[picture coming...]. So this map has general fibers ($\neq$ generic fiber) which are reduced, but over 0 we have an unreduced fiber.

When the fibers are intersected by horizontal lines we typically get two points of intersection; the point is that even at $t = 0$ the scheme theory also allows us to recover this intersection number of two. △

Here's a variant of the above example:

Example 3.10.4. Consider $\mathbb{C}[x, y, t]/\langle xy - t\rangle \to \operatorname{Spec} \mathbb{C}[t]$. But at $t = 0$, we have $\mathbb{C}[x, y]/\langle xy\rangle$, which is not irreducible. △

Let's talk about S-schemes. Recall that these were schemes X with a morphism ("structure map") $X \to S$. This is a category of S-schemes which has fiber products $X \times_S Y$ with morphisms

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

that commute.

As a special case of this, given any morphism $S' \to S$ we have the base change of the S-scheme X to S' given by

$$\begin{tikzcd} X_S \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

Example 3.10.5. With $\operatorname{Spec} K(s) \to S$, the fiber over s is just the base change to $K(s)$. For example, consider $\operatorname{Spec} \mathbb{Z}[x]/\langle f(X)\rangle \to \operatorname{Spec} \mathbb{Z}$ for $f(x) \in \mathbb{Z}[x]$. Given a prime $\langle p\rangle \in \operatorname{Spec} \mathbb{Z}$, we have $X_p \cong \operatorname{Spec} \mathbb{F}_p[x]/\langle \bar{f}(x)\rangle$. △

Here's a question. What properties of X are preserved under base change?

Lemma 3.10.6. *Let $f\colon X \to Y$ be an open (respectively closed, locally closed) immersion of S-schemes. Let $S' \to S$ be any morphism of schemes. Then the base change morphism $f'\colon X_{S'} \to Y_{S'}$ is also an open (respectively closed, locally closed) immersion of S-schemes.*

This follows from

Lemma 3.10.7. *Suppose we had maps*

$$\begin{tikzcd} X \times_S Y \arrow[r, "q"] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

such that f is an open (respectively closed) immersion, then q is an open (respectively closed) immersion.

Proof. The open case is simple.

Closed immersions correspond to a quasicoherent sheaf of ideals, namely we have

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_S \longrightarrow \iota_* \mathcal{O}_X \longrightarrow 0$$

where $\iota\colon X \to S$. The map ι fits into the commutative square

$$\begin{tikzcd} X \times_S Y \arrow[r, "q"] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "\iota"] & S \end{tikzcd}$$

We have $\operatorname{im}(g^*\mathcal{I} \to \mathcal{O}_Y) = \operatorname{im}(g^*\mathcal{I} \to g^*\mathcal{O}_S = \mathcal{O}_Y)$ is locally generated by sections, and hence it cuts out a closed subscheme $Z \subseteq Y$. Then $Z = X \times_S Y$.

By the homework we can check on affine affines. So on S-algebras, we have

$$\begin{tikzcd} A/IA & A \arrow[l] \\ R/I \arrow[u] & R \arrow[l] \arrow[u] \end{tikzcd}$$

If A is an R-algebra, then the diagram commutes; by quasicoherence we see that $q\colon Z \to Y$ is a closed immersion. $\square$

Proof of Lemma 3.10.6. We note that we have maps

$$\begin{tikzcd} X_{S'} \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y_{S'} \arrow[r] \arrow[d] & Y \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

and we see that $X_{S'}$ is a fiber product because of the universal property. We check that

$$\begin{aligned} \operatorname{Mor}(T, X_{S'}) &= \operatorname{Mor}(T, X) \times_{\operatorname{Mor}(T,S)} \operatorname{Mor}(T, S') \\ &= \operatorname{Mor}(T, X) \times_{\operatorname{Mor}(T,Y)} \left(\operatorname{Mor}(T, Y) \times_{\operatorname{Mor}(T,S)} \operatorname{Mor}(T, S') \right) \\ &= \operatorname{Mor}(T, X) \times_{\operatorname{Mor}(T,Y)} \operatorname{Mor}(T, Y_{S'}). \end{aligned}$$

Thus, by the previous lemma, if f is open (respectively closed, locally closed) then so is f. $\square$

Definition 3.10.8. A morphism of schemes is $f\colon X \to S$ is quasicompact if the map on the underlying topological spaces is quasicompact. $\triangle$

Proposition 3.10.9. *(Characterization of quasicompact morphisms) Let $f\colon X \to S$ be a morphism of schemes. Then the following are equivalent*

1. *f is quasicompact*
2. *For all $U \subseteq S$ open affine, $f^{-1}(U)$ is quasicompact*
3. *There exists an affine open covering*
$$S = \bigcup_{i \in I} U_i \text{ such that } f^{-1}(U_i) \text{ is quasicompact } \forall i$$
4. *There exists an affine open covering $S = \cup_{i \in I} U_i$ such that $f^{-1}(U_i)$ is a finite union of open affines.*

3.11 Feb 28, 2019

[Brian will be out of town next week, but there will be class. Harrison Chen will be substituting. There will be no OH (there will be usual homeworks etc.)]

Last time we talked about families of schemes, and base change (and some properties preserved under it).

Definition 3.11.1. A morphism of schemes $f: X \to S$ is quasicompact (qc) if the underlying map on topological spaces is quasicompact, i.e. for all quasicompact open $U \subseteq S$, we have $f^{-1}(U)$ quasicompact. △

We had this characterization for quasicompact morphisms, namely, it is enough to check this on an open affine covering of S such that f^{-1}(each open) is a *finite* union of open affines.

Lemma 3.11.2. *1. The base change of a quasicompact morphism is quasicompact.*

2. The composition of quasicompact morphisms is quasicompact.

3. Closed immersions are quasicompact.

These are the three main situations when our invocation of this condition is important.

Proof. For 1., we can consider the fiber product, namely for all $s \in S$, there exists an open affine neighborhood V of s such that $g(V) \subseteq U$

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

Then $f^{-1}(U)$ is covered by finitely many opens $W_1, \dots, W_n$. Then $\{V \times_U W_i\}$ is a finite affine cover of $f^{-1}(V)$.

For 2., we have quasicompact maps $X \xrightarrow{f} Y \xrightarrow{g} Z$. Thus for all $U \subseteq Z$ open, $g^{-1}(U) = \cup_{i=1}^n V_i$, with $V_i \subseteq Y$ open affine. Then $f^{-1}(V_i) = \cup_{j=1}^N W_{i,j}$, with $W_{i,j} \subseteq X$ is an open affine. Thus

$$(g \circ f)^{-1}(U) = \bigcup_i \bigcup_j W_{i,j},$$

which is a finite union of open affines.

For 3., all closed immersions are locally of the form $\operatorname{Spec} A/I \to \operatorname{Spec} A$ for some ideal $I \subset A$, and then we get our cover. □

Example 3.11.3. Open immersions are not necessarily quasicompact.

Consider the locally closed subscheme in U given as $U = \cup_{i=1}^\infty D(x_i) \subseteq \mathbb{C}[x_1, x_2, \dots]$. On each open $D(x_i) = \operatorname{Spec} \mathbb{C}[x_1, \dots][1/x_i]$, it is the closed subscheme corresponding to $I_i := \langle x_1^i, \dots, x_{i-1}^i, x_i - 1, x_{i+1}, \dots \rangle$ (that nonreduced ideal we saw at some point earlier). On $D(x_i x_j)$, for $i \neq j$, note that

$$I_i \cdot \mathbb{C}[x_1, \dots][1/(x_i x_j)] = I_j \mathbb{C}[x_1, \dots][1/(x_i x_j)] = \mathbb{C}[x_1, x_2, \dots][1/(x_i x_j)]$$

and hence by gluing we get a scheme on U. On the other hand, there is no closed subscheme on $\operatorname{Spec} \mathbb{C}[x_1, \dots]$ that gives rise to this closed subscheme structure on U. Indeed, if it did, it would correspond to an ideal $I \subset \mathbb{C}[x_1, \dots]$, but for all $f \in I$, we have $f \notin I_N$ where $N > \deg f$. △

Towards valuative criteria.

The idea of valuative criteria is that we want to make sense of "limits" relative to a base scheme S. By limits we mean with respect to the points (remember that our spaces are not Hausdorff): roughly, separatedness (the AG notion of being Hausdorff) corresponds to the fact that "limits" are unique, and properness (the AG notion of compactness) corresponds to the fact that "limits" exist.

Definition 3.11.4. Given X a scheme and $x, x' \in X$, we say that <u>x specializes to x'</u> (denoted $x \rightsquigarrow x'$) if $x' \in \overline{\{x\}}$. A subset $Z \subseteq X$ is <u>closed under specialization</u> if for all $x, x' \in X$ such that $x \in Z$ and $x \rightsquigarrow x'$, we have $x' \in Z$. △

Lemma 3.11.5. *(for algebra.) Let $f^\sharp \colon B \to A$ be a ring homomorphism. Let $T \subseteq \operatorname{Spec} A$ be closed. If $f(T)$ is closed under specialization, for any $f \colon \operatorname{Spec} A \to \operatorname{Spec} B$, then $f(T)$ is closed.*

Proof. Write $T = V(I)$ for an $I \subseteq A$. Consider $J = \ker(B \to A \to A/I)$. Then we have

$$\operatorname{Spec}(A/I) = V(I) = T \subseteq \operatorname{Spec} A \to \operatorname{Spec} B \supseteq V(J) = \operatorname{Spec}(B/J)$$

and $f(T) \subseteq V(J)$. We want this to be an equality.

We are reduced to the setting where: 1. $B \hookrightarrow A$, 2. $T = \operatorname{Spec} A$, and 3 $f(T)$ is closed under specialization. We want to show that $f(T) = \operatorname{Spec} B$. Let $\mathfrak{q} \subseteq B$ be any minimal prime. Then $B_\mathfrak{q}$ is a local ring with only 1 prime ideal. Also, $B_\mathfrak{q} \subseteq A_\mathfrak{q}$ implies $A_\mathfrak{q} \neq 0$, that is, $\mathfrak{q} \in \operatorname{im} f$. Now, any prime of B is a specialization of some *minimal* prime of B. Thus, since $f(T)$ is closed under specialization, it follows that $f(T) = \operatorname{Spec} B$. □

Definition 3.11.6. Let $f \colon X \to S$ be a map of topological spaces. We say <u>specializations lift along f</u> if for all $f(x) = s$ and $s \rightsquigarrow s'$ there exists an $x' \in X$ with $x \rightsquigarrow x'$ and $f(x') = s'$. △

Lemma 3.11.7. *(for topology.)*

1. *If specializations lift along f and T is closed under specializations, then so is $f(T)$.*
2. *Specializations lift along closed maps between topological spaces.*

Lemma 3.11.8. *Let $f \colon X \to S$ be a quasicompact morphism of schemes. Then f is closed if and only if specializations lift along f.*

Proof. The forward direction is the above Lemma 3.11.7.

The backwards direction is proven as follows: take T to be a closed set in X. We can cover S by affine opens U_i, and to show that $f(T) \cap U_i = f(T \cap f^{-1}(U_i))$ is closed in U_i. Thus we reduce to the case where S is affine.

Since f is quasicompact, we have $X = \cup_{i=1}^r X_i$ with $X_i = \operatorname{Spec} A_i$ affine opens. Since $S = \operatorname{Spec} R$, we know that A_i is an R-algebra. Since we know $f(T) \subseteq \operatorname{Spec} R$ is closed under specialization, we have

$$\coprod_{i=1}^n T \cap X_i \subseteq \coprod_{i=1}^n X_i = \operatorname{Spec}\left(\prod_{i=1}^n A_i\right) \to S \hookleftarrow f(T)$$

and now the algebra lemma (Lemma 3.11.5) applies. □

Lemma 3.11.9. *(for number theory.)*

If $B \xrightarrow{f^\sharp} A$ is a ring homomorphism and $f \colon \operatorname{Spec} A \to \operatorname{Spec} B$ then $B \to A$ satisfies going up if and only if specializations lift along f. In particular, f is closed as a map of topological spaces.

For example, if A is integral over $f^\sharp(B)$, then it satisfies going up. In particular, finite maps and surjections satisfy going up.

Definition 3.11.10. Suppose that K is a field, and suppose that $A, B \subseteq K$ are local domains (but not fields). We say that <u>A dominates B</u> if $B \subseteq A$ and $\mathfrak{m}_B = B \cap \mathfrak{m}_A$. △

Then, valuation rings (the rings we'll use for valuative criteria) are the maximal elements under the domination relation (which form a poset on local domains). An alternative, more down to earth definition is the following:

Definition 3.11.11. An integral domain A with $\mathrm{Frac}A = K$ is a valuation ring if for all $x \in K^\times$, either $x \in A$ or $x^{-1} \in A$. △

Such rings are always local.

Remark 3.11.12. Such rings admit a homomorphism $\nu\colon K^\times \to \Gamma$, where Γ is an ordered abelian group such that $x \in A$ if and only if $\nu(x) \geq 0$. △

Examples of valuation rings are $\mathbb{Z}_p$ with the p-adic valuation, and $k((x))$: the Laurent series in a field k with trivial or degree valuation.

Remark 3.11.13. If working with local Noetherian schemes, you only need to work with discrete valuation rings (e.g. most of Hartshorne has this assumption). But without this assumption, it's important to work with arbitrary valuation rings. △

If A is a valuation ring, then it has a unique generic point $(\eta) = \langle 0 \rangle$ and a unique special point $(s) = \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of A (it is always given by $\{x \in A\colon \nu(x) > 0\}$).

Valuative Criterion.

Let $f\colon X \to S$ be a morphism of schemes. We say f satisfies the existence part of the valuative criterion if, given any valuation ring A with $K := \mathrm{Frac}A$, and given any diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ A \arrow[r] \arrow[ur, dashed, "\exists"] & S \end{tikzcd}$$

the dashed arrow exists. We say f satisfies the uniqueness part of the valuative criterion if this dashed arrow is unique.

How do we map a local ring A into a scheme? Given $\operatorname{Spec} A \to S$, we have

$$\mathrm{Mor}_{\mathbf{Sch}}(\operatorname{Spec} A, S) = \{(s, \mathcal{O}_{S,s} \xrightarrow{\psi} A)\colon \psi \text{ is a local homomorphism of local rings}\}.$$

The inverse is given as follows: for all $(s, \psi\colon \mathcal{O}_{S,s} \to A)$, take an open affine neighborhood $\operatorname{Spec} R$ of s. Then s corresponds to a prime ideal $\mathfrak{p} \subseteq R$, and we have maps

$$R \to R_{\mathfrak{p}} = \mathcal{O}_{S,s} \xrightarrow{\psi} A$$

which induces $\operatorname{Spec} A \to \operatorname{Spec} R \subseteq S$.

Theorem 3.11.14. *If $f\colon X \to Y$ be a morphism of schemes and the diagonal map of f (denoted Δ_f) is quasicompact, then f is separated if and only if f satisfies the uniqueness part of the valuative criterion.*

Theorem 3.11.15. *If $f\colon X \to Y$ is a morphism of schemes and f is quasicompact, then f is universally closed if and only if it satisfies the existence part of the valuative criterion.*

(Here universally closed means closed with respect to any base change; we'll talk more about that later)

Theorem 3.11.16. *If $f\colon X \to Y$ is a morphism of schemes of finite type and Δ_f (the diagonal map of f) is quasicompact, then f is proper if and only if it satisfies the existence and uniqueness of the valuative criterion.*

4 (Affine) Algebraic Groups

4.12 Mar 5, 2019

For this week we're going to talk about a side topic.

For this week, k will denote a field; usually $\operatorname{char} k = 0$. If X is the scheme, we denote by $\mathscr{O}(X) := \Gamma(X, \mathscr{O}_x)$ the global sections, and denote by $* := \operatorname{Spec} k$, which is "a point".

For references: there is Mumford's GIT, and notes by Brion called "Introduction to Algebraic Groups", and Milne's notes called "Algebraic Groups". There is also Borel's "Linear Algebraic Groups".

Definition 4.12.1. An algebraic group is a scheme G/k with the following maps of schemes:

- Multiplication $\mu\colon G \times G \to G$
- Unit $e\colon * \to G$
- Inversion $\iota\colon G \to G$

such that the following diagrams commute:

$$\begin{tikzcd}
 & G \times G \arrow[dr, "\mu"] & \\
G \times G \times G \arrow[ur, "\mu \times \mathrm{id}"] \arrow[dr, "\mathrm{id} \times \mu"'] & & G \\
 & G \times G \arrow[ur, "\mu"'] &
\end{tikzcd}$$

(which encodes assiciativity)

$$\begin{tikzcd}
 & G \times G \arrow[dr, "\mu"] & \\
G \arrow[ur, "1 \times e"] \arrow[rr, "\mathrm{id}"] \arrow[dr, "e \times 1"'] & & G \\
 & G \times G \arrow[ur, "\mu"'] &
\end{tikzcd}$$

(which encodes the unit)

$$\begin{tikzcd}
 & G \times G \arrow[rr, "1 \times \iota"] & & G \times G \arrow[dr, "\mu"] & \\
G \arrow[ur, "\Delta"] \arrow[rr] \arrow[dr, "\Delta"'] & & * \arrow[rr, "e"] & & G \\
 & G \times G \arrow[rr, "\iota \times 1"] & & G \times G \arrow[ur, "\mu"'] &
\end{tikzcd}$$

(which encodes inverses).

This is all to say that G is a group object in the category $\mathbf{Sch_k}$. We say that G is an affine algebraic group if G is affine as a scheme. △

The slogan is that the theory of affine algebraic groups is the same as the theory of (commutative) Hopf algebras.

Definition 4.12.2. A coalgebra over k is a k-vector space C with

- Comultiplication $\Delta\colon C \times C \times C$
- Counit $\varepsilon\colon C \to k$

satisfying the dual diagrams to the associativity and unit diagrams described above. $\triangle$

Definition 4.12.3. A Hopf algebra H is an algebra and a coalgebra with an antipode map $S\colon H \to H$ satisfying the dual to the inverse diagram described above. $\triangle$

Remark 4.12.4. Antipodes are a property, not a structure (maybe.). That is to say, antipodes, if they exist are unique, and thus characterize Hopf algebras. $\triangle$

Proposition 4.12.5. *If G is a group scheme, $\mathscr{O}(G)$ are a Hopf algebra. If H is a Hopf algebra,* $\operatorname{Spec} H$ *is an affine algebraic group.*

Proof. The first part is just functoriality of taking global sections; the second part is purely formal so we won't do it. $\square$

Example 4.12.6. Any finite group $G = \operatorname{Spec}\left(\prod_{g\in G} k\right) = \coprod_{g\in G} \operatorname{Spec} k$ (finiteness is needed here; as an exercise, show that when k is algebraically closed, $\mathbb{Z} := \coprod_{n\in\mathbb{Z}} \operatorname{Spec} k$ is not affine, in particular points of $\operatorname{Spec}\left(\prod_{n\in\mathbb{Z}} k\right)$ corresponds to filters on $\mathbb{Z}$).

Then comultiplication gives a map

$$\prod_{g\in G} k_g \to \prod_{g\in G} k_g \oplus \prod_{h\in G} k_h \cong \prod_{g,h\in G} k_{g,h}$$

sending $1_g \mapsto \sum_{x,y\in G; xy=g} 1_{x,y}$. Also the counit gives a map

$$k_g \mapsto \begin{cases} 0 & \text{if } g \neq e \\ 1 & \text{if } g = e \end{cases}$$

$\triangle$

Example 4.12.7. Consider the general linear group $\mathrm{GL}_n = \operatorname{Spec} k[a_{ij}, \Delta, \Delta^{-1}]/(\det(a_{ij}) = \Delta; \Delta\Delta^{-1} = 1)$.

Definition 4.12.8. A group G is called a linear algebraic group if there is a closed embedding $cl\colon G \hookrightarrow \mathrm{GL}_n$ into GL_n. $\triangle$

We note that linear algebraic groups are affine algebraic groups. $\triangle$

Example 4.12.9. Consider $\mathrm{SL}_2 = \operatorname{Spec} k[a,b,c,d]/(ad - bc = 1)$. We get

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + b_1d_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}$$

we can read off comultiplication by pulling back across this map, that is, we get maps

$$\begin{aligned} a &\mapsto a \otimes a + b \otimes d \\ b &\mapsto a \otimes b + b \otimes d \\ c &\mapsto c \otimes a + d \otimes c \\ d &\mapsto c \otimes b + d \otimes d \end{aligned}$$

and we can read off counits

$$\begin{aligned} a &\mapsto 1 \\ b &\mapsto 0 \\ c &\mapsto 0 \\ d &\mapsto 1 \end{aligned}$$

that is just pulling back the identity, and similarly we can read off antipodes

$$\begin{aligned} a &\mapsto d \\ b &\mapsto -b \\ c &\mapsto -c \\ d &\mapsto a \end{aligned}$$

△

Example 4.12.10. We have the multiplicative group $\mathbb{G}_m = \operatorname{Spec} k[z, z^{-1}] = \mathrm{GL}_1$ with $\Delta(z) = z \otimes z$ (and $\Delta(z^n) = z^n \otimes z^n$), and $S(z) = z^{-1}$ and $\varepsilon(z) = 1$ (which is evaluation at 1). △

Example 4.12.11. We have the additive group $\mathbb{G}_a = \operatorname{Spec} k[t]$, which we can think of as a subgroup of GL_2 via

$$\mathbb{G}_a \ni x \mapsto \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_2.$$

We have $\Delta(t) = 1 \otimes t + t \otimes 1$, $\varepsilon(t) = 0$, and $s(t) = -t$. We have $\Delta(t^n) = (1 \otimes t + t \otimes 1)^n \neq 1 \otimes t^n + t^n \otimes 1$ △

Example 4.12.12. Let E be a smooth genus 1 curve, ie., an elliptic curve. So we have

$$E = \operatorname{Proj} k[x, y, z]/\langle y^2 z = x(x+z)(x-z)\rangle \subseteq \mathbb{P}^2.$$

We get the "projective completion" $\{y^2 = x(x+1)(x-1)\} \subseteq \mathbb{A}^2$. △

We have the following technical proposition:

Proposition 4.12.13. *If* $\operatorname{char} k = 0$, *then every group scheme is smooth (and is in particular reduced).*

So this says that group schemes are nice (they're varieties, for example).

Example 4.12.14. Let $G = \operatorname{Spec} k[x]/x^p \hookrightarrow \mathbb{G}_a$. Suppose we defined $\Delta \colon x \mapsto x \otimes 1 + 1 \otimes x$. Then in characteristic 0, we have

$$\Delta(x^p) = \sum \binom{p}{k} x^k \otimes x^{p-k},$$

whereas in characteristic p we have

$$\Delta(x^p) = x^p \otimes 1 + 1 \otimes x^p = 0,$$

so over characteristic p, the above proposition does not hold. △

Definition 4.12.15. A <u>G-action</u> on X is a map $\alpha \colon G \times X \to X$ with usual conditions: the diagrams

$$\begin{tikzcd} & G \times X \arrow[dr, "\alpha"] & \\ G \times G \times X \arrow[ur, "1 \times \alpha"] \arrow[dr, "\mu \times 1"'] & & X \\ & G \times X \arrow[ur, "\alpha"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[r, "e \times 1"] \arrow[rr, bend right, "1"'] & G \times X \arrow[r, "\alpha"] & X \end{tikzcd}$$

commutes. △

Dually, define a comodule for any coalgebra, for example $\mathscr{O}(G)$.

Proposition 4.12.16. *If G acts on X, then $\mathscr{O}(X)$ is a comodule for $\mathscr{O}(G)$. Likewise, if M is an algebra with a compatible comodule structure for $\mathscr{O}(G)$, then* $\operatorname{Spec} M$ *has a G-action.*

Example 4.12.17. What are the $\mathscr{O}(\mathbb{G}_m)$-comodules? We claim these are precisely the graded vector spaces. Indeed, consider $\rho\colon M \to M \otimes k[z, z^{-1}]$ given by $m \mapsto \sum m_i \otimes z^i$, where the counit gives $m = \sum m_i$.

Thus the coaction ρ gives a way to decompose $m \in M$ into homogenous parts *compatibly.* As an exercise, check the rest of the claims (that we really get a grading, so that $M = \oplus_{i\in\mathbb{Z}} M_i$, and that this is equivalent to giving the ρ comodule structure). △

Proposition 4.12.18. *(Maybe "fundamental theorem of comodules"). Every comodule is the union of its finite dimensional subcomodules.*

Proof. Let M be our comodule and C its coalgebra. We just need to show that each element is in a finite dimensional subcomodule. So take $m \in M$, and fix a basis for C, call it $\{c_i\}$. Say $\rho(m) = \sum m_i \otimes c_i$ for some (*finitely many!*) m_i. We're going to try to show that $\operatorname{Span}(m_i)$ is a submodule, and $m = \sum \varepsilon(c_i) m_i$ (where ε is the counit), so that $m \in \operatorname{Span}(m_i)$.

We have $\Delta(c_i) = \sum_{j,k} a^i_{j,k} c_j \otimes c_k$ for some constants $a^i_{j,k} \in k$. Using associativity we get:

$$\sum_{i,j,k;i\in I} m_i \otimes (a^i_{j,k} \otimes c_j \otimes c_k) = \sum_i \rho(m_i) \otimes c_i = \sum_{k\in I} \rho(m_k) \otimes c_k.$$

Equate the coefficients of c_k in the third $\otimes$-factor, so that

$$\sum_{i,j,k;i\in I} a^i_{j,k} \otimes m_i \otimes c_j = \rho(m_k).$$

So $\rho(m_i)$ is in the sum of $m_i \otimes C$. □

Proposition 4.12.19. *If C is an algebra, then its k-linear dual C^* is an algebra (note that the converse isn't true). Furthermore, if M is a left C-comodule, then M is a right C^*-module.*

Proof sketch. Suppose $f \in C^*$. Then $f \cdot m = \mathrm{ev}_f(\rho(m))$. Then everything works. □

Definition 4.12.20. Any C^*-module that comes from a C-comodule is called <u>rational</u>. Note that rational C^*-modules are unions of finite dimensional submodules. △

Note that we have a map $G \to \mathscr{O}(G)^* \to \operatorname{End}(V)$. So V is a rational G-representation if it arises from an $\mathscr{O}(G)$-comodule.

Note that if you unwind the definition of what it means to be a sheaf on BG, then what you get is a $\mathscr{O}(G)$-comodule.

Proposition 4.12.21. *Suppose G acts on X. Then $\mathscr{O}(X)$ is a rational G-representation.*

Proof. Well, $\mathscr{O}(X)$ is a $\mathscr{O}(G)$-comodule. □

As an aside, if $f \in \mathscr{O}(X)$, then $g \circ f(x) = f(g^{-1}x)$. As an application, if G is an affine algebraic group and X is affine, then there is a finite dimensional G-representation V and G-equivariant embedding of X into V.

Proof. If $f_1, \ldots, f_r \in \mathscr{O}(X)$ be generators, then let $V = \operatorname{Span}(G \cdot (f_1, \ldots, f_r))$, which is finite dimensional. Then the symmetric algebra $\operatorname{Sym}(V) \twoheadrightarrow \mathscr{O}(X)$, and taking Spec gives a closed embedding $X \hookrightarrow V^*$. □

4.13 Mar 7, 2019

Remark 4.13.1. (On rational representations.) If V is a vector space, you can consider $\operatorname{Spec}\operatorname{Sym}_k V^*$, which is V as a scheme. You get a scaling map $\mathbb{A}^1 \times V \to V$, addition $V \times V \to V$, and so on.

Now a rational representation is a vector space V with the map $G \times V \xrightarrow{\alpha} V$ with all compatibilities that is locally finite: every $v \in V$ is in a finite dimensional subrepresentation. In particular, we claim that V can be infinite dimensional, but now he's not exactly sure if the above $\operatorname{Spec}\operatorname{Sym}_k V^*$ construction is the right construction (though this is *probably* okay). △

As an exercise, one might want to show that this is equivalent to the comodule definition. Some hints:

1. The $\mathbb{A}^1$ action will induce a positive $\mathbb{Z}^{\geq 0}$ grading on $\mathscr{O}(V)$; pick out V^*.
2. Linearity and associativity gives that $V^* \subseteq \mathscr{O}(V)$ is a comodule.
3. Rationality lets us take the finite/restricted/Hopf dual of V^*.

   Definition 4.13.2. (The Hopf Dual.) Let $V^* \twoheadrightarrow W$ with W finite dimensional. Then we define $(V^*)\circ = \cup W^*$. △

4. $(V^*)^\circ$ is a $\mathscr{O}(G)^*$-module. Use the antipode to make it $\mathscr{O}(G)$-comodule.

Proposition 4.13.3. *(A "structure theorem".)*

1. *The orbits $G \cdot x \subseteq X$ are locally closed, smooth, and each component has dimension $\dim G - \dim G_x$.*
2. *$\overline{G \cdot x}$ is the union of orbits of smaller dimension.*
3. *Any orbit of minimal dimension is closed, and there is a closed orbit in $\overline{G \cdot x}$.*

Definition 4.13.4. Let X be a noetherian topological space. A subset $S \subseteq X$ is constructible if it is a finite union or intersection of open and closed subsets. △

Remark 4.13.5. Even if X is a scheme, S may not be a scheme. For example, take $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (x^2, xy)$. If the first coordinate of the image is 0 then the second one is zero too. The image is constructible, but it's not a scheme (roughly, because you can only glue schemes along opens – this is not strictly true but essentially you can't just glue the two half spaces on a point). △

Exercise: Any constructible set is the finite disjoint union of locally closed sets (Vakil 7.4A).

Theorem 4.13.6. *(Due to Chevalley) (Also in Vakil 7.4.2) Let $f\colon X \to Y$ be a morphism of finite type of noetherian schemes. Then $f(X)$ is constructible.*

Proof of Proposition 4.13.3. For 1., we'll show it is locally closed. Let $a\colon G \cdot X \to X$ be given by $(g, x) \mapsto g \cdot x$. The image of a is $G \cdot x$.

The orbit $G \cdot x$ is constructible: write

$$G \cdot x = \coprod_{Z_i \text{ locally closed}} Z_i.$$

Hence one of the Z_i must be open and dense in $\overline{G \cdot x}$. To see this, write each Z_i as an open submersion of closed spaces: we have that there is an open $Z_i \hookrightarrow Y_i$ and a closed $Y_i \hookrightarrow \overline{G \cdot x}$; we know $\cup Y_i = \overline{G \cdot x}$, and (as an exercise) there is $Y_i = \overline{G \cdot x}$. Take Z_i dense in Y_i (modulo insisting that Y_i is irreducible). Then Z_i is open in $G \cdot x$, and $\overline{Z_i} = \overline{G \cdot x}$.

So $G \cdot Z_i = G \cdot x$ is open, so $G \cdot x$ is open in $\overline{G \cdot x}$. Hence $G \cdot x$ is locally closed.

Smoothness follows from standard generic smoothness results and translation from group action; dimension results come from verifying certain flatness conditions.

For 2., show that $\overline{G \cdot x}$ is G-closed.

Exercise: Show that.

3. is immediate from 2. and finiteness (ie. Noetherianness). □

Example 4.13.7.

1. We have $\mathbb{G}_m \circlearrowleft \mathbb{A}^n$, given by $t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$.
2. We have $\mathbb{G}_m \circlearrowleft \mathbb{A}^2$ given by $t \cdot (x, y) = (tx, t^{-1}y)$, giving us $\{xy = c\}$.
3. We have $G = \mathrm{SL}_n$, or $G = \mathrm{GL}_n$, or $G = \mathrm{PGL}_n$, etc.; and $G \circlearrowleft G$ by conjugation, and orbits are conjugacy classes (for example, SL_n is classified with Jordan Normal Form).

△

Lemma 4.13.8.

1. *If $H \subseteq G$ is a subgroup, then $\bar{H}$ is a subgroup.*
2. *If H is a constructible subgroup, then $\bar{H} = H$, that is, H is closed.*
3. *If $f \colon H \to G$ is a homomorphism of algebraic groups, then $\mathrm{im}\,(f)$ and $\ker(f)$ are closed subgroups.*

Proof. For 1., If $x \in \bar{H}$, we want to show that $\overline{H \cdot x} \subseteq \bar{H}$. Well, this is true because $Hx \subseteq H\bar{H} = \bar{H}$, where the equality follows for example because $H\bar{H} = \cup_{y \in H} y\bar{H} = \cup_{y \in H} \overline{yH} = \cup \bar{H}$. Now we can take the closure of $Hx \subseteq \bar{H}$, and that is what we want.

For 2., we first claim that if $U, V \subseteq G$ are open, then $U \cdot V = G$. To see this, note that for $x \in G$, we have $xU^{-1} \cap V \neq \emptyset$, since there is u, v such that $xu^{-1} = v$, that is, $x = uv$. Now suppose H is constructible, and suppose $U \subseteq H$ is open in $\bar{H}$ (and dense). Now $U \cdot U = \bar{H} \subseteq H \cdot H = H$. So $\bar{H} = H$.

For 3., for the image use Chevalley, and for the kernel used the fact that closedness is preserved by pullback. □

Proposition 4.13.9. *Affine implies linear.*

Proof. Last class, we got a finite dimensional G-representation $G \hookrightarrow V$ that is G-equivariant, where $G \circlearrowleft G$ by left multiplication. So $G \to \mathrm{GL}(V)$ which is injective since G acts on G transitively. So it is a closed sub. □

Let's talk about homogeneous spaces. Let $H \subseteq G$ be a closed subgroups. Is there a quotient G/H?

Definition 4.13.10. Suppose $G \circlearrowleft X$. Then $p \colon X \to Y$ is a geometric quotient if

(i) p is surjective, and the fibers are precisely G-orbits

(ii) p is open

(iii) For each $U \subseteq Y$ open, then the diagram

$$\begin{tikzcd} \mathscr{O}(U) \arrow[r, "\pi^\sharp(U)"] \arrow[dr, dashed, "\cong"'] & \mathscr{O}(\pi^{-1}(U)) \\ & \mathscr{O}(\pi^{-1}(U)) \arrow[u, "G"'] \end{tikzcd}$$

commutes, where the map G is the induced representation.

△

Definition 4.13.11. Suppose $G \circlearrowleft X$. The categorical quotient (in **AffSch/Sch**) is $p \colon X \to Y$ such that for any Z in the category (ie. **AffSch/Sch**) with a trivial G-action and map, we have

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr] & Y \arrow[d, dashed, "\exists!"] \\ & Z \end{tikzcd}$$

that is, Y is initial in G-trivial schemes under X. △

We claim the following. Let $G = \mathrm{SL}_2$ and $B \in G$. In **Sch**, the categorical quotient $G/B = \mathbb{P}^1$ (for example by looking at this as a stack, showing it's realizable as a scheme, etc.). In **AffSch**, the categorical quotient G/B is just a point.

Theorem 4.13.12. *(due to Chevalley.) Let G be linear, and $H \hookrightarrow G$ be a closed submersion. There is a finite dimenisional G-representation V and $\ell \subset V$ such that $G_\ell = H$.*

The punchline is that we are looking for $G \circlearrowleft X$ transitive and $x \in X$ such that $G_x = H$.

Proof. Consider

$$I_H \longrightarrow \mathscr{O}(G) \twoheadrightarrow \mathscr{O}(H)$$

with $I_H = \langle f_1, \ldots, f_r \rangle$. Let $W = \operatorname{Span}(H \cdot \{f_1, \ldots, f_r\})$, and note that W is a finite dimensional H-representation. We claim that $\operatorname{Stab}(W) = H$. Indeed, $f \in I_H$ if and only if f vanishes on H, so $x \cdot f(-) = f(x^{-1} \cdot -)$ wich vanishes on xH. So $H \hookrightarrow \operatorname{Stab}(W)$. Also, $\operatorname{Stab}(W) \subseteq H$: if x stabilizes W, then $\langle x \cdot W \rangle$ defines a subscheme $Z = xH$. So $x \in H$.

Now take $V = \operatorname{Span}(G \cdot \{f_1, \ldots, f_r\})$. Now consider the line

$$\ell := \bigwedge^{\dim(W)} W \subseteq \bigwedge^{\dim(W)} V.$$

We claim that $\operatorname{Stab}(\ell) = H$. It is easy to show that $H \subseteq \operatorname{Stab}(\ell)$. The reverse direction is a little trickier: we have $\omega \in \ell$ if and only if $\omega \wedge w = 0$ for all $w \in W$. Take $g \in G$ and $\omega \in \wedge^{\mathrm{top}} W$, such that $g\omega = c\omega$ for some constant c. This is equivalent to saying $g\omega \wedge w = 0$ for all $w \in W$, which says that $g(\omega \wedge g^{-1} w) = 0$, which happens if and only if $g^{-1} \in \operatorname{Stab}(W) = H$. □

To get the punchline, we take $\mathbb{P}(V) \supseteq G \cdot \ell$ locally closed. This defines a subscheme X (since it's quasiprojective). This is (a candidate for) homogenous spaces.

Let's talk about the Proj construction. From this point of view, you might have $\mathbb{G}_m \circlearrowleft V - \{0\}$. The problem is that $V - \{0\}$ is not affine (though it's quasiaffine). So choose coordinates $x_0, \ldots, x_n$ of V^* and $U_i\{x_i \neq 0\}$ is affine, and $\mathscr{O}(U_i) = k[x_0, \ldots, x_n, x_i^{-1}]$. Then $U_i/\mathbb{G}_m = \operatorname{Spec} \mathscr{O}(U_i)^{\mathbb{G}_m} = \operatorname{Spec} k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$. Now, glue them together to get $\mathbb{P}(V)$.

More generally, if A is a positively graded algebra, $\operatorname{Spec} A$ is a "cone" (there's a scaling action that contracts everything to a single point). Then we can do the same thing, that is, delete the cone point and then quotient.

5 Schemes

5.14 Mar 12, 2019

Previously, we wrote $x \rightsquigarrow x'$ if $x' \in \overline{\{x\}}$, and we say x specializes to x'. We say that $Z \subseteq X$ is closed under specialization if for all $x, x' \in X$ such that $x \in Z$ and $x \rightsquigarrow x'$, we have $x' \in Z$. We had three lemmas:

Lemma 5.14.1. *(for algebra.)*
Suppose $f^\sharp \colon R \twoheadrightarrow A$ is a ring homomorphism with $T \subseteq \operatorname{Spec} A$ closed. Then if $f(T)$ is closed under specialization, then $f(T)$ is closed. See Lemma 3.11.5.

Lemma 5.14.2. *(for topology.)*

1. *If specializations lift along f and $T \subseteq X$ is closed under specializations, so is $f(T)$.*
2. *Specializations lift along closed maps.*

See Lemma 3.11.7.

Lemma 5.14.3. *(for number theory.)*
Suppose $f \colon X \to X$ is a quasicompact morphism. Then f is closed if and only if specializations lift along f. In the affine case, this corresponds to going up. See Lemma 3.11.9.

We talked about valuative criterion: let $f \colon X \to S$ be a morphism of schemes. We say f satisfies the existence part of the valuative criterion if given any diagram below for any valuation ring A (along with its fraction field K), the dashed arrow exists:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists"] & S \end{tikzcd}$$

We say it satisfies the uniqueness part if whenever the dotted arrow exists, then it's unique. You can satisfy uniqueness part without the existence part (ie., you can have a unique map for some of the valuation rings and no map for others).

We have

Lemma 5.14.4. *Let $f \colon X \to S$ be a morphism of schemes. The following are equivalent:*

(i) f satisfies the existence part of the valuative criterion.

(ii) Specializations lift along any base change of f.

We have the following terminology: if f satisfies a property P that is preserved upon all the base change morphisms, then we say it is "universally P". Thus the above lemma says that if f satisfies the existence part of the valuative criterion, then f satisfies it universally.

Proof. We have maps

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X_{S'} \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists"] \arrow[urr, dashed, "\exists"] & S' \arrow[r] & S \end{tikzcd}$$

where $\operatorname{Spec} A \to X_{S'}$ exists by the universal property fo the fiber product, and $\operatorname{Spec} A \to X$ exists by our hypotheses. Hence to show that (i) implies (ii), it is enough to show that specializations lift along f.

Let $s \rightsquigarrow s'$ in S, and let $x \in X$ such that $f(x) = s$ (assume $s \neq s'$). We have maps

$$\begin{tikzcd}
\operatorname{Spec} K(x) \arrow[rr] \arrow[d] & & X \arrow[d] \\
\operatorname{Spec} K(s) \arrow[r] & \operatorname{Spec} \mathcal{O}_{S,s'} \arrow[r] & S
\end{tikzcd}$$

Let R be the image of $\mathcal{O}_{S,s'}$ in K (algebraically). Then R is not equal to the image of $K(s)$ in K because $s \neq s'$. This implies that R is dominated by

$$\begin{tikzcd}
K(x) = K & R \arrow[l, hook'] \\
K(s) \arrow[u, hook] & \mathcal{O}_{S,s'} \arrow[l] \arrow[u]
\end{tikzcd}$$

There exists a valuation ring A with fraction field K so that

$$\begin{tikzcd}
K(x) = K & A \arrow[l, hook'] & R \arrow[l, hook'] \\
K(s) \arrow[u, hook] & & \mathcal{O}_{S,s'} \arrow[ll] \arrow[u]
\end{tikzcd}$$

where A is the natural valuation induced by the maximal ideal. Under Spec we get

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[rrr] \arrow[d] & & & X \arrow[d] \\
\operatorname{Spec} A \arrow[r] \arrow[urrr, dashed, "\exists"] & \operatorname{Spec} R \arrow[r] & \operatorname{Spec} \mathcal{O}_{S,s'} \arrow[r] & S
\end{tikzcd}$$

where the dashed arrow exists by the existence part of the valuative criterion.

Conversely, suppose we are given

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r] & S
\end{tikzcd}$$

Then by the definition of fiber product we get

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X_A = X \times_S A \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r] & \operatorname{Spec} A \arrow[r] & S
\end{tikzcd}$$

Since specializations lift along f', there is $x' \in X_A$ such that $x \rightsquigarrow x'$ where x is the image of $\operatorname{Spec} K$ and $f(x')$ is the closed point of $\operatorname{Spec} A$. Algebraically, we get maps

$$\begin{tikzcd}
K & \mathcal{O}_{X_A,x'} \arrow[l] \\
A \arrow[u] \arrow[ur]
\end{tikzcd}$$

and the ring obtained as the image of $\mathcal{O}_{X_A,x'}$ in K dominates A (and hence must equal A). Hence, the map

$$\begin{tikzcd}
\mathcal{O}_{X_A,x'} \arrow[r] \arrow[d, dashed] & K \\
A \arrow[ur]
\end{tikzcd}$$

factors, which is a local homomorphism of local rings. Hence we get a section of $\operatorname{Spec} A \to X_A$. Compose with the projection $X_A \to X$. $\square$

Definition 5.14.5. A morphism $f\colon X \to S$ is universally closed if for all $S' \to S$, the base change $X_{S'} := X \times_S S' \to S'$ is closed. △

Lemma 5.14.6. *Let $f\colon X \to S$ be quasicompact. The following are equivalent:*

(i) f is universally closed

(ii) The existence part of the valuative criterion holds for f.

Proof. By the above Lemma 5.14.4 and our topology Lemma 3.11.7 on specializations. □

Let's talk about separability. The idea is the following: For a topological space X, X is Hausdorff if and only if $\Delta\colon X \to X \times X$ given by $x \mapsto (x, x)$ (endowed with the product topology) is closed.

Lemma 5.14.7. *Let $f\colon X \to S$ be a morphism of schemes. The diagonal map $\Delta\colon X \to X \times_S X$ is an immersion.*

Proof. Define the open set

$$W = \bigcup_{U,V \text{satisfying } (*)} U \times_V U \subseteq X \times_S X.$$

where $(*)$ denotes the condition: "$U \subseteq X$ is open affine, $V \subseteq S$ is open affine, and $f(U) \subseteq V$". We want to show that $\Delta(X) \subseteq W$.

For all $x \in X$, take $V \subseteq S$ open affine such that $f(x) \in V$. Take $U \ni x$ open affine such that $f(U) \subseteq V$. Then (U, V) satisfies $(*)$ and $\Delta(x) = (x, x) \in U \times_V U$.

Now suffices to show that $\Delta\colon X \to W$ is closed. Since $(*)$ holds for (U, V), say with $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} R$, then $\Delta\colon U \to U \times_V U$ is a closed immersion, since on the algebra side we have $A \otimes_R A \to A \to 0$. □

Corollary 5.14.8. *We have that Δ is closed if and only if $\Delta(X) \subseteq X \times_S X$ is a closed subset.*

(This is because an immersion is closed if and only if its image is closed)

Corollary 5.14.9. *Given a commutative diagram*

$$\begin{tikzcd} X \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[dr] & Y \\ & S \arrow[u] \end{tikzcd}$$

we have that the equalizer Z of a and b exists (because of fiber products). It's at least a locally closed subscheme of X. Furthermore, Z is closed if and only if $\Delta_{Y/S}$ is closed.

Proof. We have $(a, b)\colon X \times_S X \to Y \times_S Y$. Then the equalizer is the fiber product

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[d, "\Delta_{Y/S}"] \\ X \times_S X \arrow[r, "{(a,b)}"] & Y \times_S Y \end{tikzcd}$$

Then use the fact that immersions are stable under base change. □

Definition 5.14.10. Let $f\colon X \to S$ be a morphism of schemes. Then

- f is separated if $\Delta_{X/S}$ is closed.
- f is quasi-separated if $\Delta_{X/S}$ is quasicompact.
- A scheme S is (quasi)-separated if $\Delta_{S/\operatorname{Spec}\mathbb{Z}}$ is (quasi)-separated.

$\triangle$

Sometimes quasi-separated is shortened to “qs”.

Lemma 5.14.11. *(Characterization of quasi-separated morphisms.) Let $f\colon X \to S$. The following are equivalent:*

(i) f is quasi-separated.

(ii) For all U, V open affines mapping into a common affine open in S, the open $U \cap V$ is quasicompact.

(iii) There exists an affine open covering $S = \cup_{i\in I} U_i$ with $f^{-1}(U_i) = \cup_{j\in J_i} V_j$ an affine open covering such that for all $j_1, j_2 \in J_i$, we have $V_{j_1} \cap V_{j_2}$ being quasicompact.

Lemma 5.14.12. *(Characterization of separated morphisms.) Let $f\colon X \to S$. The following are equivalent:*

(i) f is separated.

(ii) For all U, V mapping into common affine open in S, we have

(a) The open $U \cap V$ is affine.

(b) The map $\mathcal{O}_X(U) \otimes_{\mathbb{Z}} \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is onto.

(iii) For all $x, x' \in X$ such that $f(x) = f(x')$, there exist affine opens $U \ni x$, $V \ni x'$ in X map into common affine opens W in S such that both (a) and (b) hold.

Proof of Lemma 5.14.12. Let's show that (i) implies (ii). Indeed, since f is separated, and $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$ map into $W = \operatorname{Spec} R \subseteq S$ an open affine, say via $p\colon U \to W$ and $q\colon V \to W$, then

$$\operatorname{Spec}(A \otimes_R B) = U \times_W V = p^{-1}(U) \cap q^{-1}(V) \subseteq X \times_S X'$$

is an affine open. Since f is separated, then Δ is a closed immersion. It follows that $U \cap V = \Delta^{-1}(U \times_W V)$ is closed. This implies that $U \cap V = \operatorname{Spec}(A \otimes_R B/I)$ for some ideal $I \subseteq A \otimes_R B$. Thus we get maps $A \otimes_{\mathbb{Z}} B \twoheadrightarrow A \otimes_R B \twoheadrightarrow A \otimes_R B/I$.

That (ii) implies (iii) is immediate.

To show that (iii) implies (i), we note that the $U \times_W V$'s form an open affine cover of $X \times_S X$. Thus it is enough to show that $\Delta^{-1}(U \times_W V) = \cap V \to U \times_W V$ is closed. By (iii), $U \cap V = \operatorname{Spec} C$ and $A \otimes_{\mathbb{Z}} B \to A \otimes_R B \to C$ is surjective (since $A \otimes_{\mathbb{Z}} B \twoheadrightarrow A \otimes_R B$, $A \otimes_R B \twoheadrightarrow C$, and Δ is a closed immersion). $\square$

Corollary 5.14.13. *Any affine scheme is separated.*

Proof. $R \otimes_{\mathbb{Z}} R \twoheadrightarrow R$. $\square$

Remark 5.14.14. If $X \to S$ is separated and S is separated, then the intersection of any two affines in X is affine. $\triangle$

The composition of separated morphisms is separated. Indeed, consider the maps

$$\begin{tikzcd}
X \arrow[r, "\Delta_S"] \arrow[dr, "\Delta_T"'] & X \times_S X \arrow[r] \arrow[d] & S \arrow[d] \\
& X \times_T X \arrow[r] & S \times_T S
\end{tikzcd}$$

Since the right side is a fiber product, the composition $X \to X \times_T X$ is closed. We get that $X \to \operatorname{Spec} \mathbb{Z}$ is separated and $U \cap V = \Delta_{\mathbb{Z}}^{-1}(U \times_{\mathbb{Z}} V)$ is a closed subscheme of an affine scheme and thus affine.

Theorem 5.14.15. *(Valuative criteria for separatedness.) Let $f\colon X \to S$ be a morphism of schemes. If f is quasi-separated, and f satisfies the uniqueness part of the valuative criterion, then f is separated.*

Remark 5.14.16. If S is locally noetherian and f is locally of finite type, then f is automatically quasi-separated. $\triangle$

5.15 Mar 14, 2019

Last time, we talked about valuative criterion: for all valuation rings A with fraction field K, we have the commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

and we say that the morphism f of schemes satisfies the existence part of the valuative criterion (sometimes written (Exist)) if the dashed arrow exists, and that it satisfies the uniqueness part of the valuative criterion (sometimes written (Unique)) if whenever the dashed arrow exists, then it's unique. If f is quasicompact, then (Exist)if and only if f is universally closed. We said f is separated if $\Delta_{X/S}$ is closed, and quasi-separated if $\Delta_{X/S}$ is quasicompact. We say that the scheme X is quasi-separated if $\Delta_{X/\operatorname{Spec}\mathbb{Z}}$ is quasi-separated.

Lemma 5.15.1. *The map f is separated if and only if it satisfies an open affine matching condition and onto-ness. (cf. Lemma 5.14.12)*

Corollary 5.15.2. *We have:*

- *Affine schemes are separated.*
- *Separated compositions are separated.*

Theorem 5.15.3. *(Valuative criterion of separatedness.) Let $f\colon X \to S$ be a morphism. If f is quasi-separated, and f satisfies* (Unique)*, then f is separated.*

Remark 5.15.4. In particular, if S is locally noetherian, and f is clocally of finite type, then f is quasi-separated. △

Proof of Theorem 5.14.15. We need to show that $X \xrightarrow{\Delta} X \times_S X$ is closed. Given

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ \operatorname{Spec} A \arrow[r, "g=(a,b)"] & X \times_S X \end{tikzcd}$$

we get maps

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, shift left, "a"] \arrow[ur, shift right, "b"'] & S \end{tikzcd}$$

Consider the open U on which a and b agree. Then $\Delta \circ a = (a, a) = (a, b) = g$. So we get

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ \operatorname{Spec} A \arrow[r, "g"] \arrow[ur, "a"] & X \times_S X \end{tikzcd}$$

□

Let's see that $\mathbb{P}^1_R$ is separated.

Recall our construction of

$$\mathbb{A}^1_R = \operatorname{Spec} R[x] \supseteq D(x) \xleftrightarrow{\text{glue}} D(y) \subseteq \mathbb{A}^1_R = \operatorname{Spec} R[y]$$
$$x \mapsto y^{-1}$$
$$x^{-1} \leftarrow\!\shortmid y$$

Let's use this to check separated. We need to see that for the open covering U, V here, we get $\mathcal{O}(U)\otimes_{\mathbb{Z}}\mathcal{O}(V) \twoheadrightarrow \mathcal{O}(U\cap V)$. We have maps $R[x], R[x] \to R[x]$ and $R[x], R[y] \to R[x, x^{-1}]$ (given by $y \mapsto x^{-1}$) and $R[y], R[x] \to R[x, x^{-1}]$ given by $y \mapsto x$.

Let's see that $\mathbb{P}^1_R \to \operatorname{Spec} R$ is universally closed. It is enough to check (Exist). Consider

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathbb{P}^1_R \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

Suppose $\Im(\operatorname{Spec} K) \subseteq \operatorname{Spec} R[x]$. Then on the algebra side we have

$$\begin{tikzcd} K & R[x] \arrow[l, "q^\sharp"'] \\ A \arrow[u] & R \arrow[l] \arrow[u] \end{tikzcd}$$

If $q^\sharp(x) \in A$ then we're done. Otherwise, since A is a valuation ring we have $q^\sharp(x)^{-1} = q^\sharp(y) \in A$. Then we get

$$\begin{tikzcd} K & R[y] \arrow[l, "q^\sharp"'] \arrow[dl] \\ A \arrow[u] & R \arrow[l] \arrow[u] \end{tikzcd}$$

which gives the existence of a map $\operatorname{Spec} A \to \operatorname{Spec} R[y]$, as we wanted.

Quasicoherent sheaves.

Definition 5.15.5. Let X be a ringed space. We say that an $\mathcal{O}_X$-module is quasi-coherent (sometimes denoted quasi-coh, or QC (with capital letters to distinguish from qc = quasicompact)) if for all $x \in X$, there exists a $U \ni x$ open and an exact sequence

$$\bigoplus_{j\in J} \mathcal{O}_U \longrightarrow \bigoplus_{i\in I} \mathcal{O}_U \longrightarrow \mathcal{F}|_U \longrightarrow 0.$$

Notice that there are no finiteness restrictions on I and J. $\triangle$

Lemma 5.15.6. *Let X be a scheme, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent:*

1. *$\mathcal{F}$ is quasicoherent.*
2. *For all affine opens $U = \operatorname{Spec} R \subseteq X$, we have $\mathcal{F}|_U \cong \widetilde{M}$ for an R-module M.*
3. *There exists an affine open cover*
$$X = \bigcup_{i\in I} \operatorname{Spec} R_i$$
such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$.

We won't prove thus, but it is in EGA or in Hartshorne.

Lemma 5.15.7. *Let $X = \operatorname{Spec} R$. Let M be an R-module and let $\mathcal{G}$ be an $\mathcal{O}$-module. Then*

$$\operatorname{Mor}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{G}) = \operatorname{Hom}_R(M, \Gamma(X, \mathcal{G})),$$

given by $\beta \mapsto \beta_X : M = \Gamma(X, \widetilde{M}) \to \Gamma(X, \mathcal{G})$.

Lemma 5.15.8. *Let X be a scheme. We have* *(a) Kernels and cokernels of maps between quasicoherent sheaves are quasicoherent.*

(b) If we have a short exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

of $\mathcal{O}_X$-modules, then if two out of three are quasicoherent, then the third is also quasicoherent.

Proof. For part (a), we note that by characterization of quasicoherence on schemes, it is enough to check when $X = \operatorname{Spec} R$. Now let $\widetilde{\varphi}\colon \widetilde{M} \to \widetilde{N}$ be an $\mathcal{O}_X$-module morphism. By the previous Lemma 5.15.7, we have some R-module map $\varphi\colon M \to N$ that gives rise to $\widetilde{\varphi}$. We want to show that $\ker\widetilde{\varphi} = \widetilde{(\ker\varphi)}$ and that $\operatorname{coker}\widetilde{\varphi} = \widetilde{(\operatorname{coker}\varphi)}$. It is enough to show for $\mathcal{O}_X$-modules that

$$0 \longrightarrow \widetilde{(\ker\varphi)} \longrightarrow \widetilde{M} \xrightarrow{\widetilde{\varphi}} \widetilde{N} \longrightarrow \widetilde{(\operatorname{coker}\varphi)} \longrightarrow 0$$

is exact. We can just check this on stalks:

$$0 \longrightarrow (\ker\varphi)_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \xrightarrow{\varphi_{\mathfrak{p}}} N_{\mathfrak{p}} \longrightarrow (\operatorname{coker}\varphi)_{\mathfrak{p}} \longrightarrow 0$$

and localization is exact. This proves part (a).

For part (b), it is enough to show on $\operatorname{Spec} R$ that if we have a short exact sequence of $\mathcal{O}_X$-modules given by

$$0 \longrightarrow \widetilde{M_1} \longrightarrow \mathcal{F} \longrightarrow \widetilde{M_2} \longrightarrow 0$$

then $\mathcal{F} = \widetilde{M}$ for some R-module M. It suffices to show that

$$0 \longrightarrow \Gamma(X, \widetilde{M_1}) \longrightarrow \Gamma(X, F) \xrightarrow{\beta_X} \Gamma(X, \widetilde{M_2}) \longrightarrow 0$$

is exact, namely β_X is surjective.

Let $m_2 \in M_2$. Consider $I = \{f \in R\colon f \cdot m_2 \in \operatorname{im}\beta_X\}$. Then I is an ideal of R. We want to show that $I = R$. We have

$$X = \bigcup_{i=1}^{n} D(f_i)$$

our standard open cover such that m_2 lifts locally, ie. there are $s_i \in \mathcal{F}(D(f_i))$ with $\beta(s_i) = m_2|_{D(f_i)}$. Then $s_i|_{D(f_i f_j)} - s_j|_{D(f_i f_j)} \in \ker\beta|_{D(f_i f_j)} = \operatorname{im}\widetilde{M_1}(D(f_i f_j))$. So we get

$$s_i|_{D(f_i f_j)} - s_j|_{D(f_i f_j)} = \frac{m_{ij}}{(f_i f_j)^A}.$$

Since we have finitely many indices i and j, we can take A to be large enough for all i's and j's.

Fix i_0, and set $s'_{i_0} = (f_{i_0})^A s_{i_0}$ and $s'_i = f_{i_0}^A s_i + m_{i_0,i}/f_i^A \in \mathcal{F}(U_i)$ for all $i \neq i_0$. We have

$$\begin{aligned} s'_i - s'_{i_0} &= f_{i_0}^A s_i + \frac{m_{i_0,i}}{f_i^A} - f_{i_0}^A s_{i_0} \\ &= -f_{i_0}^A(s_{i_0} - s_i) + \frac{m_{i_0,i}}{f_i^A} \\ &= \frac{-m_{i_0,i}}{f_i^A} + \frac{m_{i_0,i}}{f_i^A} = 0. \end{aligned}$$

If $i \neq j$ with neither equal to i_0, we get

$$s'_i - s'_j = f_{i_0}^A (s_i - s_j) - \frac{m_{i_0,i}}{f_i^A} + \frac{m_{i_0,j}}{f_j^A}$$
$$f_{i_0}^A \left(\frac{m_{i,j}}{(f_i f_j)^A} \right) - \frac{m_{i_0,i}}{f_i^A} + \frac{m_{i_0,j}}{f_j^A}.$$

Note that as a section of f, we have

$$\Gamma(D(f_{i_0} f_i f_j), \widetilde{M_1}) = (M_1)_{f_{i_0} f_i f_j} = ((M_1)_{f_i f_j})_{f_{i_0}}.$$

Thus we have
$$\frac{m_{ij}}{(f_i f_j)^A} + \frac{m_{i_0,i}}{(f_{i_0} f_i)^A} - \frac{m_{i_0,j}}{(f_{i_0} f_j)^A} = (s_i - s_j) + (s_{i_0} - s_i) - (s_{i_0} - s_j) = 0.$$

So by multiplying by a large enough power B of f_{i_0}, the corresponding element on the left hand side will be killed in $M_{f_i f_j}$. Again by finiteness of i and j, we can take B sufficiently large to work for all pairs (i, j). Thus, set $s''_{i_0} = f_{i_0}^{A+B} s_{i_0}$ and $s''_i = f_{i_0}^{A+B} s_i + f_{i_0}^B m_{i_0,i}/f_i^A$, for $i \neq i_0$.

These glue to a section $s \in \Gamma(X, \mathcal{F})$, ie., $f_{i_0}^{A+B} \cdot m_2 \in \beta_X(s)$. Thus, since we can do this for an arbitrary choice of index $i_0 \in [n]$, then $R = \langle f_1^N, \ldots, f_n^N \rangle \subseteq I$ for some N sufficiently large. This means that $R = I$. $\square$

Remark 5.15.9. Later, these kinds of facts will follow because $H^i(\operatorname{Spec} R, \mathcal{F})$ vanishes for $i > 0$ and $\mathcal{F}$ quasicoherent (ie. there is no higher cohomology). $\triangle$

5.16 Mar 19, 2019

Last time, we saw that a map $f\colon X \to S$ is separated if and only if $\Delta_{X/S}$ is closed, if and only if it satisfies some algebraic criterion, and f is quasi-separated if and only if it satisfies (Unique)(ie. the uniqueness part of the valuative criterion), which states that if A is a valuation ring, the dashed arrow below is unique when it exists:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] & X \\ \operatorname{Spec} A \arrow[u] \arrow[r] \arrow[ur, dashed, "\exists!"] & S \arrow[u] \end{tikzcd}$$

We also talked about quasi-coherent schemes: we say that an $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if for all $x \in X$, there is a $U \ni x$ open in X such that we have an exact sequence

$$\bigoplus_{j\in J} \mathcal{O}_U \longrightarrow \bigoplus_{i\in I} \mathcal{O}_U \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

We want to pullback and pushforward quasicoherent sheaves. So let $f\colon X \to S$ be a morphism of schemes.

Proposition 5.16.1. *Let $\mathcal{F}$ be a quasicoherent sheaf of $\mathcal{O}_S$-modules. Then $f^*\mathcal{F}$ is quasicoherent on X. In particular, if $X = \operatorname{Spec} A$, and $S = \operatorname{Spec} R$, and $\mathcal{F} = \widetilde{M}$ on S, then $f^*\mathcal{F} = \widetilde{A \otimes_R M}$.*

Proof. For all $x \in X$, there is an open affine U on X such that $f(U) \subseteq V$ is open affine in S. Then we have

$$\bigoplus_{j\in J} \mathcal{O}_V \longrightarrow \bigoplus_{i\in I} \mathcal{O}_V \longrightarrow \mathcal{F} \longrightarrow 0$$

by quasicoherence of $\mathcal{F}$. We apply f^* to get

$$\bigoplus_{j\in J} \mathcal{O}_U \longrightarrow \bigoplus_{i\in I} \mathcal{O}_U \longrightarrow f^*\mathcal{F} \longrightarrow 0$$

where exactness follows because $(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{S,f(x)}} \mathcal{O}_{X,x}$, and $\otimes$ is right exact.

For the second part, we note that

$$\begin{aligned} \operatorname{Mor}_{\mathcal{O}_X}(f^*\mathcal{F}, \mathcal{G}) &= \operatorname{Mor}_{\mathcal{O}_S}(\mathcal{F}, f_*\mathcal{G}) \\ &= \operatorname{Hom}_R(M, \Gamma(S, f_*\mathcal{G})) \\ &= \operatorname{Hom}_R(M, \Gamma(X, \mathcal{G})) \\ &= \operatorname{Hom}_A(M \otimes_R A, \Gamma(X, \mathcal{G})) \\ &= \operatorname{Mor}_{\mathcal{O}_X}(\widetilde{M \otimes_R A}, \mathcal{G}). \end{aligned}$$

□

For pushforwards, note that on affines $f\colon \operatorname{Spec} A \to \operatorname{Spec} R$, if N is an A-module, then $f_*(\widetilde{N}) = \widetilde{N_R}$ by considering N as an R-module.

Remark 5.16.2. Pushforwards along proper maps behave well, but along general morphisms, quasicoherence might not be preserved. △

Example 5.16.3. Let k be a field (for simplicity). Consider

$$S = \coprod_{n=1}^{\infty} \operatorname{Spec} k[x] \xrightarrow{f} \operatorname{Spec} k[x].$$

We get $\mathcal{F} = \mathcal{O}_X$ with global sections $\Gamma(S, f_*\mathcal{F}) = \Gamma(X, \mathcal{F}) = \prod_{n=1}^{\infty} k[x]$.

On the other hand, $\Gamma(D(x), f_*\mathcal{F}) = \Gamma(f^{-1}(D(x)), \mathcal{F}) = \prod_{n=1}^\infty k[x]_{\langle x\rangle}$. But we have $\Gamma(S, f_*\mathcal{F}) \to \Gamma(D(x), f_*\mathcal{F})$, which induces a map

$$\left(\prod_{n=1}^\infty k[x]\right)_{\langle x\rangle} \to \prod_{n=1}^\infty k[x]_{\langle x\rangle}$$

and if $f_*\mathcal{F}$ were quasicoherent this would be an isomorphism. But this is not: for example, $1, 1/x, 1/x^2, \dots) =$ does not lie in the image. △

Proposition 5.16.4. *If $f\colon X \to S$ is quasiseparated and quasicompact, then f_* preserves the property of being quasicoherent.*

Proof. We reduce immediately to the case where S is affine.

Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module, and let $X = \cup_{i=1}^n X_i$ be an open affine cover of X. Since X is quasiseparated, we have

$$X_i \cap X_j = \bigcup_{k=1}^{N_{i,j}} X_{i,j,k},$$

with each $X_{i,j}$ affine opens. Then we get an exact sequence

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \bigoplus_i (f|_{X_i})_*(\mathcal{F}|_{X_i}) \longrightarrow \bigoplus_{i,j,k} (f|_{X_{i,j,k}})_*(\mathcal{F}|_{X_{i,j,k}})$$

Hence $f_*\mathcal{F}$ is the kernel of the map between quasicoherent sheaves. By Lemma 5.15.8, this says that $f_*\mathcal{F}$ is quasicoherent. □

We want to talk about properties that can be detected locally. Let X be a scheme and let $\mathcal{P}$ be some property of rings. We want to study the notion of "X being locally $\mathcal{P}$", that is, for all $x \in X$, there exists $U \ni x$ affine and open such that $\mathcal{O}(U)$ satisfies $\mathcal{P}$.

We say "$\mathcal{P}$ is local" if

(a) $\mathcal{P}$ for R implies $\mathcal{P}$ for R_f for all $f \in R$.

(b) If $f_1, \dots, f_n$ generate R and R_f satisfies $\mathcal{P}$, then R satisfies $\mathcal{P}$.

Lemma 5.16.5. *If $\mathcal{P}$ is a local property of rings and X is a scheme, then the following are equivalent:*

1. *X is locally $\mathcal{P}$.*
2. *For all $U \subseteq X$ open affine, $\mathcal{O}(U)$ satisfies $\mathcal{P}$.*
3. *There is an open affine covering $X = \cup_i U_i$ such that $\mathcal{O}(U_i)$ satisfy $\mathcal{P}$.*
4. *There is an open affine covering $X = \cup_i U_i$ such that each each X_i is locally $\mathcal{P}$.*

If any of these equivalent conditions hold, then any open subscheme $Y \subset X$ is locally $\mathcal{P}$.

Proof. The only nontrivial part is the implication 3. implies 2..

For all $X = \cup_i U_i$ affine open we have $\mathcal{O}(U_i)$ satisfy $\mathcal{P}$ for all i, by the previous lemma $U = \cup_{j=1}^m W_j$ of standard opens in U and in some subindexed open U_{ij}. This means that $\mathcal{O}(U_{ij})$ satisfies $\mathcal{P}$, and hence $\mathcal{O}(W_j)$ satisfy $\mathcal{P}$, and hence $\mathcal{O}(U)$ satisfy $\mathcal{P}$. □

Example 5.16.6. Being "Noetherian" $=: \mathcal{P}$ is a local property.

Indeed, if R is Noetherian then R_f is Noetherian, and given the sequence

$$0 \longrightarrow R \longrightarrow \prod_{i=1}^n R_{f_i} \longrightarrow \prod_{i,j}^n R_{f_i f_j}$$

Since R_{f_i} is Noetherian we see that $R_{f_if_j}$ is Noetherian. Since R is the kernel of a map between Noetherian rings, it is Noetherian. △

Definition 5.16.7. A scheme X is said to be Noetherian if X is locally Noetherian and quasi-compact. △

Proposition 5.16.8. *If $j\colon U \hookrightarrow X$ is an immersion and X is locally Noetherian, then j is quasicompact.*

Proof. We know that X admits a cover by affine opens which are Spec s of Noetherian rings and affine opens are thus quasicompact as topological spaces. Hence their subspaces are also quasicompact. □

Definition 5.16.9. Let $\mathcal{P}$ be a property of ring homomorphisms. We say "$\mathcal{P}$ is local" if the following conditions hold:

(a) For all $f \in R$, if $R \to A$ satisfies $\mathcal{P}$ then $R_f \to A_f$ satisfies $\mathcal{P}$.

(b) For all $f \in R$ and $a \in A$ and $R_f \to A$, if $R_f \to A$ satisfies $\mathcal{P}$, then $R \to A_a$ satisfies $\mathcal{P}$.

(c) For all $R \to A$, if $R \to A_{a_i}$ satisfies $\mathcal{P}$ and the a_i generate A, then $R \to A$ satisfies $\mathcal{P}$. △

Remark 5.16.10. Usually the first two conditions are almost immediate; condition (c) is hardest to verify in practice. △

Here are some examples of local properties of ring homomorhpisms:

Example 5.16.11. $R \to A$ being finite type, that is to say, A being a finite type R-algebra. △

Example 5.16.12. $R \to A$ being finite presentation, that is to say, A is of finite presentation over R (as an R-algebra, A has finitely many generators and relations). △

Example 5.16.13. $R \to A$ being flat, that is to say, A is a flat R-module. △

Example 5.16.14. We'll see later that "smoothness" is also a local property of ring homomorphisms. △

Let's check that Example 5.16.11 is a local property of ring homomorphisms.

We get condition (a), since $R \to A$ satisfies $\mathcal{P}$ if and only if $A = R[x_1, \dots, x_n]/I$, which implies $A_f = R_f[x_1, \dots, x_n]/IR_f[x]$, which happens if and only if $R_f \to A_f$ satisfies $\mathcal{P}$.

We get condition (b) since $R_f \to A$ satisfies $\mathcal{P}$ precisely when

$$A = R_f[x_1, \dots, x_n]/I = R[x_1, \dots, x_n, y]/\langle I, yf - 1\rangle.$$

This says that $A_a = R[x_1, \dots, x_n, y, z]/\langle I, yf - 1, za - 1\rangle$, which happens if and only if $R \to A_a$ satisfies $\mathcal{P}$.

We get condition (c), since if $R \to A_{a_i}$ satisfies $\mathcal{P}$ then $A_{a_i} = R[x_1, \dots, x_{ik_i}]/I_i$. Each $\overline{x_{ij}}$ in A_{a_i} must be of the form h_{ij}/a_i^N for some sufficiently large N. Since $\langle a_1, \dots, a_n\rangle = A$, then the $D(a_i)$ cover Spec A, and $1 = \sum a_i g_i$ for some g_i. We claim that $R[u_i, v_{ij}, z_k] \twoheadrightarrow A$ given by $U_i \mapsto a_i, v_{ij} \mapsto h_{ij}, z_k \mapsto g_k$. Indeed, for all $a \in A$, there are sufficiently large N, M so that

$$a = a \cdot 1 = \sum a_i^{N+M} \tilde{g}_i \cdot a = \sum a_i^M h_{ij} \tilde{g}_i$$

where $\tilde{g}_i$ is some combination of a_j's and g_i's. Take M sufficiently large such that $a_i^M(a \cdot a_i^N - h_{ij}) = 0$. This shows us that A has property $\mathcal{P}$.

5.17 Mar 21, 2019

Last time, we talked about the following question: under what operations is the category of quasicoherent sheaves well behaved? If $f\colon X \to S$ is quasicompact and quasiseparated, then f_* preserves quasicoherentness.

We also talked about local properties in schemes:

- Geometrically, we said that X is "locally $\mathcal{P}$" if for all $x \in X$ there exists $U \ni x$ open affine such that $\mathcal{O}(U)$ is $\mathcal{P}$.
- Algebraically, we said that "$\mathcal{P}$ is local" if R is $\mathcal{P}$ implies R_f is $\mathcal{P}$ for all $f \in R$, and if $R = \langle f_1, \dots, \rangle$ and R_{f_i} is $\mathcal{P}$ for all i, then R is $\mathcal{P}$.
- If "$\mathcal{P}$ is local", then we can check locally $\mathcal{P}$ on an affine covering of X.

Definition 5.17.1. Let $\mathcal{P}$ be a property of ring homomorphisms. We say $\mathcal{P}$ is local if

(a) for all $f \in R$, if $R \to A$ is $\mathcal{P}$, then $R_f \to A_f$ is $\mathcal{P}$,

(b) for all $f \in R$, $a \in A$, and $R_f \to A$, if $R_f \to A$ is $\mathcal{P}$ then $R \to A_a$ is $\mathcal{P}$,

(c) for all $R \to A$, if $R \to A_{a_i}$ is $\mathcal{P}$ and $A = \langle a_1, \dots, a_n \rangle$, then $R \to A$ is $\mathcal{P}$.

△

Examples include being finite type, being finitely presented (sometimes abbreviated "fp"), and being flat.

Definition 5.17.2. We say $\mathcal{P}$ is stable under base change if for all $R \to A$ and $R \to R'$, if $R \to A$ satisfies $\mathcal{P}$ then $R' \to R' \otimes_R A$ satisfies $\mathcal{P}$. We say $\mathcal{P}$ is stable under composition if for all $A \to B$ and $B \to C$ satisfying $\mathcal{P}$, then $A \to C$ also satisfies $\mathcal{P}$. △

Definition 5.17.3. Let $\mathcal{P}$ be a property of ring homomorphisms. Let $f\colon X \to S$ be a morphism of schemes. We say f is locally $\mathcal{P}$ (or "locally of type $\mathcal{P}$") if for all $x \in X$ there exists $U \ni x$ affine open in X, and $V \subseteq S$ affine open, such that

$$f(U) \subseteq V \text{ is } \mathcal{P}. \quad \triangle$$

Remark 5.17.4. For properties $\mathcal{P}$ considered, we usually only take conditions in which $\mathcal{P}$ is local. △

For example, being locally of finite type is an example.

Lemma 5.17.5. *Let $f\colon X \to S$ are maps of schemes. The following are equivalent:*

(a) f is locally (of type) $\mathcal{P}$.

(b) For all open affines $U \subseteq X$ and $V \subseteq S$ such that $f(U) \subseteq V$, we have $\mathcal{O}_S(V) \to O_X(U)$ satisfying $\mathcal{P}$.

(c) There is an open covering $S = \cup_i V_i$ and open coverings $f^{-1}(V_i) = \cup_{j \in I} U_j$ such that $f|_{U_j} \to V_i$ is locally of type $\mathcal{P}$.

(d) There exists an affine open covering $S = \cup_i V_i$ and affine open coverings $f^{-1}(V_i) = \cup_{j \in I_j} V_j$ such that $\mathcal{O}_S(V_i) \to \mathcal{O}_X(U_j)$ satisfies $\mathcal{P}$.

Proof. 2. implies 4. implies 3. implies 1. are all trivial. Let's do 1. implies 2. There exists $U = \cup_i U_i$ with $U_i = \operatorname{Spec} A_i$. Then $f(U_i) \subseteq V_i \subseteq S$ where $V_i \subseteq S$ is open. Then $V_i = \operatorname{Spec} R_i$ and $R_i \to A_i$ satisfies $\mathcal{P}$.

Note that V_i is not necessarily in V. Let $x \in U_i$ and $f(x) \in V \cap V_i$. Thus, there is $V_{ij} \subseteq V \cap V_i$ with $f(x) \in V_{ij}$, where the V_{ij} are some standard opens in V and V_i. Write $V_{ij} = \operatorname{Spec}(R_i)_{h_j}$ for some $h_j \in R_i$. Now $f^{-1}(V_{ij}) \cap U_i = U_i \times_{V_i} V_{ij} = \operatorname{Spec}(A_i \otimes_{R_i} (R_i)_{h_j})$. Since $\mathcal{P}$ is local (by Property (a)) we have $(R_i)_{h_j} \to A_i \otimes_{R_i} (R_i)_{h_j}$ satisfies $\mathcal{P}$. Now take $U_i' = \operatorname{Spec}(\mathcal{O}_X(U) a_i) \ni x$ standard open affine in U and $U_i \cap f^{-1}(V_{ij})$. Since $V_{ij} = \operatorname{Spec}(\mathcal{O}_S(V) t_j)$ for some t_j, $f\colon \mathcal{O}_S(V)_{t_j} = (R_i)_{h_j} \to A_i \otimes_{R_i} (R_i)_{h_j}$ satisfy $\mathcal{P}$, by property (b) of "$\mathcal{P}$ is local".

Since U is quasicompact, we know that finitely many $\mathcal{O}_X(U)$ a_i's suffice. Thus, by (c) of "$\mathcal{P}$ is local", we get that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ satisfies $\mathcal{P}$. □

Definition 5.17.6. A morphism of schemes is locally of finite type if it's locally $\mathcal{P}$ as above with $\mathcal{P}$ being "finite type" $\triangle$

Definition 5.17.7. The morphism f is of finite type if it is locally finite type and quasicompact. $\triangle$

Example 5.17.8. Let k be a field. Then an algebraic variety over k, is an integral, separated, scheme of finite type over k. Here's an interesting phenomenon: varieties are not stable under base change! One can check that $\operatorname{Spec}\mathbb{Q}(i)$ is a variety over $\mathbb{Q}$. We get

$$\begin{tikzcd} \operatorname{Spec}\mathbb{Q}(i)\times_{\mathbb{Q}}\operatorname{Spec}\mathbb{C} \arrow[r] \arrow[d] & \operatorname{Spec}\mathbb{Q}(i) \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[r] & \operatorname{Spec}\mathbb{Q} \end{tikzcd}$$

$\triangle$

Goals. In the immediate future, our goals are the following. We want to show that quasiseperated is automatic in many contexts, and more importantly if f is of finite type and S is locally noetherian, then f_* preserves quasicoherentness.

Lemma 5.17.9. *Let $\mathcal{P}$ be a property of ring homomorphisms.*

1. *If $\mathcal{P}$ is local and stable under base change, then a morphism locally of type $\mathcal{P}$ is stable under base change.*
2. *If $\mathcal{P}$ is local and stable under composition, then a morphism locally of type $\mathcal{P}$ is stable under compositions.*

Proof. Let $g\colon S'\to S$ be a map between schemes, so we get a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Then for all $s'\in S'$, there is $U'\subseteq S'$ affine open and $U\subseteq S$ affine open such that $g(U')\subseteq U$. Now $f^{-1}(U)=\cup_{i\in I}V_i$ and $f^{-1}(U)=\cup_i U'\times_U V_i$ is an affine open cover, and $\mathcal{O}_S(U)\to\mathcal{O}_X(V_i)$ satisfies $\mathcal{P}$. This means that

$$\mathcal{O}_{S'}(U')\to\mathcal{O}_{X'}(U'\times_U V_i)=\mathcal{O}_{S'}(U')\times_{\mathcal{O}_S(U)}\mathcal{O}_X(V_j)$$

satisfies $\mathcal{P}$.

For part ii, we consider morphisms $f\colon X\to Y$ and $g\colon Y\to Z$ locally of type $\mathcal{P}$. For all $U\subseteq Z$ affine open, $g^{-1}(U)=\cup_{i\in I}V_i$, we have $\mathcal{O}_Z(U)\to\mathcal{O}_Y(V_i)$ satisfying $\mathcal{P}$. We also have $f^{-1}(V_i)=\cup_{j\in I_i}W_{ij}$ and $\mathcal{O}_Y(V_i)\to\mathcal{O}_X(W_{ij})$ satisfies $\mathcal{P}$. Since $\mathcal{P}$ is stalbe under compositions, it follows that $\mathcal{O}_X(U)\to\mathcal{O}_X(W_{ij})$ satisfies $\mathcal{P}$. $\square$

For example, $\mathcal{P}$ = "finite type" is stable under base change and composition. Thus, morphisms being locally of finite type are as well.

Lemma 5.17.10. *If $f\colon X\to S$ is locally of finite type, and S is locally noetherian. Then X is locally Noetherian. (Thus, f is quasi-separated.)*

This is why "quasiseparated is automatic in many contexts" – see the goals we outlined earlier.

Proof. For all $x\in X$, there is $x\in U\subseteq X$ affine open with $V\subseteq S$ affine open such that $f(U)\subseteq V$. Since S is locally Noetherian, then $\mathcal{O}_S(V)$ is Noetherian. Since $\mathcal{O}_S(V)\to\mathcal{O}_X(U)$ is of finite type, $\mathcal{O}_X(U)$ is Noetherian.

Consider

$$
\begin{tikzcd}
X \arrow[r, "\Delta"] & X \times_S X \arrow[dl, "\pi_1"'] \arrow[dr] & \\
X \arrow[rr, "f"] & & S
\end{tikzcd}
$$

It's enough to show $X \times_S X \to X$ is locally Noetherian because $\Delta_{S/S}$. Since$X \times_S X \to X$ is local and $f \colon X \to S$ is locally of finite type, $X \times_S X \to S$ is locally of finite type by composition. $\square$

Note that if $f \colon X \to S$ and $g \colon Y \to S$ are both locally of finite type, so is $X \times_S Y \to S$.

Hence a key fact is that f is finite type and S is locally Noetherian, then f_* sends quasicoherent sheaves to quasicoherent sheaves.

Projective schemes. Let $S = \oplus_{d \geq 0} S_d$ be a graded commutative ring, and define $S_+ = \oplus_{d>0} S_d$ its irrelevant ideal. Instead of taking Spec we are taking Proj, so that

$$\operatorname{Proj}(S) := \{\text{prime ideals } \mathfrak{p} \subseteq S \colon \mathfrak{p} \text{ graded, and } S_+ \subsetneq \mathfrak{p}\}.$$

Let M be a graded S-module, so that $M = \oplus_{d \in \mathbb{Z}} M_d$. Here graded means that $S_a \cdot M_b \subseteq M_{a+b}$.

Note that $\operatorname{Proj}(S) \subseteq \operatorname{Spec} S$. So we can give Proj the subspace topology, and our functions $f \in S_+$ are to be the homogenous polynomials. We define $D_+(f) := D(f) \cap \operatorname{Proj} S$, and

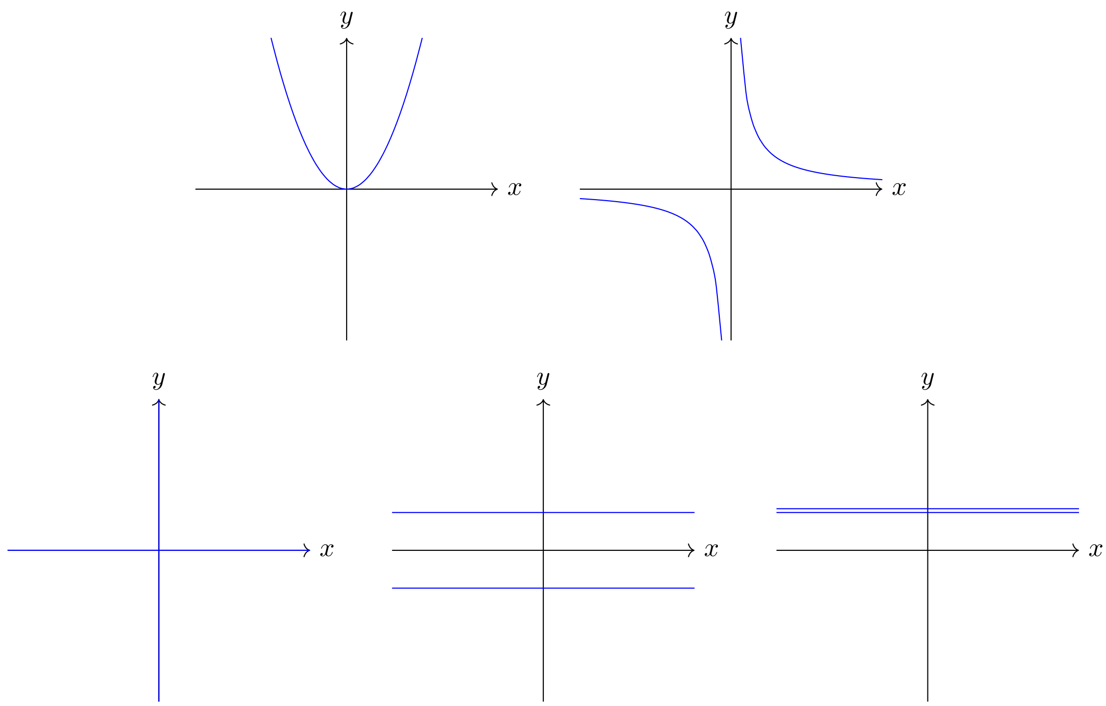
$$M(f) := \left\{ \frac{x}{f^n} \colon x \in M \text{ homogenous, } \deg x = n \cdot \deg f \right\} \subseteq M_f.$$

Hence $D+(f)$ is open in $\operatorname{Proj} S$.

5.18 Mar 26, 2019

We'll talk about projective schemes. The technical motivation for this is that projective schemes are the most fertile source of proper maps. But the proto-motivation for this is that projective schemes are the ultimate realization of the "projective perspective" in classical geometry. This projective perspective roughly says that most essential geometry properties should be invariant under projection. Projective schemes preserve algebra data under projection, too.

There is a classification of quadrics (degree 2 curves) in the plane. For example, there are parabolas and hyperbolas, there are degenerate cases like double-lines:



and in general they are given by $ax^2 + bxy + cy^2 + dx + ey + f = 0$. Homogenizing this equation yields an equation $ax^2 + bxy + cy^2 + dxy + eyz + fz^2 = 0$, and this gives a degree two curve in projective space.

Over an algebraically closed field k, we have Bezout's theorem: given two distinct curves C and D in $\mathbb{P}^2_k$, then they meet at exactly $\deg C \cdot \deg D$ points, counted with multiplicity. Sheafy perspectives generalize this very well (one can interpret this result as a statement in sheaf cohomology).

Our setup consisted of a graded ring $S = \oplus_{d \geq 0} S_d$ with its irrelevant ideal $S_+ := \oplus_{d>0} S_d$. We defined

$$\operatorname{Proj} S = \{\mathfrak{p} \subseteq S \text{ prime} \colon \mathfrak{p} \text{ graded}, \, S_+ \not\subseteq \mathfrak{p}\} \subseteq \operatorname{Spec} S.$$

It is endowed with the induced Zariski topology. If $f \in S_+$ is a homogenous polynomial, we define $D_+(f) := D(f) \cap \operatorname{Proj} S$. Thus $D_+(f)$ is open in $\operatorname{Proj} S$.

If M is a graded S-module, that is, $M = \oplus_{d \in \mathbb{Z}} M_d$ such that $S_a M_b \subseteq M_{a+b}$. We can define

$$M_+(f) := \{x/f^n \colon x \in M \text{ homogenous}, \; \deg x = n \deg f\} \subseteq M_f$$

and sometimes it is denoted $M(f)$ if it is clear from context.

Some easy properties:

(a) The $D_+(f)$ form a basis for the topology on $\operatorname{Proj} S$.

(b) There is a natural bijection of sets $D_+(f) \leftrightarrow \operatorname{Spec} S_{(f)}$, where

$$S_{(f)} := \{x/f^k \colon x \text{ homogenous},\ \deg x = k \deg f\} \subseteq S_f.$$

Hence we have the following picture:

$$\begin{tikzcd}
\operatorname{Spec} S & D(f) \arrow[l, hook', "\subseteq"'] \arrow[r, leftrightarrow] & \operatorname{Spec} S_f \arrow[dr] & \\
\operatorname{Proj} S \arrow[u, hook, "\subseteq"] & D_+(f) \arrow[l, hook', "\subseteq"'] \arrow[u, hook, "\subseteq"] \arrow[r, leftrightarrow] & (??) \arrow[u, hook, "\subseteq"] \arrow[r, leftrightarrow] & \operatorname{Spec} S_{(f)}
\end{tikzcd}$$

What should go in the (??) part of the diagram?

Lemma 5.18.1. *Let $S = \oplus_{d\in\mathbb{Z}} S_d$ be a $\mathbb{Z}$-graded ring. Assume there is $d > 0$ and $f \in S_d$ such that f is invertible. We have*

$$\operatorname{Spec} S \supset \{\mathbb{Z}\text{-graded prime ideals of } S\}.$$

We also have a map

$$\begin{aligned} \operatorname{Spec} S &\to \operatorname{Spec} S_0 \\ \mathfrak{q} &\mapsto \mathfrak{q} \cap S_0 \\ \sqrt{\mathfrak{p}S} &\leftarrow\!\shortmid \mathfrak{p} \end{aligned}$$

There is a homeomorphism

$$\varphi\colon \{\mathbb{Z}\text{-graded prime ideals of } S\} \leftrightarrow \operatorname{Spec} S_0.$$

Example 5.18.2. Suppose $S = S_0[x, x^{-1}]$ so that

$$\operatorname{Spec} S \cong \operatorname{Spec} S_0 \times \underbrace{\mathbb{G}_m}_{=\operatorname{Spec} \mathbb{Z}[x,1/x]}$$

where $\mathbb{G}_m$ is as we saw in Example 4.12.10. The lemma says that characters of $\mathbb{G}_m$ are in correspondence with $\mathbb{Z}$. △

If $\mathfrak{p} \subseteq S$ is a homogenous prime, then

$$M(\mathfrak{p}) := \{x/f \colon x, f \text{ are homogenous},\ \deg x = \deg f, f \notin \mathfrak{p}\} \subseteq M_{\mathfrak{p}}.$$

In particular, $S(\mathfrak{p})$ is defined by considering S as a homogenous S-module.

Observation 5.18.3. If $D_+(f) \subseteq D_+(f)$, then $g^e = af$ where $e \geq 1$ and a is homogenous in H. Hence we have the commutative diagram

$$\begin{tikzcd}
S_f \arrow[r] & S_g \\
S(f) \arrow[r] \arrow[u, hook] & S(g) \arrow[u, hook]
\end{tikzcd}$$

where the horizontal maps are localization (by g at the top row, and $g^{\deg f}/f^{\deg g}$ at the bottom row). We also get the commutative diagram

$$\begin{tikzcd}
M_f \arrow[r] & M_g \\
M(f) \arrow[r] \arrow[u, hook] & M(g) \arrow[u, hook]
\end{tikzcd}$$

We also have

$$\begin{tikzcd} D_+(f) & D_+(g) \arrow[l, hook', "\subseteq"'] \\ \operatorname{Spec} S(f) \arrow[u, "\varphi_f"] & \operatorname{Spec} S(g) \arrow[l] \arrow[u, "\varphi_g"'] \end{tikzcd}$$

and for any $h \in S(f)$, there is $g \in S_+$ homogenous such that $D_+(g) = \varphi_f(D(h)) \in \operatorname{Spec} S(f)$. △

Hence we get

Proposition-Definition 5.18.4. *Let S be a graded ring and M a graded S-module. Then*

(a) The structure sheaf $\mathcal{O}_{\operatorname{Proj} S}$ is the (unique) sheaf of rings on $\operatorname{Proj} S$ such that

$$\mathcal{O}_{\operatorname{Proj} S}(D_+(f)) = S(f)$$

with restriction maps which commute

$$\begin{tikzcd} \mathcal{O}_{\operatorname{Proj} S}(D_+(f)) = S(f) \arrow[r] \arrow[d, hook] & \mathcal{O}_{\operatorname{Proj} S}(D_+(g)) = S(g) \arrow[d, hook] \\ S_f \arrow[r] & S_g \end{tikzcd}$$

This defines a $\mathscr{B}$-sheaf which upgrades uniquely to a sheaf.

(b) $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ is a scheme, where $D_+(f)$ opens are all affine and isomorphic to $\operatorname{Spec} S(f)$.

(c) There exists a unique sheaf of $\mathcal{O}_{\operatorname{Proj} S}$-modules $\widetilde{M}$ such that $\widetilde{M}(D_+(f)) = M(f)$ with restriction maps

$$\begin{tikzcd} \widetilde{M}(D_+(f)) = M(f) \arrow[r] \arrow[d, hook] & \widetilde{M}(D_+(g)) = M(g) \arrow[d, hook] \\ M_f \arrow[r] & M_g \end{tikzcd}$$

(d) $\widetilde{M}$ is a quasicoherent sheaf of $\mathcal{O}_{\operatorname{Proj} S}$-modules, ie. $\widetilde{M}|_{D_+(f)} \cong \widetilde{M(f)}$.

(e) There is a canonical map $M_0 \to \Gamma(\operatorname{Proj} S, \widetilde{M})$ which when restricted to $D_+(f)$ is given by $M_0 \to M(f)$, where $x \mapsto x/1$.

(f) There is a canonical morphism of schemes $\operatorname{Proj} S \to \operatorname{Spec} S_0$ induced by

$$S_0 \to \Gamma(\operatorname{Proj} S, \widetilde{S}) = \Gamma(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S}).$$

Remark 5.18.5. The map in part (f) of Proposition-Definition 5.18.4 is in general neither injective nor surjective, and this makes commutative algebra and algebraic geometry hard/interesting. △

The fact that global sections of $\operatorname{Proj} S$ are small, but that $\operatorname{Proj} S$ still encodes a lot of information, is reflected in the fact that Proj has higher cohomology. This is in stark contrast to Spec; affine schemes have vanishing higher cohomology [More on this next semester, I guess].

Definition 5.18.6. For $n \in \mathbb{Z}$, write $M(n)$ to be the graded S-module such that $M(n)_d = M_{d+n}$. In particular, $\mathcal{O}_{\operatorname{Proj} S}(n) := \widetilde{S(n)}$. So we shift the grading, and shifting gives various maps into global sections, that is, we have $S_n = S(n)_0 \to \Gamma(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S}(n))$. △

Given graded S-modules M, N, we have a canonical $\mathcal{O}_{\operatorname{Proj} S}$-module map

$$\widetilde{M} \otimes_{\mathcal{O}_{\operatorname{Proj} S}} \widetilde{N} \to \widetilde{M \otimes_S N}.$$

For the same reason as in Remark 5.18.5, this map is neither injective nor surjective in general. But people study the kernel and cokernel of this map (and also the map in part (f) of 5.18.4), there's a whole theory.

Anyways, on $D_+(f)$, this restricts to

$$\begin{gathered} M(f) \otimes_{S(f)} N(f) \to (M \otimes_S N)(f) \\ \frac{m}{f^k} \otimes \frac{n}{f^\ell} \mapsto \frac{m \otimes n}{f^{k+\ell}} \end{gathered}$$

so we have maps

$$\mathcal{O}_{\operatorname{Proj} S}(n) \otimes_{\mathcal{O}_{\operatorname{Proj} S}} \mathcal{O}_{\operatorname{Proj} S}(m) \to \mathcal{O}_{\operatorname{Proj} S}(m+n)$$

and

$$\mathcal{O}_{\operatorname{Proj} S}(n) \otimes_{\operatorname{Proj} S} \widetilde{M} \to \widetilde{M(n)}.$$

Warning. $\operatorname{Proj} S$ is not generally quasicompact.

Example 5.18.7. Let $S = \mathbb{C}[x_1, x_2, \dots]$. Then $\operatorname{Proj} S$ is not compact, because for example we can cover with $D_+(x_i)$ for $i = 1, 2, \dots$, but no finite subset will cover (since we now need to respect the grading, in contrast to the affine case). △

Lemma 5.18.8. *The morphism* $\operatorname{Proj} S \to \operatorname{Spec} S_0$ *is separated.*

Proof. It suffices to show that:

- $D_+(f) \cap D_+(g) = D_+(fg)$ is affine. This is because $D_+(fg) \cong \operatorname{Spec} S(fg)$.
- The map $S(f) \otimes_{\mathbb{Z}} S(g) \twoheadrightarrow S(fg)$. This is because for all $a/(f^n g^m) \in S(fg)$ with $\deg a = n \deg f + m \deg g$, we see that

$$\frac{ag^\ell}{f^n + k} \otimes \frac{af^n}{g^{n+\ell}} \mapsto \frac{a}{f^n g^m}$$

for some $k = (\deg g)(\deg f)^r$ and $\ell = (\deg f)^{r+1} - m$ for r sufficiently large.

□

Example 5.18.9. Let R be a ring, and let $S = R[x_0, x_1, \dots]$ with $\deg x_i = 1$. With

$$\operatorname{Proj} S = \mathbb{P}^n_R \to \operatorname{Spec} R$$

we get another (easy and general) proof that $\mathbb{P}^n_R$ is separated. △

5.19 Mar 28, 2019

Last time, for S a graded ring with irrelevant ideal S_+, we defined

$$\operatorname{Proj} S := \{\mathfrak{p} \in \operatorname{Spec} S \colon \mathfrak{p} \text{ graded}, \ S_+ \not\subseteq \mathfrak{p}\}.$$

We also had distinguished opens $D_+(f) := D(f) \cap \operatorname{Proj} S$, and for graded S-modules M we defined

$$M(f) := \left\{ \frac{x}{f^n} \colon x \in M \text{ homogenous}, \ \deg x = n \deg f \right\} \subseteq M_f.$$

There is a scheme $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$, where $\mathcal{O}_{\operatorname{Proj} S}(D_+(f)) = S(f)$. Geometrically, projective schemes are the right thing to look at – they're just harder algebraically.

The analogous $\widetilde{M}$'s on $\operatorname{Proj} S$ are always quasicoherent. We also defined, for $n \in \mathbb{Z}$, these graded S-modules $M(n)$; they shift indices, so that $M(n)_d = M_{d+n}$. Then $\mathcal{O}_{\operatorname{Proj} S}(n) = \widetilde{S(n)}$ is a twisted structure sheaf. We have multiplication maps

$$\mathcal{O}_{\operatorname{Proj} S}(n) \otimes_{\mathcal{O}_{\operatorname{Proj} S}} \mathcal{O}_{\operatorname{Proj} S}(m) \to \mathcal{O}_{\operatorname{Proj} S}(m+n)$$

and

$$\mathcal{O}_{\operatorname{Proj} S}(n) \otimes \widetilde{M} \to \widetilde{M}(n).$$

Lemma 5.19.1. *Let $Y = \operatorname{Proj} S$, and assume that $Y = \cup_{f \in S_1} D_+(f)$. Then each $\mathcal{O}_Y(n)$ is invertible (as modules) and are locally free of rank 1 over primes. Furthermore, these multiplication maps are isomorphisms.*

Proof. Pick $f \in S_1$, and note that $\mathcal{O}_Y(n)|_{D_+(f)} = \widetilde{S(n)_{(f)}} = \widetilde{(S_f)_n}$, viewing $(S_f)_n$ as an $S(f)$-module. But $f^n \in (S_f)_n$ so $S(f) \to (S_f)_{(n)}$ given by $x \mapsto f^n x$ is an isomorphism. For the isomorphism on multiplication, note that we have maps

$$\begin{gathered}(S_f)(n) \otimes_{S(f)} (S_f)(m) \to (S_f)(m+n) \\ x f^n \otimes y f^m \mapsto x y f^{m+n}\end{gathered}$$

and similarly

$$\begin{gathered}(M_f)(n) \otimes_{S(f)} (S_f)(m) \to (S_f)(m+n) \\ m f^n \otimes y f^m \mapsto y m f^{m+n}\end{gathered}$$

Hence, these are isomorphisms. ☐

Observation 5.19.2. If we can view $Y = \cup_{f \in S_1} D_+(f)$ then

$$S \to \bigoplus_{d \geq 0} \Gamma(Y, \mathcal{O}_Y(d)) =: \Gamma_*(Y, \mathcal{O}_Y(1))$$

is a map of graded rings. △

Corollary 5.19.3. *In this situation, $\mathcal{O}_Y(n) \cong \mathcal{O}_Y(1)^{\otimes n}$, and $\widetilde{M}(n) \cong \widetilde{M} \otimes_{\mathcal{O}} \mathcal{O}(1)^{\otimes n}$, since $M \otimes_S S(n) \cong M(n)$.*

Definition 5.19.4. We have

$$\Gamma_*(Y, \widetilde{M}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(Y, \widetilde{M}(n)).$$

This is a $\Gamma_*(Y, \mathcal{O}_Y(1))$-module. △

The main question is: we have maps

$$\begin{tikzcd} M \arrow[r, dashed, leftrightarrow, "?"] & \Gamma_*(Y, \widetilde{M}) \\ M_d \arrow[u] \arrow[r] & \Gamma(Y, \widetilde{M(d)}) \arrow[u] \end{tikzcd}$$

and we want to understand the relationship between M and $\Gamma_*(Y, \widetilde{M})$. In particular, we want to know whether the dashed arrow is an isomorphism (and if ever, then when). Also we had a correspondence between quasicoherent sheaves on $\operatorname{Spec} R$ and modules over R, but for Y a Proj, is every quasicoherent $\mathcal{O}_Y$-module an $\widetilde{M}$ or vise versa?

Lemma 5.19.5. *The morphism $\mathbb{P}^n_R \to \operatorname{Spec} R$ is quasicompact, of finite type, and universally closed.*

Proof. To get quasicompactness, note that $\mathbb{P}^n_R = \cup_{i=0}^n D_+(x_i) = \cup_{i=0}^n \operatorname{Spec} R[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$. Using that if $\mathfrak{p} \in \operatorname{Proj} S$ then $\mathfrak{p} \not\supseteq S_+$, then $\mathfrak{p} \ni x_i$ for some i. Since $R[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$ are finitely generated R-algebras, we get finite type. We've seen that they're separated in Lemma 5.18.8 [...right?]. Universally closed comes from the valuative criterion. ☐

Definition 5.19.6. A morphism $f: X \to S$ is <u>proper</u> if it is finite type, separated, and universally closed. △

Remark 5.19.7. Usually, the easiest way to check this is by valuative criterion (Exist) and (Unique). △

An important algebra lemma to prove is the following:

Lemma 5.19.8. *The canonical maps $R[x_0, \ldots, x_n]_d \to \Gamma(\mathbb{P}^n_R, \mathcal{O}(d))$ are isomorphisms for all $d \in \mathbb{Z}$.*

Proof. By the sheaf condition, we have

$$0 \longrightarrow \Gamma(\mathbb{P}^n, \mathcal{O}(d)) \longrightarrow \bigoplus_{i=0}^n \Gamma(D_+(x_i), \mathcal{O}(d)) \xrightarrow{\varphi} \bigoplus_{i,j=0}^n \Gamma(D_+(x_i x_j), \mathcal{O}(d))$$

We want to look at the kernel of

$$\bigoplus_{i=0}^n (R[x_0, \ldots, x_n]_{x_i})_d \to \bigoplus_{i,j=0}^n (R[x_0, \ldots, x_n]_{x_i x_j})_d.$$

Indeed, given $(F_i/x_i^{n_i})$ for $i = 0, \ldots, n$ with $x_i \nmid F_i$, with $\deg F_i - n_i = d$, note that we have

$$\frac{F_i}{x_i^{n_i}} - \frac{F_j}{x_j^{n_j}} = 0, \quad \text{i.e.,} \quad x_j^{n_j} F_i = x_i^{n_i} F_j,$$

which implies $x_j | F_j$ or $n_j = 0$. But we assumed $x_i \nmid F_i$, thus, $n_i = 0$ for all i. Hence, all the $(F_i/x_i^{n_i})$'s are just polynomials F_i. So $F_i - F_j = 0$ for all $i \neq j$ and now all the $F_i = F$ for some $F \in R[x_0, \ldots, x_n]_d$. ☐

Let's talk about maps into $\mathbb{P}^n_R$. The idea from topology is that we can think about the classifying space $\mathbb{P}^\infty = B(\mathbb{C}^\times)$ and homotopy classes of maps $X \to \mathbb{P}^\infty$, denoted $[X, \mathbb{P}^\infty]$, is in correspondence with line bundles on X.

Some key facts: on $\operatorname{Proj} S$, suppose S is generated by degree 1 elements of R. Then:

- $\mathcal{O}_{\operatorname{Proj} S}(1)$ is an invertible $\mathcal{O}_{\operatorname{Proj} S}$-module.
- $\mathcal{O}_{\operatorname{Proj} S}(n) = \mathcal{O}_{\operatorname{Proj} S}(1)^{\otimes} n$.

Example 5.19.9. Take $\Gamma(\mathbb{P}^n, \mathcal{O}(1)) = R\langle x_1, \ldots, x_n \rangle$, where $\{x_1, \ldots, x_n\}$ generate $\mathcal{O}_{\mathbb{P}^n}(1)$ over $\mathcal{O}_{\mathbb{P}^n}$. △

Definition 5.19.10. Let X be as scheme and $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module (these are line bundles). Given $s \in \Gamma(X, \mathcal{L})$, we set $X_s := \{x \in X : s_x \notin \mathfrak{m}_x \mathcal{L}_x\}$. This is an open set. Given sections $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$, we say $\{x_0, \ldots, x_n\}$ generates $\mathcal{L}$ over X if $X = \cup_{i=0}^n X_{s_i}$. △

Observation 5.19.11. We have

- Given $F \in R[x_0, \dots, x_n]_d$ for $d > 0$, if we think of F as a global section, then $(\mathbb{P}^n_R)_F = D_+(F)$.
- If $f\colon Y \to X$ is a morphism of schemes, then
$$f^{-1}(X_s) = Y_{f^*(s)}$$
where $f^*(s) \in \Gamma(Y, f^*\mathcal{L})$.
- Let $\varphi\colon X \to \mathbb{P}^n_R$ be a morphism. Then we have $\mathcal{L} := \varphi^*\mathcal{O}_{\mathbb{P}^n}(1)$. This is an invertible sheaf on X and $s_i := \varphi^*(x_i)$, for $i = 0, \dots, n$, are sections of $\Gamma(X, \mathcal{L})$ that generate $\mathcal{L}$ over X.

△

Theorem 5.19.12. *Given a scheme X over R, an invertible sheaf $\mathcal{L}$ on X, and $n+1$ many global sections $s_0, \dots, s_n$ of $\mathcal{L}$ which generate $\mathcal{L}$ over X, there exists a unique morphism*
$$\varphi_{(\mathcal{L}, s_0, \dots, s_n)}\colon X \to \mathbb{P}^n_R$$
such that

(i) $\varphi^*_{(\mathcal{L}, s_0, \dots, s_n)}(\mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{L}$, *and*

(ii) $\varphi^*(x_i) = s_i$.

Proof. Let $p \in X$. Pick i such that $p \in X_{s_i}$, and choose an affine open $p \in U \subseteq X_{s_i}$ with $U = \operatorname{Spec} A$, where A is an R-module. Then $\mathcal{L}|_U \cong \widetilde{M}$ where $s_j|_U = m_j \in M$. Since $U \subseteq X_{S_i}$, we have $M = A \cdot m_i$. Write $m_j = f_j \cdot m_i$ for some unique $f_j \in A$. Define $\operatorname{Spec} A = U \to D_+(x_i) = \operatorname{Spec} R[x_0/x_i, \dots] \subseteq \mathbb{P}^n_R$ to be the map corresponding to the algebra map sending $x_j/x_i \to f_j \in A$. □

Example 5.19.13. Let $X = \operatorname{Spec} B$ be some R-algebra, and let $\mathcal{L} = \mathcal{O}_X$. Then $s_0, \dots, s_n$ corresponds to $f_0, \dots, f_n \in B = \Gamma(X, \mathcal{O}_X)$ such that $\langle f_0, \dots, f_n \rangle = B$. The theorem says that this gives a morphism $\operatorname{Spec} B \to \mathbb{P}^n_R$. Conceretely, we have maps

$$\begin{tikzcd}
\mathbb{A}^{n+1}_R \setminus V(x_0, \dots, x_n) \arrow[d, "\pi"] & D(x_i) \arrow[l, hook', "\subseteq"'] \arrow[d] \\
\mathbb{P}^n_R & D_+(x_i) \arrow[l, hook']
\end{tikzcd}$$

Given $f_0, \dots, f_n$, since $\langle f_0, \dots, f_n \rangle = B$, we get a morhpism $\operatorname{Spec} B \to \mathbb{A}^n_R$ that avoids $V(x_0, \dots, x_n)$. Then just apply π. △

Example 5.19.14. Consider $\mathbb{P}^1_k \to \mathbb{P}^n_k$, for k a field. Define $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(d)$.

- When $d < 0$, this is not possible because $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(d)) = 0$.
- When $d = 0$, then we have $s_0, \dots, s_n \in \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}^1}) = k$. We claim that $\varphi_{(\mathcal{O}_{\mathbb{P}^1}, s_0, \dots, s_n)}$ is constant, since we have the commutative diagram

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[r] \arrow[dr] & \mathbb{P}^n \\
& \operatorname{Spec} k \arrow[u]
\end{tikzcd}$$

- When $d > 0$, then we have $s_0, \dots, s_n$ corresponding to $F_0, \dots, F_n \in k[x_0, \dots, x_n]_d$ with $F_0, \dots, F_n$ with no common zeros in $\mathbb{A}^2_k$ (except for $(0,0)$). We have $[x_0 : x_i] \mapsto [F_0, \dots, F_n]$. On $D_+(x_0)$ we have $f_i = F_i/x_0^d = F(1, x_i/x_0) \in k[x_i/x_0]$. Thus

$$\begin{tikzcd}
\operatorname{Spec} k[x_i/x_0] \arrow[r] \arrow[dr, "\varphi|_{D_+(x_0)}"'] & \mathbb{A}^{n+1}_k \setminus \{0\} \arrow[d, "\pi"] \\
& \mathbb{P}^n_R
\end{tikzcd}$$

△

6 (Affine) Algebraic Groups

6.20 Apr 9, 2019

[Harrison lectured today.]

Recall that we were talking about geometric quotients. Suppose $G \circlearrowright X$. A map $\pi\colon X \to Y$ is called a projection if it is surjective and the fibers are exactly G-orbits (think of Y as "X/G"). The map should be open, and it should satisfy $\pi^\sharp\colon \mathscr{O}(U) \to \mathscr{O}(\pi^{-1}(U))$ should factor through the invariants, that is, the diagram commutes:

$$\begin{tikzcd} \mathscr{O}(U) \arrow[r] \arrow[dr, dashed, "\cong"'] & \mathscr{O}(\pi^{-1}(U)) \\ & \mathscr{O}(\pi^{-1}(U))^G \arrow[u, hook] \end{tikzcd}$$

Geometric quotients don't always exist (we saw examples last time). On the other hand, categorical quotients always exist. These are maps $\pi\colon X \to Y$ such that for $Z \in \mathscr{C}$ (where $\mathscr{C}$ might be **AffSch**, or just **Sch** [and the quotient depends on the category you are taking quotients in]) with trivial G action, the map

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "G-\text{eq.}"'] & Y \\ & Z \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

that is to say, Y is initial amongst (affine) schemes with trivial G-action and a map from X.

Example 6.20.1. We constructed these homogeneous spaces. Let $X = G$, and suppose $H \subseteq G$ acts on X by left/right multiplication. We claim that $X/H = G/H$ has a scheme-relation, and it is a geometric quotient.

There is a natural question here: is it a categorical quotient? This isn't always going to work because these homogeneous spaces aren't always going to be schemes. But let's see this concretely with an example.

Suppose $\mathscr{C} = \textbf{AffSch}$. Suppose $G = \mathrm{SL}_2$ and $H \subseteq G$ consist of upper triangular matrices. Then $H\backslash G = \{H\text{-cosets}\}$, which are in bijection to lines in $\mathbb{C}^2$. Explicitly, the bijection is given by

$$\begin{aligned} \{H\text{-cosets}\} &\leftrightarrow \text{lines in } \mathbb{C}^2 (= \mathbb{P}^1) \\ H \cdot \begin{bmatrix} * & * \\ a & b \end{bmatrix} &\leftarrowtail \ell = \begin{bmatrix} a \\ b \end{bmatrix} \end{aligned}$$

△

Let's go on an aside and talk about the functor of points. The Yoneda lemma says the following: suppose $\mathscr{C}$ is a locally small category, so that $\mathrm{Hom}_{\mathscr{C}}$ are sets. Define a functor $h_A = \mathrm{Hom}_{\mathscr{C}}(-, A)\colon \mathscr{C}^{\mathrm{op}} \to \textbf{Set}$ sending X to $\mathrm{Hom}(X, A)$. The Yoneda lemma says that natural transformations from h_A to F is $F(A)$, that is, if $F = h_B$, then natural transformations from h_A to h_B is $h_B(A) = \mathrm{Hom}(A, B)$. Actually, $h\colon \mathscr{C} \to \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \textbf{Set})$ given by $A \mapsto h_A$ is fully faithful (sometimes called an "embedding").

Now let $X \in \mathscr{C} = \textbf{Sch}$. The functor of points of X is $h_X(-) = \mathrm{Hom}_{\textbf{Sch}}(-, X)$. Often, we just call $X(S) = h_X(S) = \mathrm{Hom}_{\textbf{Sch}}(S, X)$ the "S-points of X". Given a functor $F\colon \textbf{Sch}^{\mathrm{op}} \to \textbf{Set}$, when is $F \simeq h_X$ for a scheme X? That is to say, when is F is represented by a scheme X? We should look for properties that representable functors h_X satisfy.

1. We have a sheaf condition.

The intuition is that if we cover S by some opens U_i, and we provide maps $f_i\colon U_i \to X$, we can glue to a function$S \to X$ (if they are equal on intersections).

So fix an affine scheme S, and an (affine) open cover $\mathcal{U} = \sqcup U_i \to S$. We get a truncated Cech diagram

$$\mathcal{U} \times_S \mathcal{U} \rightrightarrows \mathcal{U} \longrightarrow S$$

$$\coprod U_i \times U_j \rightrightarrows \coprod U_i$$

and so the sheaf condition says

$$X(\mathcal{U} \times_S \mathcal{U}) \leftleftarrows X(\mathcal{U}) \longleftarrow X(S)$$

should be a limit (this is just a rephrasing of the intuition above).

This condition is not enough. Namely, we can take $F(S)$ to be a constant sheaf valued on two points. The claim is that this is not representable (that this really is not representable will be left as an exercise).

2. We have an atlas.

There should be $\mathcal{U}$ and $p\colon \mathcal{U} \to X$ which is an open cover (when X is a scheme). So, ask if there is $\mathcal{U}$, and $p\colon h_{\mathcal{U}} \to F$ such that for any scheme S, the map $F \times_S h_U \to S$ is an open immersion, and $F \times_S h_U$ is representable by a scheme. This is hard to check in practice, but it's certainly necessary.

So we generalize what it means to be an open cover. We'll allow $\mathcal{U} \to X$ that are not just disjoint unions of open immersions, but étale map (we'll black box this, but it's the algebraic geometry analogue of a covering space).

Definition 6.20.2. An algebraic space is a functor $F\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ which satisfy 1. and 2. above for étale covers. Furthermore, the diagonal $\Delta\colon F \to F \times F$ is representable. △

For a morphism to be representable, the functor $S^1 \mapsto X(S^1) \times_{(X\times X)(S')} \mathrm{Mor}(S', S)$ should be representable. The point is that we want a presentation of X, that is, we want schemes

$$U \times_X U \rightrightarrows U \longrightarrow X$$

and if Δ is representable then $U \times_X U = (U \times U) \times_{X\times X} \Delta$ is a scheme.

Hence an algebraic space is "étale locally" a scheme. Think about X as a "classifying space" and h_X as a "classifying functor" of the moduli problem.

Example 6.20.3. Suppose $X = \mathbb{A}^1$, and $h_{\mathbb{A}^1}(S) = \mathrm{Mor}(S, \mathbb{A}^1) = \mathscr{O}(S)$.

Suppose $X = \mathbb{G}_m$. Then $h_{\mathbb{G}_m}(S) = \mathrm{Mor}(S, \mathbb{G}_m) = \mathscr{O}(S)^X$.

Suppose $X = \mathbb{P}^1$. Then $h_{\mathbb{P}^1}(S) = \mathrm{Mor}(S, \mathbb{P}^1) = \{\text{line bundles on } S \text{ plus globally generated sections on } S\}$ △

Stacks try to classify $F(S) = \{G\text{-torsors on } S\}/\simeq$. F has to be a sheaf, but torsors are locally trivial, so

$$F(X) = \lim \prod F(U_i) \rightrightarrows F(U_i \cap U_j).$$

Since $F(U_i)$ is a point for small enough U_i, we see also that $F(X)$ is also a point. The solution to this is to replace sets with groupoids (there is a lot of work in making this precise). We can define the quotient stack $X/G(S)$ to be the set of

$$\begin{tikzcd} P \arrow[r, "G\text{-eq}"] \arrow[d, "G\text{-tor}"'] & X \\ S \end{tikzcd}$$

The idea is we are resolving S by P which has a free and transitive G action mapping out of this "resolution". We can ask: if $G \circlearrowleft X$ freely and properly, is there a categorical quotient? The answer is yes in stacks (we take X/G), and even in algebraic spaces (we can still take X/G), but it depends on representability for schemes.

Let's talk about reductive groups. Let G be an affine algebraic group over a field k (at some point we'll insist that $k = \mathbb{C}$, but not yet)

Definition 6.20.4. The algebraic group G is reductive if every smooth connected unipotent normal subgroup of $G \times_k \bar{k}$ is trivial. This happens to be equivalent to G having no normal subgroups isomorphic to $\mathbb{G}_{a,\bar{k}}$ △

Definition 6.20.5. A group H is unipotent if it is isomorphic to a closed subgroup of the strictly triangular matrices

$$\begin{bmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

△

Definition 6.20.6. The group G is linearly reductive if every rational representation of V is isomorphic to

$$\bigoplus_{V_i \text{ irred}} V_i,$$

that is, every representation is completely reductible. △

Example 6.20.7. The group $\mathbb{G}_a$ is not reductive, as $\mathbb{G}_a \hookrightarrow \mathrm{GL}_2$ given by

$$x \mapsto \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_2 = \mathrm{GL}(k^2) = V$$

shows (details left as exercise).

Also $\mathbb{G}_a$ is not linearly reductive because V is 2-dimensional, and all irreducibles of abelian groups are 1-dimensional, and we can check that V cannot be decomposed (since for generic x, the matrix $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ will not be diagonalizable). △

Theorem 6.20.8. *Suppose* $\operatorname{char} k = 0$. *Then G is linearly reductive if and only if it is reductive.*

Example 6.20.9. Suppose $\operatorname{char} k = p$. Then for $G = C_p = \mathbb{Z}/p\mathbb{Z}$, and $V = \mathbb{F}_p^2 = k^2$, we have

$$\sigma^n \mapsto \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

△

The proof of Theorem 6.20.8 can be found in more generality in Milne's book *Algebraic Groups*, where it is Theorem 22.42 in the published version [which we can access] and 22.138 in the free online version.

Proof of Theorem 6.20.8 for $k = \mathbb{C}$. The strategy is to construct the Reynolds operator, which is a splitting of the inclusion of G-invariants:

$$\begin{tikzcd} V^G \arrow[r, hook] & V \arrow[l, bend right, "R_V"'] \end{tikzcd}$$

The following is a fact: $G(\mathbb{C})$ is a $\mathbb{C}$-Lie group, and has a compact real form $K \subseteq G(\mathbb{C})$ that is dense in the Zariski topology.

Recall that for finite groups,

$$R_V(x) = \frac{1}{|G|} \sum_{g \in G} g \cdot x.$$

For compact groups, we can pick a Haar measure

$$R_V(x) = \frac{1}{\mu(K)} \int_K g \cdot x \, d\mu(g).$$

Assume we have a splitting, say $U \in \operatorname{Irr}(G)$ and $U \subseteq V$. We want to split off all representations isomorphic to U. What we do is use the adjunction property on $R_{V \otimes U^*} \colon V \otimes U^* \to (V \otimes U^*)^G$ to get φ as below:

$$\begin{tikzcd} V \arrow[r, "\varphi"] & (V \otimes U^*)^G \otimes U = \bigoplus U \arrow[l, bend left, dashed, "\psi"] \end{tikzcd}$$

We claim that ψ, given by evaluation on $(U^* \otimes U)$, is a splitting, and that ψ is injective. This will be left as an exercise. We'll omit the proof of the converse. $\square$

Examples include $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}, \mathrm{O}_n$, but we'll focus on the first two.

Theorem 6.20.10. *Suppose G is reductive and acts on $X = \operatorname{Spec} A$. Then*

(i) A^G is finitely generated

(ii) $X//G := \operatorname{Spec} A^G$ is the categorical quotient in **AffSch**

(iii) The fibers of $\pi \colon X \to X//G$ contain a unique closed orbit.

Proof. To check (i), we define $R_X \colon A = \mathscr{O}(X) \to \mathscr{O}(X^G) = A^G$. We claim that R_X is A^H-linear, and we can cehck that R_V is natural with respect to G-equivariant morphisms, that is,

$$\begin{tikzcd} V \arrow[r, "G\text{-eq}"] \arrow[d, "R_V"'] & W \arrow[d, "R_W"] \\ V^G \arrow[r] & W^G \end{tikzcd}$$

commutes.

Now, note that if $z \in A^G$, $\mu_z(f) = zf$ is a G-morphism. So $z \cdot R_V(f) = R_V(zf)$.

We'll continue this proof next time. $\square$

6.21 Apr 11, 2019

Let's restate what we were proving.

Theorem 6.21.1. *Suppose G is reductive (over $\mathbb{C} = k$) and acts on $X = \operatorname{Spec} A$. Then*

(i) A^G is a finitely generated algebra over k

(ii) $X//G := \operatorname{Spec} A^G$ is the categorical quotient in **AffSch**

(iii) The fibers of $\pi\colon X \to X//G$ contain a unique closed orbit.

Proof. There is a correspondence

$$\begin{aligned} \{\text{ideals in } \mathscr{O}(X)^G\} &\rightleftarrows \{\text{saturated } G\text{-closed ideals in } \mathscr{O}(X)\} \\ I &\mapsto \mathscr{O}(X) \cdot I \\ I \cap \mathscr{O}(X)^G &\leftarrow\!\shortmid I \end{aligned}$$

in particular the backwards map is the Reynolds operator. Indeed, if $I \subseteq \mathscr{O}^G$ is an ideal, then $I \cdot \mathscr{O}(X)$ is G-stable because I is G-invariant, and $I \cap \mathscr{O}(X)^G = (I \cdot \mathscr{O}(X))^G = R_X(I \cdot \mathscr{O}(X)) = I \cdot R_X(\mathscr{O}(X)) = I \cdot \mathscr{O}(X)^G$.

The point is that ideals in $\mathscr{O}(X)^G$ corresponds to closed subschemes of $X//G$, and saturated G-closed ideals in $\mathscr{O}(X)$ corresponds to saturated G-closed subschemes.

We claim that A^G is noetherian. If $I \subseteq A^G$ is an ideal, then $I = R_X(I \cdot \mathscr{O}(X))$, and since $\mathscr{O}(X)$ is noetherian we are done.

Next, choose an embedding $X \hookrightarrow V$ where V is a finite dimensional G-representation. It suffices to show that $\mathscr{O}(V)^G$ are finitely generated. So we choose an embedding

$$\begin{array}{c} \mathscr{O}(V) = \bigoplus_{n \geq 0} \operatorname{Sym}^n V^* \\ \big\uparrow \\ \mathscr{O}(V)^G = \bigoplus_{n \geq 0} (\operatorname{Sym}^n V^*)^G \end{array}$$

Choose homogenous generators of $\mathscr{O}(V)^G$ to get a chain of sets $S_1 \subseteq S_2 \subseteq S_3 \subseteq \dots$ of generators in degree at most d, where S_d is finite. Then $\langle S_1 \rangle \subseteq \langle S_2 \rangle \subseteq \dots$ terminates at d, and we can take S_d to be our set of generators. $\square$

The quotient is almost never geometric.

Example 6.21.2. Let $t \cdot (x_1, \dots, x_n) = (tx_1, \dots, tx_n)$, with $\theta(x) = k[x_1, \dots, x_n]$, and the weight of $x_i = -1$. We have $\theta(x)^G = k$ and $\mathbb{A}^n = X \to X//G = \mathrm{pt}$ is not geometric. $\triangle$

Example 6.21.3. Let $t \cdot (x, y) = (tx, t^{-1}y)$ with $\theta(x) = k[x, y]$ with the weight of $x = 1$ and $y = -1$. We have $\theta(X)^G = k[xy] \simeq \mathbb{A}^1$ so $\pi\colon \mathbb{A}^2 \to \mathbb{A}^1$ is not geometric. $\triangle$

Proposition 6.21.4. *If the action of G on X is proper (that is, $X \times G \to X \times X$, so orbits are closed), then $\pi\colon X \to X//G$ is geometric.*

Let's talk about G-equivariant sheaves on X.

Definition 6.21.5. A $\underline{G\text{-equivariant structure}}$ on a quasicoherent sheaf $\mathscr{F}$ on a scheme X is an isomorphism $\alpha_{\mathscr{F}}\colon a^*\mathscr{F} \xrightarrow{\simeq} p^*\mathscr{F}$ where

$$\begin{tikzcd} G \times X \arrow[r, bend left, "a"] \arrow[r, bend right, "p"'] & X \end{tikzcd}$$

is given by $a(g,x) = g \cdot x$ and $p(g,x) = x$. These maps satisfy

$$\begin{tikzcd} G \times G \times X \arrow[r, bend left, "(m\times 1)"] \arrow[r, bend right, "(1\times a)"'] & G \times X \end{tikzcd}$$

giving $(m \times 1)^* \alpha_{\mathscr{F}} = (1 \times a)^* \alpha_{\mathscr{F}}$ and

$$X \xrightarrow{e\times 1} G \times X$$

giving $(e \times 1)^* \alpha_{\mathscr{F}} = \mathrm{id}_{\mathscr{F}}$. △

The idea is that if $\mathscr{E}$ is a locally free sheaf on X, think of $\mathscr{E}$ as a sheaf of sections of E. The claim is that a G-equivariant structure on $\mathscr{E}$ is the same as having for every (gx) an identification $E_x \simeq E_{G\cdot x}$. Indeed, fix $g \in G$ and consider

$$\begin{tikzcd} X \arrow[r, "i_g"] & G \times X \arrow[r, bend left, "a"] \arrow[r, bend right, "p"'] & X \end{tikzcd}$$

$$x \longmapsto (g,x)$$

induces

$$a_g^* \mathscr{F} \xrightarrow[\simeq]{i_g^* \alpha_{\mathscr{F}}} \mathscr{F}$$

Example 6.21.6. Let $X = \operatorname{Spec} k = *$. A G-equivariant sheaf on X is a rational G-representation. △

Example 6.21.7. If $X = G$ and G acts by left multiplication, then a G-equivariant sheaf on X is determined by its fiber at any point (in particular, maybe at $e \in G$). The idea is that given the fiber E_e at the identity, we have a unique identification via a_g^* with E_g. This says that $\mathbf{QCoh}^G(G) = \mathbf{QCoh}(\mathrm{pt})$. △

Example 6.21.8. Let $X = \operatorname{Spec} A$ and $G \circlearrowright X$. Then A is a G-representation. This means that we have $\mathbf{QCoh}^G(X) = A - \mathrm{Mod}_{\mathbf{Rep}(G)}$ with $g \cdot (a \cdot m) = (g \cdot a) \cdot (g \cdot m)$. △

Example 6.21.9. If $G \circlearrowright X$ transitively, choose $x \in X$. Then we get an equivalence

$$\mathbf{QCoh}^G(X) \simeq \mathbf{Rep}(\mathrm{Stab}(G,x)).$$

△

We'll leave the details of these examples as exercises.

Example 6.21.10 (SL_2-equivariant sheaves on $\mathbb{P}^1$)**.** Think of $\mathbb{P}^1$ as lines in $\mathbb{C}^2$ (with the standard SL_2-representation). We have a transitive action, and the stabilizer of the base point $x = \mathrm{Span}\{(1,0)\}$ is

$$B = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

which gives

$$\mathbb{P}^1 \simeq G/B$$
$$g \cdot (1,0) \leftarrow\!\shortmid gB$$

This says $\mathbf{QCoh}^G(G/B) \simeq \mathbf{Rep}(B)$. This means that B is a Borel subgroup and a solvable group. This says that

$$U = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \simeq \mathbb{G}_a \xhookrightarrow{\text{normal}} B \twoheadrightarrow H \simeq \mathbb{G}_m \ni \begin{bmatrix} z & \\ & z^{-1} \end{bmatrix}$$

We claim that $\mathrm{Irr}(U)$ are all trivial. This means that $\mathrm{Irr}(B) = \mathrm{Irr}(H) = \mathrm{Irr}(\mathbb{G}_m) = \mathbb{Z}$. Let's prove this. V be a rational $\mathbb{G}_a$-representation, with dimension 1. We get maps

$$V \xrightarrow{\ \rho\ } k[x] \otimes V \xrightarrow[\rho]{1} k[x] \otimes k[y] \otimes V$$

$$v \longmapsto (\sum a_i x^i) \otimes v \longmapsto (\sum a_i x^i)(\sum a_i g^i)v$$

$$(\sum a_i x^i) \otimes v \longmapsto \sum a_i (x+y)^i v$$

so that

$$\sum a_i a_j x^i y^i = \sum \binom{i+j}{i} a_{i+j} x^i y^j.$$

Hence we arrive at $a_i a_j = \binom{i+j}{i} a_{i+j}$ with $a_0 = 1$. This gives $a_n = a_1^n/n!$, which implies $a_1 = 0$. △

Here's a weird thing. Take

$$b = \begin{bmatrix} -x * y & \\ 0 & -x \end{bmatrix},$$

and let B act by conjugation. Then we have

$$0 \longrightarrow \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = V_{-2} \hookrightarrow b \longrightarrow \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} = V_0 \longrightarrow 0$$

and we claim that this sequence does not split G-equivariantly. We have Birkhoff factorization: every vector bundle on $\mathbb{P}^1$ splits into a sum of line bundles. We cover $\mathbb{P}^1$ with two affine opens $U_0 = k[t]$ and $U_\infty = k[t^{-1}]$. Then $\mathscr{E}|_{U_0} k[t] \otimes \{e_1, \ldots, e_n\}$ and $\mathscr{E}|_{U_\infty} = k[t^{-1}] \otimes \{f_1, \ldots, f_n\}$. To get gluing, find an $A \in \mathrm{GL}_n(k[t, t^{-1}])$ with $A \colon \mathscr{E}_0|_{\mathbb{G}_m} \xrightarrow{\simeq} \mathscr{E}_\infty|_{\mathbb{G}_m}$.

The sequence does not split G-equivariantly, but if we forget, then it does split. In particular, it splits into $\mathscr{O}(-1) \oplus \mathscr{O}(1-1)$. We'll leave the details for an exercise, but let's set it up a little. This is a G-equivariant vector bundle V where the fiber at $(1,0) \leftrightarrow B \in G/B \cong \mathbb{P}^1$ is the B-representation b. So the fiber at $g \cdot (1,0) \leftrightarrow gB \in G/B$ is the gBg^{-1}-representation $gbg^{-1} \subseteq g$. So we can take open charts U_0 to be the coordinate

$$\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ t \end{bmatrix}$$

where t is a coordinate of U_0 coming from $\mathbb{A}^1 \hookrightarrow \mathrm{SL}_2$. For U_∞ we can take

$$\begin{bmatrix} s & -1 \\ 1 & - \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} s \\ 1 \end{bmatrix}.$$

The exercise is to write down a basis in each chart, find a gluing matrix, and reduce to the diagonal (Birkhoff).

7 Schemes

7.22 Apr 16, 2019

We were talking about projectivity. There was this Proj construction. They're the correct setting to phrase algebro-geometric questions in. Given a scheme X/S, if you have an invertible sheaf $\mathcal{L}$ (also called a line bundle) with $n+1$ global sections $s_0, \dots, s_n$ that generate $\mathcal{L}$, then there is a unique morphism $\varphi\colon X \to \mathbb{P}^n_S$ such that $\varphi^*(\mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{L}$ and $\varphi^*(x_i) = s_i$. We also saw examples of the importance of projectivity (for example, some of Harrison's lectures).

On the homework we did a problem about closed subschemes of $\mathbb{P}^n_R$.

Proposition 7.22.1. *Let $Z \hookrightarrow \mathbb{P}^n_R$ be a closed subscheme. Then there exists a graded ideal $I \subseteq R[x_0, \dots, x_n]$ such that*

1. *$Z = V_+(I)$ as subsets of topological spaces.*
2. *$Z \cong \operatorname{Proj}(R[x_0, \dots, x_n]/I)$*
3. *The ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^n_R}$ of Z is $\widetilde{I} \subseteq \widetilde{R[x_0, \dots, x_n]} = \mathcal{O}_{\mathbb{P}^n}$.*

This will follow from

Proposition 7.22.2 ([H], II.5.15)**.** *Let $\mathcal{F}$ be a quasicoherent sheaf on $\mathbb{P}^n_R$. Then $\mathcal{F} = \widetilde{M}$ for some graded $R[x_0, \dots, x_n]$-module M. Indeed, we can take*

$$M = \Gamma_*(\mathbb{P}^n_R, \mathcal{F}) = \bigoplus_{d \in \mathbb{Z}} \Gamma(\mathbb{P}^n_R, \mathcal{F}(d)).$$

Remark 7.22.3. Suppose $\varphi\colon S_1 \to S_2$ is a homomorphism of graded rings. The map $\operatorname{Proj} S_2 \to \operatorname{Proj} S_1$ is only well defined as a morphism on an open set

$$U(\varphi) := \bigcup_{f \in (S_1)_+} D_+(\varphi(f))$$

where on $D_+(\varphi(f)) \to D_+(f)$ the map is given by $\operatorname{Spec}(S_2)_{\langle \varphi(f) \rangle} \to \operatorname{Spec}(S_1)_{\langle f \rangle}$ △

Remark 7.22.4. The open set $U(\varphi)$ is equal to all of $\operatorname{Proj} S_2$ if and only if $(S_2)_+ \subseteq \sqrt{\varphi((S_1)_+)S_2}$. In particular, if $S_1 \twoheadrightarrow S_2$ then this holds. Here, $\operatorname{Proj} S_2 \hookrightarrow \operatorname{Proj} S_1$ and this is a closed immersion. △

Example 7.22.5. The map $\varphi\colon \mathbb{C}[X, Y, Z] \to \mathbb{C}[X, Y]$ given by $Z \mapsto 0$ corresponds to the closed immersion $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$. △

Example 7.22.6. Consider $\mathbb{C}[X, Y] \overset{\varphi}{\hookrightarrow} \mathbb{C}[X, Y, Z]$. Then we get a map from

$$U(\varphi) = D_+(X) \cup D_+(Y) \subseteq \mathbb{P}^2 = \operatorname{Proj} \mathbb{C}[X, Y, Z]$$

to $\operatorname{Proj} \mathbb{C}[X, Y] = \mathbb{P}^1$. For example, on $D_+(X)$ we have $\mathbb{C}[Y/X] \hookrightarrow \mathbb{C}[Y/X, Z/X]$ corresponding to $\mathbb{A}^2 \to \mathbb{A}^1$ given by projection onto the Y-axis. △

Let's prove that Proposition 7.22.2 implies Proposition 7.22.1.

Proof. We have a short exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^n_R} \longrightarrow \iota_* \mathcal{O}_Z \longrightarrow 0$$

where $\iota\colon Z \hookrightarrow \mathbb{P}^n_R$ is a closed immersion. By Proposition 7.22.2, since $\mathcal{I}$ is a quasicompact [quasicoherent...?] sheaf of ideals, we have $\mathcal{I} = \widetilde{I}$ for some ideal

$$I = \bigoplus_{d\in\mathbb{Z}} \Gamma(\mathbb{P}^n, \mathcal{I}(d)) \subseteq \bigoplus_{d\in\mathbb{Z}} \Gamma(\mathbb{P}^n, \mathcal{O}(d)) = R[x_0, \ldots, x_n].$$

The inclusion here comes by tensoring the short exact sequence with the invertible sheaf $\mathcal{O}_{\mathbb{P}^n}(d)$ and taking Γ.

Since $\iota\colon Z \hookrightarrow \mathbb{P}^n_R$ is a closed immersion, we have $\mathcal{I}|_{D_+(x_j)} = \widetilde{I_j}$ for some ideal $I_j \subseteq R[x_i/x_j]$. Then $Z \cap D_+(x_j) = \operatorname{Spec} R[x_i/x_j]/I_j$. By definition of $\widetilde{I}$, we have

$$I_j = I_{\langle x_j\rangle} = \left\{\frac{f}{x_j^d}, f \in I, \deg f = d\right\}.$$

Hence $(R[x_0, \ldots, x_n]/I)_{\langle x_i\rangle} = R[x_i/x_j]/I_j$. Hence $D_+(x_j) \cap Z = D_+(\bar{x}_j)$ where $\bar{x}_j$ is the image of x_j in $R[x_0, \ldots, x_n]/I$. This means that on an affine cover we have an injective map from

$$\operatorname{Proj}(R[x_0, \ldots, x_n]) \supseteq Z \supseteq D_+(x_j \cap Z)$$

to

$$\operatorname{Proj}(R[x_0, \ldots, x_n]/I) \supseteq D_+(\bar{x}_j)$$

Thus $Z \cong \operatorname{Proj}(R[x_0, \ldots, x_n]/I)$. □

Example 7.22.7. Consider $\mathbb{P}^2_R = \operatorname{Proj} k[X, Y, Z]$ and the ideals $I_1 = \langle X(X+Y+Z), Y(X+Y+Z), Z(X+Y+Z)\rangle$ and $I_2 = \langle X + Y + Z\rangle$. We claim that $\operatorname{Proj}(k[X, Y, Z]/I_1) = \operatorname{Proj}(k[X, Y, Z]/I_2)$ as closed subschemes of $\mathbb{P}^2_k$. We can check this on $D_+(X), D_+(Y), D_+(Z)$; for example on $D_+(Z)$ we see that

$$(I_1)_{\langle Z\rangle} = \left\langle \frac{X(X+Y+Z)}{Z^2}, \frac{Y(X+Y+Z)}{Z^2}, \frac{Z(X+Y+Z)}{Z^2} \right\rangle = \langle x(x+y+1), y(x+y+1), x+y+1\rangle = \langle x+y+1\rangle$$

whereas

$$(I_2)_{\langle Z\rangle} = \left\langle \frac{X+Y+Z}{Z}\right. = \langle x+y+1\rangle = (I_1)_{\langle Z\rangle}$$

where $x = X/Z$ and $y = Y/Z$. △

Generally, $I \subseteq k[x_0, \ldots, x_n]$ homogenous defines the same closed subscheme as $I \cdot k[x_0, \ldots, x_n]_+$.

Proof of Proposition 7.22.2. We want to show that if $\mathcal{F}$ is a $\mathbb{P}^n_R$ then

$$\Gamma_+(\mathbb{P}^n_R, \mathcal{F})_{\langle x_0\rangle} \xrightarrow{\cong} \Gamma(D_+(x_0), \mathcal{F}),$$

where $\Gamma_+(\mathbb{P}^n_R, \mathcal{F}) = \{s/x_0^m \colon s \in \Gamma(\mathbb{P}^n, \mathcal{F}(d)\})/\sim$.

For this, it suffices to show that

(a) Given $s_1, s_2 \in \Gamma(\mathbb{P}^n_R, \mathcal{F})$ such that $s_1|_{D_+(x_0)} = s_2|_{D_+(x_0)}$ then there is N sufficiently large so that $x_0^N s_1 = x_0^N s_2 \in \Gamma(\mathbb{P}^n_R, \mathcal{F}(N))$.

(b) Given $s \in \Gamma(D_+(x_0), \mathcal{F})$, there exists $d \geq 0$ and $\tilde{s} \in \Gamma(\mathbb{P}^n_R, \mathcal{F}(d))$ such that $\tilde{s}|_{D_+(x_0)} = x_0^d s$.

Let's prove (a). It's enough to show that if $s \in \Gamma(\mathbb{P}^n, \mathcal{F})$ and $s|_{D_+(x_0)} = 0$ then $0 = x_0^N s \in \Gamma(\mathbb{P}^n, \mathcal{F}(N))$. On each $D_+(x_i)$, we have $\mathcal{F}|_{D_+(x_i)} \cong \widetilde{M_i}$ for some $R[x_j/x_i]$-module M_i. So $s|_{D_+(x_i)} = m_i \in M_i$. Note that $D_+(x_0) \cap D_+(x_i) = \operatorname{Spec}(R[x_0/x_i, \ldots, x_n/x_i, (x_0/x_i)^{-1}])$ implies $m_i|_{D_+(x_0)\cap D_+(x_i)} = 0$, which implies $(x_0/x_i)^{N_i} m_i = 0$, or $x_0^{N_i} m_i / x_i^{N_i} = 0$. So if we let $N \geq \max\{N_i\}$ we get $x_0^N|_{D_+(x_j)} = \frac{x_0^N m_i}{x_j^N} = 0$.

Let's prove (b). We have $\mathcal{F}|_{D_+(x_j)} = \widetilde{M_j}$ for some $R[x_i/x_j]$-module M_j. Then we have a diagram

$$
\begin{tikzcd}
\mathcal{F}|_{D_+(x_0)} \arrow[r] \arrow[d, leftrightarrow, "\cong"] & \mathcal{F}|_{D_+(x_0x_j)} \arrow[d, leftrightarrow, "\cong"] & \mathcal{F}|_{D_+(x_j)} \arrow[l] \arrow[d, leftrightarrow, "\cong"] \\
M_0 \arrow[r, "\alpha"] & (M_0)_{\langle x_j/x_0\rangle} \cong (M_j)_{\langle x_0/x_j\rangle} & M_j \arrow[l, "\beta"']
\end{tikzcd}
$$

Then s corresponds to an element $m_0 \in M_0$. Wrote $\alpha(m_0)$ for the image of $s|_{D_+(x_0x_j)}$ through this map. This gives $\alpha(m_0)\cdot(x_0/x_j)^d = \beta(m_j)$ for some of the $m_j \in M_j$ and integer d. Choose $s_j = x_j^d m_j \in \Gamma(D_+(x_j), \mathcal{F}(d))$ and $s_0 = x_0^d m_0 \in \Gamma(D_+(x_j), \mathcal{F}(d))$ for d sufficiently large such that $\alpha(m_0)\cdot(x_0/x_j)^d = \beta(m_j)$ for all j. If the s_j's glue to give $\tilde{s} \in \Gamma(D_+(x_j), \mathcal{F}(d))$ then we're done. But we only know that $s_J|_{D_+(x_0x_j)} = s_0|_{D_+(x_0x_j)}$, and we need $s_i|_{D_+(x_ix_j)} = s_+|_{D_+(x_ix_j)}$. But on $D_+(x_ix_ix_j)$ we have

$$s_i|_{D_+(x_0x_ix_j)} - s_j|_{D_+(x_ix_ix_j)} = s_0|_{D_+(x_0x_ix_j)} - s_0|_{D_+(x_0x_ix_j)} = 0.$$

By part (a), multiply $s_0, \dots, s_n$ by $x_0^{d''}$ such that $x_0^{d''}|_{D_+(x_ix_j)} - x_0^{d''} s_j|_{D_+(x_ix_j)} = 0$. Now take $d = d' + d'' \geq d', d''$. This means that $\tilde{s}|_{D_+(x_i)} = x_0^{d''} s_i$. $\square$

Now let $\mathcal{F}$ be quasicoherent on $\mathbb{P}^n_R$, and assume R is noetherian (like in Hartshorne). Then $\Gamma_*(\mathbb{P}^n_R, \mathcal{F})$ is an $R[x_0, \dots, x_n]$-module.

If we assume that $\mathcal{F}$ is coherent, is this module finitely generated? The answer is yes, sometimes: the general setting for finiteness of H^0 is given by:

- R is a noetherian ring
- $S = \operatorname{Spec} R$
- $\pi\colon X \to S$ is proper
- $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module (importantly, $\mathcal{O}_X$, along with invertible sheaves and more generally locally free sheaves)

Then $H^0(X, \mathcal{F})$ is a finite R-module. In this setting you get to approximate functions, use Nakayama's lemma, etc.; otherwise, you need to compactify things, or approximate R with noetherian rings, etc.

We can't prove this in this course, but the proof sketch via dévissage is as follows:

1. Use the cohomology and prove this for $H^i(X, \mathcal{F})$ for all $i > 0$ (next semester).
2. Use Chow's lemma, which allows us to reduce to the projective case.
3. Now for $\mathbb{P}^n_S \to S$, just show $\mathcal{F} = \widetilde{M}$, and reduce to $M = R[x_0, \dots, x_d]_{\langle x_d\rangle}$ and just compute the cohomology of $\mathcal{O}_{\mathbb{P}^n}(d) = \widetilde{R[x_0, \dots, x_d]}$.

We'll see Chow's lemma next time.

7.23 Apr 18, 2019

Today we'll talk about projective morphisms. It will be a little technical. We saw that projective morphisms are always proper; a partial converse is given by Chow's Lemma, which roughly states that a proper morphism is not that far from being a projective morphism.

Theorem 7.23.1 (Chow's Lemma). *Let S be a noetherian scheme, and let $f\colon X \to S$ be finite type and separated. Suppose we have a commutative diagram*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r] \arrow[d] & X' \arrow[r, hook] \arrow[d, "\pi"] \arrow[dr] & \mathbb{P}^n_S \arrow[d] \\ U \arrow[r] & X \arrow[r] & S \end{tikzcd}$$

Then:

- *π is proper and surjective,*
- *$X' \hookrightarrow \mathbb{P}^n_S$ is an immersion*
- *there exists some dense open $U \subseteq X$ so that $\pi^{-1}(U) \to U$ is an isomorphism.*

An important special case is the following classical case that Chow looked at. If X and S are reduced, we can take X' to be reduced. Let $\overline{X'} \to \mathbb{P}^n_S$ be the closure of X' in $\mathbb{P}^n_S$. Then we have

$$\begin{tikzcd} X' \arrow[r, hook, "\text{open}"] \arrow[d, "\pi"'] & \overline{X'} \arrow[r, hook, "\text{closed}"] \arrow[d] & \mathbb{P}^n_S \arrow[dl] \\ X \arrow[r] & S & \end{tikzcd}$$

This is one reason why the constructible set stuff is so convenient.

Definition 7.23.2. We say a morphism of schemes $X \xrightarrow{f} S$ is <u>projective</u> (in the sense of Hartshorne) if there exists a closed immersion $X \hookrightarrow \mathbb{P}^n_S$ over S. △

Remark 7.23.3. This is not the same as that in EGA. In EGA, a map $f\colon X \to S$ is projective if X is isomorphic (as an S-scheme) to a closed subscheme of a projective bundle $\mathbb{P}(\mathcal{E})$ for some quasicoherent finite type $\mathcal{O}_S$-module $\mathcal{E}$. △

Both these definitions are useful (and used in practice).

Lemma 7.23.4. *We have*

(a) Closed immersions are proper and projective

(b) Projective implies proper

(c) Compositions of projective (respectively proper) morphisms are projective (respectively proper)

(d) Base changes of projective (respectively proper) morphisms are projective (respectively proper)

(e) Fiber products of projective (respectively proper) morphisms are projective (respectively proper)

Proof. Part (a) should be clear. We also saw (b) in previous lectures (when we were doing the Proj construction). Let's prove part (c).

Let $Y \to X, X \to S$ be projective morphisms. We have maps

$$\begin{tikzcd} Y \arrow[r] \arrow[d, hook] & X \arrow[r] \arrow[d, hook] & S \\ \mathbb{P}^n_X \arrow[ur] & \mathbb{P}^m_S \arrow[ur] & \end{tikzcd}$$

and by base change we get

$$\begin{tikzcd}
\mathbb{P}^n_X \arrow[r, hook] \arrow[d] & \mathbb{P}^n_{\mathbb{P}^m_S} \arrow[d] \\
X \arrow[r, hook] & \mathbb{P}^m_S
\end{tikzcd}$$

where

$$\mathbb{P}^n_{\mathbb{P}^m_S} = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} (\mathbb{P}^m_{\mathbb{Z}} \times_{\mathbb{Z}} S) = (\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^m_{\mathbb{Z}}) \times_{\mathbb{Z}} S$$

and $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^m_{\mathbb{Z}}$ is projective over $\operatorname{Spec} \mathbb{Z}$. We have

$$\begin{aligned}
\mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^m_{\mathbb{Z}} &\to \mathbb{P}^{nm+n+m}_{\mathbb{Z}} \\
([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]) &\mapsto [x_i y_j]_{i,j}
\end{aligned}$$

given by the Segre embedding. This gives a closed immersion

$$Y \hookrightarrow \mathbb{P}^n_X \hookrightarrow (\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^m_{\mathbb{Z}}) \times_{\mathbb{Z}} S \hookrightarrow \mathbb{P}^{nm+n+m}_{\mathbb{Z}} \times_{\mathbb{Z}} S = \mathbb{P}^{nm+n+m}_S.$$

Let's prove part (d). Suppose $X \to Y$ is projective, so we have

$$\begin{tikzcd}
X \arrow[r] \arrow[d, hook] & Y \\
\mathbb{P}^n_Y \arrow[ur] &
\end{tikzcd}$$

and $Y' \to Y$. Then we can take the fiber product to get

$$\begin{tikzcd}
X \times_Y Y' \arrow[r, hook] \arrow[d] & \mathbb{P}^n_{Y'} = \mathbb{P}^n_Y \times_Y Y' \arrow[d] \\
X \arrow[r, hook] & \mathbb{P}^n_Y
\end{tikzcd}$$

To prove part (e), if $X \to S$ and $Y \to S$ are projective, then so is $X \times_S Y \to X$ by part (d). Then so is the composition $X \times_S Y \to X \to S$. $\square$

In the variety setting we had this diagram

$$\begin{tikzcd}
X' \arrow[r, hook, "\text{open}"] \arrow[d, "\pi"'] & \overline{X'} \arrow[r, hook, "\text{closed}"] \arrow[d] & \mathbb{P}^n_S \arrow[dl] \\
X \arrow[r] & S &
\end{tikzcd}$$

with a map $h\colon X' \hookrightarrow \mathbb{P}^n_S$.

Lemma 7.23.5. *Given maps*

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& S &
\end{tikzcd}$$

where f is proper and g is separated, then $h(X)$ is closed.

Proof. Indeed, we can take a section

$$\begin{tikzcd}
& X \times_S Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X \arrow[ur, bend left, "h'"] \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& S &
\end{tikzcd}$$

given by h. Since g is separated we get $\pi_X \colon X \times_S Y \to X$ is also separated. Now $h'(X)$ is closed, as it's the equalizer of

$$X \times_S Y \underset{h \circ \pi_X}{\overset{\pi_Y}{\rightrightarrows}} Y$$

Since f is proper, π_Y is closed. and hence $h(X) = \pi_Y \circ h'(X)$ is also closed. $\square$

We'll apply this observation to in the following setting. Let X be a proper variety and let $k = \bar{k}$ be an algebraically closed field. Then $\Gamma(X, \mathcal{O}_X) = \bar{k}$. Moreover, if $X \to \operatorname{Spec} k$ is proper,

$$\Gamma(X, \mathcal{O}_X) \cong \operatorname{Mor}(X, \underbrace{\mathbb{A}^1_k}_{\cong \mathbb{G}_a})$$

and we get

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathbb{A}^1_k \arrow[r, hook] \arrow[d] & \mathbb{P}^1_k \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}$$

Lemma 7.23.5 says that $f(X)$ is closed in both $\mathbb{A}^1_k$ and $\mathbb{P}^1_k$. Then $f(X)$ must be a closed point of $\mathbb{A}^1_k$, and hence $\Gamma(X, \mathcal{O}_X) = \bar{k} = k$. Note that if we assume X is proper over S, Chow's lemma (Theorem 7.23.1) implies π' is proper, and hence so is $X' \xrightarrow{\pi'} X \to S$. Lemma 7.23.5 says that $X' \to \overline{X'} \hookrightarrow \mathbb{P}^n_S$ has closed image. Then, since X, S is reduced, we have a closed immersion $X' \hookrightarrow \mathbb{P}^n_S$. Conversely, if $X' = \overline{X}$, then X' is projective over S, so X' is proper over S, and hence for all T closed in X, we have $f(T) = f(\pi(\pi^{-1}(T)))$ is closed. This holds for any base change $S' \to S$ and $X'_S \to X_S$. Thus $X \to S$ is proper.

Summary. Given $f \colon X \to S$ separated and finite type, then $X \to S$ is proper if and only if there exists a projective $X' \to S$ with a surjective morphism $X' \to X$ over S.

Let's prove Chow's Lemma (Theorem 7.23.1). We recall the setup first.

We have S a noetherian scheme and $f \colon X \to S$ finite type and separated, along with a commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r] \arrow[d] & X' \arrow[r, hook] \arrow[d, "\pi"] \arrow[dr] & \mathbb{P}^n_S \arrow[d] \\ U \arrow[r] & X \arrow[r] & S \end{tikzcd}$$

We need to show that π is proper and surjective, $X' \hookrightarrow \mathbb{P}^n_S$ is an immersion and that there is a dense open $U \hookrightarrow X$ such that $\pi^{-1}(U) \xrightarrow{\sim} U$.

Proof of Theorem 7.23.1 for varieties. Suppose $S = \operatorname{Spec} k$, with $k = \bar{k}$ a field, and suppose X is a variety. Write $X = U_1 \cup \cdots \cup U_k$ where $U_i = \operatorname{Spec} A_i \subseteq X$; and let $A_i = \operatorname{Spec} k[x_{i,0}, \ldots, x_{i,n_i}]/I_i$. Let Z_i be the closure of U_i in $\mathbb{P}^{n_i}$ and consider the maps

$$\begin{tikzcd} & \mathbb{A}^{n_i} \arrow[dr, "\text{open}"] & \\ U_i \arrow[ur, "\text{closed}"] \arrow[rr, "j_i"] \arrow[dr, "\text{open}"'] & & \mathbb{P}^{n_i} \\ & Z_i \arrow[ur, "\text{closed}"'] & \end{tikzcd}$$

Set $U = U_1 \cap \cdots \cap U_k \subseteq X$, which is dense and open. We have the map

$$\begin{tikzcd}
U \arrow[rr, "j=(j_1,\dots,j_k)"] \arrow[dr, "\text{open}"'] & & \mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k} \\
& Z \arrow[ur, "\text{closed}"'] &
\end{tikzcd}$$

where Z denotes the closure of $j(U)$. We have

$$\begin{tikzcd}
U \arrow[r, hook, "\text{open}"] \arrow[d, hook, "\text{open}"'] & Z \arrow[r, hook] \arrow[dr, "p_i"] & Z_1\times\cdots\times Z_k \arrow[d, "pr_i"] \\
U_i \arrow[rr, hook, "\text{open}"] & & Z_i
\end{tikzcd}$$

where the pr_i is proper because it's the restriction of $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}\to\mathbb{P}^{n_i}$ to a closed subshceme. This means that p_i is proper. Now let $V_i=p_i^{-1}(U_i)$. Let $X':=p_1^{-1}(U_1)\cup\cdots\cup p_k^{-1}(U_k)=V_1\cup\cdots\cup V_k$. We want to map X' to X.

We claim that $p_i|_{V_i\cap V_j}=p_j|_{V_i\cap V_j}$, and thus glue to a morphism $\pi\colon X'\to X$. Indeed, since X is separated and contains U as a dense open, the locally closed subscheme in $V_i\cap V_j$ where $p_i\colon V_i\cap V_j\to V_i\hookrightarrow X$ and $p_j\colon V_i\cap V_j\to V_j\hookrightarrow X$ agree is closed, it agrees on all of $V_i\cap V_j$. This proves the claim.

We also claim that $\pi^{-1}(U_i)=V_i$. Indeed, consider the diagram

$$\begin{tikzcd}
V_i \arrow[rr, hook] \arrow[dr, "p_i|_{V_i}"'] & & \pi^{-1}(U_i)\subseteq Z \arrow[dl, "\pi|_{\pi^{-1}(U_i)}"] \\
& U_i &
\end{tikzcd}$$

Since Z is separated, it follows that $\pi|_{\pi^{-1}(U_i)}$ is separated. Then $p_i|_{V_i}$ is proper, because it's a base change of a proper morphism to U_i. Thus the image of V_i is closed in $\pi^{-1}(U_i)$, and thus it must be equal because it is dense (it contains U).

We also claim that π is proper. This is because $X=U_1\cup\cdots\cup U_n$, and each restriction of $|pi$ to $\pi^{-1}(U_i)$ is now identified with $p_i|_{V_i}$, a proper map, and being proper is local on the base. It remains to show that $\pi^{-1}(U)\xrightarrow{\sim}U$. This follows from the argument applied to $\pi^{-1}(U)=V$, that is,

$$\begin{tikzcd}
U \arrow[rr, hook] \arrow[dr, "\text{id}"] & & \pi^{-1}(U) \arrow[dl] \\
& U &
\end{tikzcd}$$

This proves Chow's Lemma for varieties. □

Definition 7.23.6. A scheme X of finite type over a field k is said to be quasi-projective if X has an immersion into $\mathbb{P}^n_k$ for some N. △

Lemma 7.23.7. *A proper quasi-projective variety is projective.*

Proof. We have

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "\text{proper}"'] & & \mathbb{P}^n_k \arrow[dl, "\text{separated}"] \\
& \operatorname{Spec} k &
\end{tikzcd}$$

implies $h(X)$ is closed. □

Corollary 7.23.8 (Chow's Lemma for varieties)**.** *For all varieties X, there is a quasiprojective variety X and a surjective morphism $X'\xrightarrow{\pi}X$ which is an isomorphism over a nonempty open $U\subseteq X$. Moreover, X is proper if and only if X' is projective.*

7.24 Apr 23, 2019

Today we'll talk about applications of scheme theory to curves.

By a curve we mean a variety over a field k of dimension 1. Our varieties are irreducible. A basic fact is the following. For varieties X, we have $\dim X = d$ if and only if for all $x \in |X|$, we have $\dim \mathcal{O}_{X,x} = d$, if and only if $\operatorname{trdeg}_k k(X) = d$, where $k(X)$ denotes the ring of rational functions, and trdeg denotes transcendence degree.

Definition 7.24.1. Let S be an integral scheme. The function field $k(S) := \operatorname{Frac}\mathcal{O}_S(U)$ for $\emptyset \neq U \subseteq S$. We have $k(S) = \mathcal{O}_{S,\eta}$ for η generic point of S. $\triangle$

Definition 7.24.2. A morphism $f\colon X \to Y$ of varieties over k is said to be:

(a) dominant if $f(X)$ is dense in Y, or equivalently if $f(\eta_X) = \eta_Y$, or equivalently if $f(X)$ contains a nonempty open subset of Y. (This is why irreducibility is an important assumption)

(b) birational if it is dominant and furthermore $\mathcal{O}_{\eta_Y} = k(Y) \to k(X) = \mathcal{O}_{\eta_X}$ is an isomorphism, and equivalently (by Chevalley) that there exists a nonempty $U \subseteq Y$ open such that $f^{-1}(U) \xrightarrow{\sim} U$.

$\triangle$

Lemma 7.24.3. *Suppose $f\colon X \to Y$ is a proper biraitonal morphism of curves. Assume that Y is regular (that is, $\mathcal{O}_{Y,y}$ is regular for all $y \in Y$). Then f is an isomorphism.*

Proof. Recall that if A is a noetherian local ring with dimension 1, then A is regular if and only if A is a DVR. Pick $x \in X$. We have

$$\begin{tikzcd}
\mathcal{O}_{X,x} \arrow[d, hook] & \mathcal{O}_{Y,y} \arrow[l, "f^*"'] \arrow[d, hook] \\
k(X) & k(Y) \arrow[l, "\cong"']
\end{tikzcd}$$

which implies that f^* is an isomorphicm by definition of a valuation ring (maximal under inclusion).

Suppose $x, x' \in |X|$ such that $f(x) = f(x')$. Then $\mathcal{O}_{X,x} = \mathcal{O})X,x' = \mathcal{O}_{Y,f(x)}$. We have maps

$$\begin{tikzcd}
\operatorname{Spec}\mathcal{O}_{X,x} \arrow[rrd, "\iota_x"] \arrow[rd] & & \\
& \operatorname{Spec}\mathcal{O}_{X,x'} \arrow[r, "\iota_{x'}"] \arrow[d] & X \arrow[d, "f"] \\
& \operatorname{Spec}\mathcal{O}_{Y,f(x)} \arrow[r] \arrow[luu, dashed, bend left, "s_1"] \arrow[u, dashed, bend left, "s_2"] & Y
\end{tikzcd}$$

Since $\mathcal{O}_{Y,f(x)} \cong \mathcal{O}_{X,x} \cong \mathcal{O}_{X,x'}$, we have sections s_1 and s_2. By composing with ι_X and $\iota_{x'}$ we get two morphisms

$$\operatorname{Spec}\mathcal{O}_{f(x)} \underset{\iota_{x'} \circ s_2}{\overset{\iota_x \circ s_1}{\rightrightarrows}} X$$

By the valuative criterion of properness, we have $\iota_x \circ s_1 = \iota_{x'} \circ s_2$ which ipmlies $x = x'$.

Since f is proper, we know that $f(X)$ is closed in Y. Since it contains the generic point $f(X) = Y$. Hence f is injective, surjective, and closed. We get $f_x\colon \mathcal{O}_{X,x} \xrightarrow{\sim} \mathcal{O}_{Y,f(x)}$ so $X \cong Y$. $\square$

Proposition 7.24.4. *Any regular curve is quasiprojective.*

Proof. Given a regular curve X, by Chow's lemma (Theorem 7.23.1) we have $\pi\colon X' \to X$ proper and birational and X' quasiprojective. Now we apply the previous Lemma 7.24.3. $\square$

Lemma 7.24.5. *If X is a regular curve and Y is a proper variety, any morphsim $f\colon U \to Y$ for nonempty open $U \subseteq X$ extends to a full morphism $f\colon X \to Y$.*

Proof. Let Z be the closure of $\iota_U \times f\colon U \times U \to X \times Y$. Then Z is a variety (we're using Lemma 3.8.21 here) and $U \subseteq Z$ is open and dense: we have

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr, "\sim"] & X \times Y \arrow[r] \arrow[d] & Y \\ & X \arrow[ul, bend left] & \end{tikzcd}$$

By Lemma 7.24.3 we have $Z \cong X$, and we get an inverse $X \to Z$. Now we compose with $Z \to Y$. $\square$

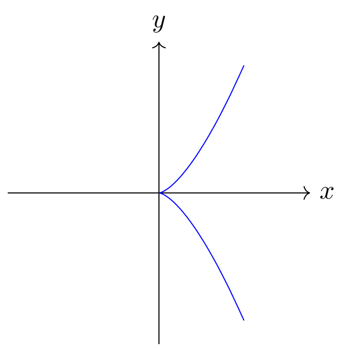
Recall that an integral scheme S is normal if all affine open $\mathcal{O}_S(U)$ are integrally closed in $\bar{k}(S)$ (that is, these are normal domains).

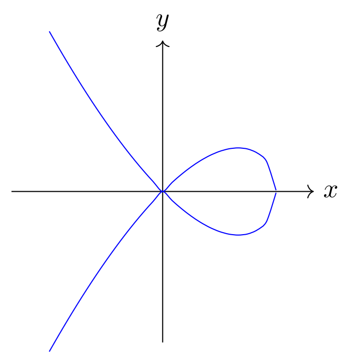
Lemma 7.24.6. *For any variety X, there exists a canonical morphism of varieties $\nu\colon X^\nu \to X$ called the normalalization of X, where ν is birational, finite, and X^ν is a normal variety.*

Proof. This follows from the existence of an integral closure: if A is a finitely generated domain over k, then the integral closure A^{int} of A in $\operatorname{Frac} A$ is a finite A-module. $\square$

Recall from homework that we showed finite morphisms are proper.

Example 7.24.7 (Classical cases of normalization)**.** If we have a cuspidal singularity $\operatorname{Spec} k[x,y]/\langle y^2 - x^3\rangle$ or a nodal singularity $\operatorname{Spec} k[x,y]/\langle y^2 - x^2(1-x)\rangle$, as in





[The curve on the right is supposed to be connected/smooth with a single nodal singularity, and "any other singularities are due to the author" - Jake]

respectively. In the respective function fields, we have $(y/x)^2 - x = 0$ and $(y^2/x) - (1-x) = 0$, which implies y/x is integral. We have

Exercise: Show that $A[y/x] \subseteq k(C)$ is normal. These are precisely the normalizations. $\triangle$

Warning. The integral closure of a domain that is finitely generated over k in its fraction field is finite over itself. This is not true for finitely generated algebras over k (with nilpotents, say). For example, for $k[x,\varepsilon]/\langle \varepsilon^2\rangle$ in $\operatorname{Frac} k[x,\varepsilon]/\langle \varepsilon^2\rangle = k(x)[\varepsilon]/\langle \varepsilon^2\rangle$, then the integral closure is not finitely generated because $(\varepsilon/x^n)^2 = 0$.

Definition 7.24.8. For X and Y varieties over k, a <u>rational map</u> $X \dashrightarrow Y$ is an equivalence class of morphisms $f\colon U \to Y$ where $U \subseteq X$ is nonempty and open, and $(f\colon U \to Y) \sim (g\colon V \to Y)$ if there is a nonempty open $W \subseteq U, V$ such that $f|_W = g|_W$.

We say a rational map $X \dashrightarrow Y$ is <u>dominant</u> if for any representative $f\colon U \to Y$, it is dominant in the usual sense. (That this is well defined follows from Chevalley). $\triangle$

Observation 7.24.9. Let X and Y be varities over k. If $(f\colon U \to Y) \sim (g\colon V \to Y)$ then $f|_{U\cap V} = g|_{U\cap V}$. Since Y is separated, the set of points where the maps agree is closed in $U \cap V$. Since $U \cap V$ is nonempty and open, it is dense. Hence the maps f and g glue to give a morphism $U \cap V \to Y$. Hence, there's some maximal open where the rational map is defined as a morphism. △

Henceforth, if $f\colon X \dashrightarrow Y$ is a rational map between varieites, take it with a chosen map.

Observation 7.24.10. Let $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow Z$ with f dominant. Then we get its composition

$$\begin{tikzcd}
U \cap f^{-1}(V) \arrow[r, "\subseteq"] \arrow[dr, "f|_{U\cap f^{-1}(V)}"'] & U \arrow[r, "\subseteq"] \arrow[dr, "f"] & X \arrow[d, dashed] \\
 & V \arrow[r, "\subseteq"] \arrow[dr, "g"] & Y \arrow[d, dashed] \\
 & & Z
\end{tikzcd}$$

Thus if $R(X)$ denotes the set of rational maps $X \dashrightarrow \mathbb{A}^1_k$, or equivalently the set of rational functions on X, then

$$R(X) = \operatorname{colim}_{U \subseteq X} \mathrm{Mor}_{\mathbf{Var}}(U, \mathbb{A}^1_k) = \operatorname{colim}_{U\subseteq X} \mathcal{O}_X(U).$$

Compare this to the homework problem on varieties. △

Proposition 7.24.11. *The category of varieties with objects varieties over k and morphisms dominant rational maps is antiequivalent to the category of finitely generated field extensions $K \supseteq k$ with k-algebra homomorphisms*

$$X \mapsto k(X)$$
$$\varphi\colon X \dashrightarrow Y \mapsto (\varphi^*\colon k(Y) \to k(X))$$

Corollary 7.24.12. *There is an anti-equivalence between the subcategory of regular (equivalently, normal) projective curves with dominant (equivalently, non-constant) morphisms of varieties and the subcategory of finitely field extensions $K \supseteq k$ with $\mathrm{trdeg}_k K = 1$ with k-algebra homomorphisms.*

Remark 7.24.13. In the correspondence above, being finitely generated over k is crucial. If, say, K is not finitely generated over k but $\mathrm{trdeg}_k K = 1$, then the corresponding curve is a "Riemann surface with infinite genus" (this is not a variety, but it's a scheme). △

Example 7.24.14. Consider $C_n\colon y^n = x(x-1)(x-\lambda)$ over k. This is an n-sheeted branched cover of $\mathbb{P}^1$. The genus of C_n is $n-1$. We have a family of field extensions and corresponding curves

$$k(C_n) \hookrightarrow k(C_{2n}) \hookrightarrow k(C_{4n}) \hookrightarrow \dots$$

$$\dots \xrightarrow{y \mapsto y^2} C_{4n} \xrightarrow{y\mapsto y^2} C_{2n} \xrightarrow{y \mapsto y^2} C_n$$

Then $\lim_k k(C_{2n})$ is not finitely generated but of transcendence degree 1 over k. This is not the function field of an algebraic curve, but it's a k-algebra, so a scheme over k. △

7.25 Apr 25, 2019

[I am grateful for Joseph's and Jake's notes, from which I copied.]
We'll talk about divisors today.

The motivation comes from arithmetic. Let L/K be a finite field extension, and let $\mathcal{O}_K$ and $\mathcal{O}_L$ be their rings of integers (these are by definition the integral closures of $\mathbb{Z}$ in the respective fields). Take a point $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$. Then we have primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ in $\operatorname{Spec} \mathcal{O}_L$ that lie over $\mathfrak{p}$, with $\mathfrak{q}_i \neq \mathfrak{q}_j$, that is to say, we have

$$\mathfrak{p}\mathcal{O}_L = \mathfrak{q}_1^{e_1} \ldots \mathfrak{q}_r^{e_r}.$$

The e_i are called the ramification degree. We can define $f_i = [k(\mathfrak{q}_i) : k(\mathfrak{p})]$, where $k(\mathfrak{q}_i)$ and $k(\mathfrak{p})$ are the residue fields of $\mathcal{O}_{L,\mathfrak{q}_i}$ and $\mathcal{O}_{K,\mathfrak{p}}$. We have the degree formula

$$[L : K] = \sum_{i=1}^{r} e_i f_i.$$

We have a map $f \colon \operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_K$ [corresponding to $\mathcal{O}_K \hookrightarrow \mathcal{O}_L$], and the preimage of a point $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ is precisely the $\mathfrak{q}_i$ lying over $\mathfrak{p}$.

The takeaway is that the degree of f over the generic point η of $\operatorname{Spec} \mathcal{O}_K$ is the same as that above $\mathfrak{p}$. In general, we let A be a DVR, $K = \operatorname{Frac} A$, and L be a finite extension of K. We further assume that the integral closure B of A in L is finite over A (this holds for "reasonable rings", such as Japanese or Nagata rings in EGA). Then $\mathfrak{m}_A B = \mathfrak{m}_a^{e_1} \ldots \mathfrak{m}_r^{e_r}$, and

$$[L : K] = \sum_{i=1}^{r} e_i [k(\mathfrak{m}_i) : k(\mathfrak{m}_A)].$$

Proof. Since B is a finite A module and is torsion-free (as an A-module), then B is free, and $B \cong A^{\oplus n}$. Thus, $n = [L : K]$. Since B is the integral closure of A in L, B is normal. Since B is finite over A, we have $\dim B = \dim A = 1$. Hence B is a finite type algebra over a Neotherian ring, and so is Noetherian, and hence a Dedekind domain. All local rings at closed points are DVRs, and $\mathfrak{m}B = \mathfrak{m}_1^{e_1} \ldots \mathfrak{m}_r^{e_r}$. Now

$$\begin{aligned} n &= \operatorname{length}_A(B/\mathfrak{m}_A B) \\ &= \sum \operatorname{length}_A(B/\mathfrak{m}_i e^{e_i} B) \\ &= \sum_i e_i \operatorname{length}_A(B/\mathfrak{m}_i B) \\ &= \sum_i e_i [k(\mathfrak{m}_i) : k(\mathfrak{m}_A)]. \end{aligned}$$

□

Definition 7.25.1. Let X be a Noetherian scheme. (More general machinery is developed in EGA, but we follow Hartshorne's treatment).

1. An effective Cartier divisor on X is a closed subscheme $D \hookrightarrow X$ so for all $x \in D$ there is an affine open $\operatorname{Spec} A \ni x$, so $D \cap \operatorname{Spec} A = \operatorname{Spec} A/\langle f \rangle$ for f a nonzerodivisor $f \in A$. Equivalently, $\mathcal{I}_D \subseteq \mathcal{O}_X$ the ideal sheaf of D is an invertible sheaf of $\mathcal{O}_X$-modules.

2. A Weil divisor on X is a finite formal sum $D = \sum n_Z [Z]$ with $n_Z \in \mathbb{Z}$ and $Z \subseteq X$ is a subscheme of codimension 1.

3. A prime divisor $Z \subseteq X$ is an irreducible, reduced closed subscheme such that $\dim \mathcal{O}_{X,\eta} = 1$ where $\eta \in Z$ is the generic point of Z.

△

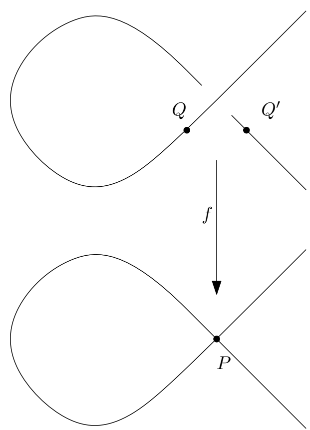
The way to think about these discussion is the following: the effective Cartier divisor is the correct technical notion for general schemes, but in nice settings (eg. smooth algebraic varieties and especially curves, over a field) the notion of effective Cartier divisor and Weil divisor agree. The question is: how do we pass between Weil divisors and Cartier divisors?

Let D be an effective Cartier divisor, so that

$$[D] = \sum_{\zeta \in Z \subseteq X} \operatorname{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{D,\xi})[Z]$$

where Z is a prime divisor and $\xi \in D$. Here $\xi \in D$ if and only if $\overline{\{\xi\}} = Z \subseteq D$; this works because if $\xi \in D$, then $\mathcal{O}_{D,\xi} = \mathcal{O}_{X,\xi}/\langle f \rangle$ for some non zerodivisor $f \in \mathcal{O}_{X,\xi}$, and thus $\dim \mathcal{O}_{D,\xi} = \dim \mathcal{O}_{X,\xi} - 1 = 0$.

Let's talk about pulling back divisors. Unfortunately, this is not always possible, as in the picture below:



We cannot pull back the Weil divisor $[P]$ of the singular point P (since it should be $f^*[P] = \frac{1}{2}[Q] + \frac{1}{2}[Q']$). However, you can pull back in the following setting:

Example 7.25.2. Let $f\colon X \to Y$ be a morphism of Noetherian schemes, and let $D \hookrightarrow Y$ be an effective Cartier divisor; we ["have"?] $f^{-1}(D)$ as a fiber product:

$$\begin{array}{ccc} f^{-1}(D) & \longrightarrow & D \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Here $f^{-1}(D)$ is an effective Cartier divisor. An important special case is when $f\colon X \to Y$ is a dominant morphism of varieties, since $k(Y) \hookrightarrow k(X)$. Here $f^*D = f^{-1}(D)$. △

Remark 7.25.3. Cartier divisors and invertible sheaves form a cohomology theory. Weil divisors form a homology theory. Cohomology naturally pulls back. Homology naturally pushes forward, but doing the reverse is tricky. △

Lemma 7.25.4. *Let Y be a regular (normal) curve. Any Weil divisor D on Y can be written uniquely as $D = [D_1] - [D_2]$ where $D_i \hookrightarrow Y$ are effective Cartier divisors and $D_1 \cap D_2 = \emptyset$.*

Proof. Let

$$D = \sum_{y \in |Y|} n_y[y] = \sum_{\substack{n_y > 0 \\ y \in |Y|}} n_y[y] + \sum_{\substack{n_y < 0 \\ y \in |Y|}} n_y[y]$$

and setting $D_1 = \sum_{n_y > 0} n_y[y]$ and $D_2 = \sum_{n_y < 0} n_y[y]$. Note that each $[y]$ is associated to an effective Cartier divisor $\mathcal{I}(y)$. So

$$\sum n_y[y] \leftrightarrow I(y)^{\otimes n_y}.$$

Note that

$$\mathcal{I}(y)_x = \begin{cases} \mathcal{O}_{Y,x} & x \neq y \\ \mathfrak{m}_y & x = y \end{cases}$$

□

Example 7.25.5. Let $\mathbb{P}^1_k = \operatorname{Proj} k[T_0, T_1]$ and let $t = T_0/T_1$. Consider the Weil divisor $3[t = 0] + 5[t = 17]$. We get

$$\operatorname{Spec} k[t]/\langle (t-17)^5 t^3 \rangle \xhookrightarrow{\text{closed}} \operatorname{Spec} k[t] \xhookrightarrow{\text{open}} \mathbb{P}^1_{\mathbb{C}}$$

and since the image set is closed (only true for curves!), the composition is a closed immersion. Hence D is a closed subscheme of $\mathbb{P}^1_k$ and so is a Cartier divisor. △

Consider the special case of Weil divisors on curves. Let $f\colon X \to Y$ be a dominant (ie. non constant) morphism of curves, and Y is regular. Given any Weil divisor D on Y, we have $D = D_1 - D_2$ by lemma. We have $f^*D = [f^*D_1] - [f^*D_2]$, where f^*D_i are pullbacks of Cartier divisors.

Example 7.25.6. Consider the map

$$\begin{aligned} \operatorname{Proj} \mathbb{C}[S_0, S_1] = \mathbb{P}^1_{\mathbb{C}} &\xrightarrow{\varphi} \mathbb{P}^1_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[T_0, T_1] \\ T_0 &\mapsto S_0^2 \\ T_1 &\mapsto S_1^2 \end{aligned}$$

On an affine open $\operatorname{Spec} \mathbb{C}[t]$, φ corresponds to a finite morphism

$$\operatorname{Spec} \mathbb{C}[s] = \mathbb{A}^1_{\mathbb{C}} \xrightarrow{\varphi} \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[t]$$

corresponding to a finite ring extension $\mathbb{C}[t] \hookrightarrow \mathbb{C}[s]$ given by $t \hookrightarrow s^2$. We have $\varphi^{-1}(\operatorname{Spec} \mathbb{C}[t]) = \operatorname{Spec} \mathbb{C}[s]$ because $\mathbb{C}[s]$ is the integral closure of $\mathbb{C}[t]$ in $\mathbb{C}(s)$. For D as in the previous Example 7.25.5 we have

$$\begin{aligned} \varphi^*(D) &= [\varphi^{-1} D] \\ &= [\operatorname{Spec} \mathbb{C}[s]/\langle s^6 (s^2 - 17)^5 \rangle] \\ &= 6[s = 0] + 5[s = \sqrt{17}] + 5[s = -\sqrt{17}]. \end{aligned}$$

△

Definition 7.25.7. Given a Weil divisor $D = \sum_{x \in |X|} n_x [x]$ on a curve X, set

$$\deg D = \sum_{x \in |X|} n_x [k(x) : k].$$

△

The finiteness of $[k(x) : k]$ is not obvious, for example over $\mathbb{C}$ it is Hilbert's Nullstellensatz. We have

Lemma 7.25.8. *Any nonconstant proper morphism between curves over a field k is finite. More generally, any proper morphism with finite fibers is finite.*

See Hartshorne for a proof.

Theorem 7.25.9. *Let $f\colon X \to Y$ be a nonconstant (equivalently, dominant) morphism of projective regular (equivalently, proper regular) curves over a field k. Let $n = [k(X) : k(Y)] = \deg f$. Then for all $y \in |Y|$, we have*

$$\deg(f^*[y]) = n \cdot \deg[y].$$

So by linearity, for all Weil divisors D on Y, we have

$$\deg(f^*D) = n \cdot \deg D.$$

Proof. Let $y \in |Y|$, and choose an affine open neighborhood Y of y. By Lemma 7.25.8 we have $f^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B$ affine, and $A \to B$ is finite. Since X is regular, B is integrally closed. Hence we have

$$\begin{array}{ccc} A_{\mathfrak{m}_y} & \hookrightarrow & k(Y) \\ \downarrow & & \downarrow \\ B_{\mathfrak{m}_y} & & k(X) \end{array}$$

with $B_{\mathfrak{m}_y}$ containing maximal ideals $\mathfrak{m}_{x_1}, \ldots, \mathfrak{m}_{x_r}$.This gives $\mathfrak{m}_y B_{\mathfrak{m}_y} = \mathfrak{m}_{x_1}^{e_1} \ldots \mathfrak{m}_{x_r}^{e_r}$. Then

$$\begin{aligned}
\deg(f^*[y]) &= \deg[f^{-1}(y)] \\
&= \deg\left(\sum_{i=1}^{r} \operatorname{length}_{\mathcal{O}_{X,x_i}}(k(y) \otimes \mathcal{O}_{X,x_i})[x_i]\right) \\
&= \sum_{i=1}^{r} \operatorname{length}_{B_{\mathfrak{m}_{x_i}}}\left(\frac{B_{\mathfrak{m}_{x_i}}}{\mathfrak{m}_y B_{\mathfrak{m}_{x_i}}}\right)[k(x_i):k] \\
&= \sum_{i=1}^{r} \dim_{k(x_i)}\left(\frac{B_{\mathfrak{m}_{x_i}}}{\mathfrak{m}_{x_i}^{e_i} B_{\mathfrak{m}_{x_i}}}\right)[k(x):k(y)][k(y):k] \\
&= \sum_{i=1}^{r} e_i[k(x):k(y)][k(y):k] \\
&= [k(X):k(Y)] \cdot [k(y):k] \\
&= n \cdot \deg([y]).
\end{aligned}$$

□

7.26 Apr 30, 2019

Today we will flesh out the correspondences

$$\begin{tikzcd}
 & \{\text{divisors}\} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\{\text{line bundles}\} \arrow[rr, leftrightarrow] & & \{\text{invertible sheaves}\}
\end{tikzcd}$$

Let X be a noetherian scheme and $\mathcal{L}$ an invertible sheaf. Pick an $s \in \Gamma(X, \mathcal{L})$.

Definition 7.26.1. We say s is a <u>regular section</u> if $\mathcal{O}_X \xrightarrow{\cdot s} \mathcal{L}$ is injective. △

We let $Z(s) :=$ largest closed subscheme Z of X such that $s|_Z \equiv 0$. In other words, this is kind of the max of the $\{\iota \colon Z \hookrightarrow X$ such that $0 = \iota^* s \in \Gamma(Z(s), \iota^*\mathcal{L})\}$. The subtlety is in the schematic structure here.

Locally, if we choose a trivialization $\varphi_U \colon \mathcal{L}_U \cong \mathcal{O}_U$, then $\varphi_U(s) = f \in \Gamma(U, \mathcal{O}_X)$ and $Z(s) \cap U = Z(f)$.

Example 7.26.2. If $U = \operatorname{Spec} A$, then $Z \cap U = \operatorname{Spec} A/\langle f \rangle$. △

Note that s regular implies that f is a nonzerodivisor. Hence s is a regular section if and only if $Z(s)$ is an effective Cartier divisor.

Example 7.26.3. If we know X is integral, then s is regular if and only if $s \neq 0$. △

Lemma 7.26.4. *Let $X, \mathcal{L}, s$ be as above, with s regular. Let $D = Z(s)$. Then $\mathcal{L} \cong \mathcal{I}_D^{-1} \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}_D, \mathcal{O}_X)$.*

Proof. Equivalently, this is the statement that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{I}_D \cong \mathcal{O}_X$. We see that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{I}_D \to \mathcal{O}_X$. Then locally, $(s', f) \mapsto (fs'/s)$ is an isomorphism. □

Conversely, if $D \hookrightarrow X$ is an effective Cartier divisor, we have

$$\mathcal{O}_X(D) := \mathcal{I}_D^{-1} \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}_D, \mathcal{O}_X)$$

is an invertible sheaf; this has a canonical section associated with D, denoted 1_D, where $Z(1_D) = D$.

Here's a question: given D and D', when is $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$?

Lemma 7.26.5. *If X is integral, then $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ if and only if there is $f \in k(X)^*$ such that for all $\operatorname{Spec} A \subseteq X$ affine open with $D \cap \operatorname{Spec} A = \operatorname{Spec} A/\langle a \rangle$ and $D' \cap \operatorname{Spec} A = \operatorname{Spec} A/\langle a' \rangle$, then $f = u \cdot a/a'$ with $u \in A^\times$.*

Proof. If φ is an isomorphism $\mathcal{O}_X(D) \xrightarrow{\cong} \mathcal{O}_X(D')$, then $\varphi(1_D) = f \cdot 1_{D'}$. □

Definition 7.26.6. Let X be integral and Notherian, and let $f \in k(X)^*$. We have a Weil divisor

$$\operatorname{div}(f) := \sum_{\xi \in Z \subseteq X} \operatorname{ord}_\xi(f)[Z]$$

where ξ is a generic point in a prime divisor Z, and $\operatorname{ord}_\xi(f) := \operatorname{length}(\mathcal{O}_{X,\xi}/\langle a \rangle) - \operatorname{length}(\mathcal{O}_{X,\xi}/\langle b \rangle)$ and $a, b \in \mathcal{O}_{X,\xi}$ is such that $f = a/b \in \operatorname{Frac}(\mathcal{O}_{X,\xi}) = k(X)$. △

We define the notation $\operatorname{Cl} X := \{\text{Weil divisors on } X\}/\{\text{principal divisors on } X\}$. Now we have

Lemma 7.26.7. *On a regular curve X,*

$$\begin{aligned} \operatorname{Cl} X &\xrightarrow{\cong} \operatorname{Pic} X = \{\text{line bundles}\}/\sim \\ D &\mapsto \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1} \end{aligned}$$

where $D = D_1 - D_2$, for D_1, D_2 effective Cartier divisors.

Proof. By Lemma 7.26.5, this map is well defined and injective. It remains to show surjectivity. There's some group structure also preserved but we won't dwell too much on it.

Observation 7.26.8. Any $\mathcal{L}$ which has a nonzero section is isomorphic to $\mathcal{O}_X(D_1)$ for some effective divisor D_1. So it is enough to show that given an arbitrary $\mathcal{L}$, there exists some D_2 effective such that $\mathcal{L}(D_2)$ has a nonzero global section.

So pick a nonempty affine $U \subseteq X$ such that $\Gamma(U, \mathcal{L}) \neq 0$. Pick any nonzero section $s \in \Gamma(U, \mathcal{L})$. Then $X \setminus U = \{x_1, \ldots, x_n\}$ consists of finitely many closed points. Then for N sufficiently large, s extends to a section of $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(N([x_1] + \cdots + [x_r]))$ (for example, say s is a rational section of $\mathcal{L} \otimes_{\mathcal{O}_X} k(X)$; then any $N \geq \max_{i \in [r]}\{\mathrm{ord}_{x_i}(s)\}$ works). △

So here D_1 corresponds to s and D_2 corresponds to $N([x_1] + \cdots + [x_r])$. □

Lemma 7.26.9. *On a nonsingular projective curve X, the degree of any principal divisor is 0.*

Proof. Pick any $f \in k(X)^*$; we may assume f is finite over k: if not, it's convention to set $\mathrm{div}(f)$.

We have a morphism $f\colon X \to \mathbb{P}^1$ corresponding to the map $k(t) \hookrightarrow k(X)$ on the algebra side given by $t \mapsto f$. Here, note that $\mathrm{div}(f) = f^*([0] - [\infty])$, and $\deg(\mathrm{div}(f)) = \deg f^*([0]) - \deg^*([\infty]) = \deg f - \deg f = 0$. □

As a consequence of this discussion, the degree of an invertible sheaf $\mathcal{L}$ on a regular projective curve is well defined, so that if

$$\mathcal{L} \cong \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}$$

then

$$\deg(\mathcal{L}) := \deg(D_1) - \deg(D_2).$$

Example 7.26.10. We have $\operatorname{Pic} \mathbb{P}^1_k \cong \mathbb{Z}$ with isomorphism given by degree. △

Proof. Since $\operatorname{Pic} X \cong \operatorname{Cl} X$, it is enough to show that every degree 0 Weil divisor is principal. There are two cases:

- If k is algebraically closed, then any degree 0 divisor is of the form

  $$\sum_i a_i[\alpha_i] - \sum_j b_j[\beta_j]$$

  with $a_i, b_j > 0$ and $\sum_i a_i = \sum_j b_j$. By a linear change of coordinates we can assume without loss of generality that $[\alpha_i], [\beta_j] \neq \infty$, and now just take $\mathrm{div}(f)$ for

  $$f = \frac{\prod (t - \alpha_i)^{\alpha_i}}{\prod (t - \beta_j)^{\beta_j}},$$

  which is regular at ∞ because $\sum a_i = \sum b_j$.

- If k is not algebraically closed, then $k(\alpha_i) \cong k[t]/\langle f_i \rangle$ for some $f \in k[t]$ monic and irreducible (and similarly $k(\beta - j) \cong k[t]/\langle g_j \rangle$ for $g \in k[t]$ monic and irreducible). Thend egree 0 means that

  $$\sum a_i \deg f_i - \sum b_j \deg g_j = 0$$

  and now we can take $\mathrm{div}(f)$ for

  $$f = \frac{\prod_i f_i^{a_i}}{\prod_j g_j^{b_j}}.$$

□

Example 7.26.11. Let X be nonsingular and projective over algebraically closed k, and suppose $\operatorname{Pic} X \cong \mathbb{Z}$. Then $X \cong \mathbb{P}^1_k$. △

Proof. Pick $x_1, x_2 \in X$ closed points Then $\operatorname{Pic} X \cong \mathbb{Z}$ means that $[x_1] - [x_2] = \operatorname{div}(f)$ for some f. Consider $f\colon X \to \mathbb{P}^1_k$. Note that $f^*([0]) = [x_1]$ implies that f is of degree 1. We have $k(X)$ is a degree 1 extension of $k(\mathbb{P}^1)$. Since X is regular and projective, we have $\mathbb{P}^1_k \cong X$. $\square$

Warning. In general, $\operatorname{Pic} \mathbb{A}^1_R \not\cong \operatorname{Pic} \operatorname{Spec} R$ and $\operatorname{Pic} \mathbb{P}^1_R \not\cong \operatorname{Pic} R \times \mathbb{Z}$. But for "nice" R, for example if it is regular in codimension 1, these isomorphisms do hold.

Motivation for next semester. Why is it natural to introduce cohomology in scheme theory?

A natural question that one can ask is the following. Suppose X is some scheme so that the reduced scheme X^{red} is affine. Is X itself affine?

The answer is yes, but the proof is extremely tough. Even in the Noetherian case, the proof uses Serre's cohomological criterion for being affine: If X is affine, coherent sheaves over $\mathcal{O}_X$ have no higher cohomology because global sections allows you to recover your $\widetilde{M}$; Serre proved the converse of this.

Here's another question. Let X be a scheme over a field k. If X^{red} is projective, then is X projective?

The answer here is no (this is also proven via cohomology).

7.27 May 2, 2019

Today we'll talk about ample invertible sheaves. The motivation comes from the fact that very ample line bundles have enough global sections to embed into some $\mathbb{P}^n$. But from the point of view of sheaf theory, it is more natural to study ample line bundles: roughly, an ample line bundle $\mathcal{L}$ is a line bundle where $\mathcal{L}^{\otimes n}$ for some $n \geq 1$ is very ample.

Definition 7.27.1 (EGA)**.** [different from Hartshorne, as we'll see below]

1. We say that an invertible sheaf $\mathcal{L}$ on X is ample if
   (a) X is quasicompact
   (b) for all $x \in X$, there is $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ for some $n \geq 1$, so that
   $$X_s = \{x \in X : s \text{ generates } \mathcal{L}_X \text{ as an } \mathcal{O}_{X,x}\text{-module}\}$$
   is affine.
2. We say $f: X \to S$ is projective if there is
   $$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \mathbb{P}(\mathcal{E}) = \operatorname{Proj}_S(\operatorname{Sym}^* \mathcal{E}) \arrow[d] \\ & S \end{tikzcd}$$
   where $\mathcal{E}$ is a quasicoherent $\mathcal{O}_S$-module of finite type (locally on S, $\mathcal{E}|_{\operatorname{Spec} A} = \widetilde{M}$ for M a finite type A-module, then $\mathbb{P}(\mathcal{E})|_{\operatorname{Spec} A} = \operatorname{Proj}(\operatorname{Sym}^*_A M)$).
3. We say $\mathcal{L}$ is relatively ample if for all $V \subseteq S$ open affine $\mathcal{L}|_{f^{-1}(V)}$ is ample on $f^{-1}(V)$.
4. We say $\mathcal{L}$ is relatively very ample if there exists an immersion
   $$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr] & \mathbb{P}(\mathcal{E}) = \operatorname{Proj}_S(\operatorname{Sym}^* \mathcal{E}) \arrow[d] \\ & S \end{tikzcd}$$
   such that $\mathcal{L} \cong \iota^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ for some quasicoherent sheaf $\mathcal{E}$ of finite type.
5. We say $f: X \to S$ is quasi-projective if f is of finite type and admits a relatively ample invertible $\mathcal{O}_X$-module.

$\triangle$

Contrast this with Hartshorne's definitions (who most importantly assumes Noetherianity):

Definition 7.27.2. Let X be noetherian and let $\mathcal{L}$ be an invertible sheaf on X.

1. We say $\mathcal{L}$ is ample if for all coherent sheaves $\mathcal{F}$ on X, there is $d_0(\mathcal{F})$ such that $\mathcal{F} \otimes_{\mathcal{O}_n} \mathcal{L}^{\otimes d}$ is globally generated for all $d \geq d_0$
2. We say $f: X \to S$ is projective if there exists a closed immersion
   $$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \mathbb{P}^n_S \arrow[d] \\ & S \end{tikzcd}$$

3. We say $f\colon X\to S$ is quasiprojective if f factors

$$X \xhookrightarrow{\text{open immersion}} X' \xrightarrow{\text{projective as in (2)}} S$$

△

Lemma 7.27.3. *Let $S = \operatorname{Spec} R$ with R Noetherian, and let $f\colon X\to S$ is proper and $\mathcal{L}$ is an invertible sheaf on X. Suppose for all $\mathcal{F}$ coherent, there is $d(\mathcal{F})\in\mathbb{Z}$ such that for all $d\geq d(\mathcal{F})$ we have $H^1(X,\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^d)=0$. Then $\mathcal{L}$ is ample.*

The converse is not true, for example $\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}\setminus\{0\}}$ is ample, but has nonzero $H^1(\mathbb{A}^2_{\mathbb{C}}\setminus\{0\},\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}\setminus\{0\}})$.

Proof of Lemma 7.27.3. Pick a closed point $x\in|X|$. Then we have $j_x\colon\{x\}\hookrightarrow X$ giving

$$0\longrightarrow\mathcal{I}_x\longrightarrow\mathcal{O}_X\longrightarrow(j_x)_*k(x)\longrightarrow 0$$

and twisting by $\mathcal{L}^{\otimes d}$ gives

$$0\longrightarrow\mathcal{I}_x\otimes\mathcal{L}^{\otimes d}\longrightarrow\mathcal{L}^{\otimes d}\longrightarrow\mathcal{L}^{\otimes d}\otimes k(x)\longrightarrow 0$$

and we have $\Gamma(X,\mathcal{L}^{\otimes d}k(x))\cong k(x)$, though this isomorphism is non-canonical.

By assumption, we can pick $s_i\in\Gamma(X,\mathcal{L}^{\otimes i})$ for all $i\in\{d(\mathcal{I}_x),2d(\mathcal{I}_x)-1\}$ with $s_i(x)\neq 0$. Then

$$U_x=\bigcap_{i=d(\mathcal{I}_x)}^{2d(\mathcal{I}_x)-1}X_{s_i}.$$

We see that for all $x'\in U_x$ and for all $d\geq d(\mathcal{I}_x)$, there is $s\in\Gamma(X,\mathcal{L}^{\otimes d})$ such that $s(x')\neq 0$ (for example, we can take $s=s_{d(\mathcal{I}_x)}^m\cdot s_{d-md(\mathcal{I}_x)}$). Since X is quasicompact,

$$X=U_{x_1}\cup\cdots\cup U_{x_t}\qquad\text{for } x_i\in|X|.$$

Then $d_0(\mathcal{O}_X)=\max_{i=1,\dots,t}\{d(\mathcal{I}_{x_i})\}$. Hence for all $d\geq d_0(\mathcal{O}_X)$, $\mathcal{L}^{\otimes d}$ is globally generated.

Since we have this for $\mathcal{O}_X$, for a general coherent sheaf, apply the same argument to

$$0\longrightarrow\mathcal{I}_x\mathcal{F}\longrightarrow\mathcal{F}\longrightarrow\mathcal{F}\otimes_{\mathcal{O}_X}k(x)\longrightarrow 0$$

□

Lemma 7.27.4. *Suppose X is Noetherian. Then if $\mathcal{L}$ is ample in the sense of Hartshorne (Defnition 7.27.2), then $\mathcal{L}$ is ample in the sense of EGA (Definition 7.27.1).*

Proof. Pick $x\in X$ and an affine open $U\subseteq X$ containing x. Let $\mathcal{I}$ be an ideal sheaf of $X\setminus U$. Since X is Noetherian, $\mathcal{I}$ is coherent. By assumption, there is sufficiently large d and $s\in\Gamma(X,\mathcal{I}\otimes\mathcal{L}^{\otimes d})$ with $s(x)\neq 0$. Then $X_S\subseteq U$ by construction, and we can check directly that it's affine. Alternatively, we could have assumed $\mathcal{L}|_U\cong\mathcal{O}_U$, and then $X_S=D(f)$ where $s\leftrightarrow f\in\Gamma(U,\mathcal{O}_U)$. □

Lemma 7.27.5. *Let X be Noetherian and $\mathcal{L}$ is ample in the sense of EGA (Definition 7.27.1). Then*

$$X\xhookrightarrow{\text{open immersion}}\mathbb{P}(\Gamma_*(X,\mathcal{L}))$$

where $\Gamma_*(X,\mathcal{L}) =: \oplus_{d\geq 0}\Gamma(X,\mathcal{L}^{\otimes d})$.

Proof. For all $s\in\Gamma(X,\mathcal{L}^{\otimes d})$, we consider $D_+(s)=\operatorname{Spec}(\Gamma_*(X,\mathcal{L})_{\langle s\rangle})\subseteq\mathbb{P}(\Gamma_*(X,\mathcal{L}))$. We want

$$\begin{tikzcd} X \arrow[r, hook] & \mathbb{P}(\Gamma_*(X,\mathcal{L})) \\ X_s \arrow[u, hook] \arrow[r, hook] & D_+(s) \arrow[u, hook] \end{tikzcd}$$

Pick an s such that X_s is affine, and by assumption such X_s's cover X. By construction, $\mathcal{O}_X(X_s) = \Gamma_*(X,\mathcal{L})_{\langle s \rangle}$. Then $\psi_s\colon X_s \to D_+(s)$, then all these agree on overlaps. $\square$

Proposition 7.27.6. *If X is proper over $S = \operatorname{Spec} A$, with A noetherian, and if we have an invertible line bundle $\mathcal{L}$ on X such that for all coherent $\mathcal{F}$ on X, there exists $d(\mathcal{F})$ such that for all $d \geq d(\mathcal{F})$, we have $H^1(X, \mathcal{F}\otimes\mathcal{L}^d) = 0$, then $X \cong \operatorname{Proj}(\Gamma_*(X,\mathcal{L}))$, and for d sufficiently large, this is just $\psi^*\mathcal{O}(d)$.*

Proof. By the lemma above, ψ is an open immersion. We have

$$\begin{tikzcd} X \arrow[rr, hook, "\psi"] \arrow[dr, "\text{proper}"'] & & \operatorname{Proj}(\Gamma_*(X,\mathcal{L})) \arrow[dl, "\pi"] \\ & \operatorname{Spec} A & \end{tikzcd}$$

Since we are separated, $\operatorname{im}(\psi)$ is closed implies $\operatorname{Proj}(\Gamma_*(X,\mathcal{L})) = \psi(X)\coprod Y$. Suppose $Y \neq \emptyset$. Then $D_+(s) \subseteq Y$ for some s. Then $X_s = \emptyset$ implies s is a nilpotent. So $D_+(s) = D_+(s^N) = D_+(0) = \emptyset$, which is a contradiction. $\square$

Remark 7.27.7. Each $H^0(X,\mathcal{L}^d)$ is a finite A-module. But more importantly, $\Gamma_*(X,\mathcal{L})$ is a finitely generated A-algebra. But note the importance of this last result, eg. $\mathbb{C}[x,y]$ is finitely generated in each degree, but $\mathbb{C}[x, xy, xy^2, xy^3, \dots]$ is not finitely generated). $\triangle$

Summary. Suppose we have $f\colon X \to S = \operatorname{Spec} R$ which is proper, and R is noetherian. Let $\mathcal{L}$ be an invertible sheaf on X, and consider the following three conditions for all coherent sheaves on X

(a) $H^1(X, \mathcal{F}\otimes\mathcal{L}^n) = 0$ for n sufficiently large

(b) $\mathcal{F}\otimes\mathcal{L}^n$ is globally generated for n sufficiently large (ample in the sense of Hartshorne (Definition 7.27.2))

(c) For all $x \in X$, there is $s \in \Gamma(X,\mathcal{L}^n)$, for some $n \geq 1$ such that $x \in X_s$ and X_s is affine (ie. ample in the sense of EGA (Definition 7.27.1))

We showed that (a) implies (b) implies (c). In fact (c) implies (a), so these three are all equivalent, but this is much more involved to show. We showed that in this case, we have $X \cong \operatorname{Proj}(\Gamma_*(X,\mathcal{L}))$.

Proposition 7.27.8. *If $f\colon X \to S = \operatorname{Spec} R$ is proper, and R is Noetherian, and any of the (equivalent) conditions (a), (b), (c), then $\Gamma_*(X,\mathcal{L})$ is a finitely generated R-algebra.*

Warning. If X is a quasicompact scheme with $\mathcal{L}$ an invertible sheaf on X such that (c) is true, then:

(i) $X \subseteq \operatorname{Proj}(\Gamma_*(X,\mathcal{L}))$ is an open immersion, and

(ii) $\mathcal{L}$ is an ample invertible sheaf in the sense of EGA.

However, here $\Gamma_*(X,\mathcal{L})$ is not necessarily finitely generated. For example, one can take k to be a field, $X = \operatorname{Proj}(k[u,v,z_1,z_2,z_3,\dots]/I)$ where $I = \langle z_i^2 - u^{2i}\rangle$, and $\deg u = \deg v = 1$ and $\deg z_i = i$. Then:

1. $X = D_+(u) \cup D_+(v)$
2. $\mathcal{O}_X(1)$ is an invertible sheaf on X, and $\mathcal{O}_X(n) \cong \mathcal{O}_X(1)^{\otimes n}$
3. $\Gamma(X, \mathcal{O}_X(n)) = (k[x,y,z_i]/I)_n$ is the degree n part of $k[x,y,z_i]/I$

However, $k[u,v,z_1,z_2,\dots]$ is not finitely generated, and $X \to \operatorname{Spec} k$ is not proper and not of finite type.

Proof of Proposition 7.27.8. We first try to find a closed immersion $X \hookrightarrow \mathbb{P}^m_S$. To do this, choose $s \in \Gamma(X, \mathcal{L}^{d_i})$, with $i = 0, \ldots, n$ such that

$$X = \bigcup_{i=0}^{n} X_{s_i}.$$

The quasicompactness of X says we can choose this finite cover. Since X/S is of finite type, write $A_i = R[a_{i1}, \ldots, a_{in_i}/I$ with $A_i = \mathcal{O}_X(X_i) \cong \Gamma_*(X, \mathcal{L})_{\langle x_{ij} \rangle}$. Choose $s_{ij} \in \Gamma(X, \mathcal{L}^{e_j d_i})$ such that $s_i^{e_j} a_{ij}$ extends to a global section s_{ij} for $j = 1, \ldots, n_i$. Let $N = \operatorname{lcm}_{i,j}(d_i, e_{ij} d_i)$.

Consider $\varphi \colon \varphi_{\mathcal{L}^N} : X \to \mathbb{P}^m_S$ defined by $\langle s_0^{N/d_0}, \ldots, s_n^{N/d_n}, s_{ij} s_i^{N/d_i - e_{ij}}, \ldots \rangle$, where $m := n + \sum_i n_i$. Since $X_{s_i} = X_{s_i^{N/d_i}}$ for $i = 0, \ldots, n$ cover X, we get that φ is a morphism. It remains to show that φ is a closed immersion. Since X/S is proper, $\varphi(X)$ is closed. Let $\mathbb{P}^m_S = \operatorname{Proj}(R[T_i, T_{ij}])$, and note that $\varphi^{-1}(D_+(T_i)) = X_{s_i}$. These cover X.

So on the algebra side we have

$$R\left[\frac{T_0}{T_{i_0}}, \ldots, \frac{T_n}{T_{i_0}}, \frac{T_{ij}}{T_{i_0}}\right] \to \mathcal{O}_X(X_{i_0})$$

$$\frac{T_{i_0 j}}{T_{i_0}} \mapsto \frac{(s_{i_0 j} \ldots s_{i_0}^{\frac{N}{i_0} - e_{i_0 j}})}{s_{i_0}^{\frac{N}{i_0}}} = \frac{s_{i_0 j}}{s_{i_0}^{e_{i_0 j}}} = a_{i_0 j}$$

and so this is surjective, and hence φ is closed in $\bigcup_{i=0}^{n} D_+(T_i)$ and hence is an immersion. So there is a closed immersion such that $\iota^* \mathcal{O}_{\mathbb{P}^m_S(1)} \cong \mathcal{L}^{\otimes N}$ for some $N > 0$. Finally, we have

$$\begin{aligned}
\Gamma_*(X, \mathcal{L}) &= \bigoplus_{n \geq 0} \Gamma_*(X, \mathcal{L}^n) \\
&= \bigoplus_{n \geq 0} \left(\bigoplus_{n_i = 0}^{N} \Gamma(X, \mathcal{L}^{\otimes(n_i + nN)}) \right) \\
&= \bigoplus_{n \geq 0} \left(\bigoplus_{n_i = 1}^{N-1} \Gamma(\mathbb{P}^m_R, \iota_* \mathcal{L}^{\otimes(n_i + nN)}) \right) \\
&= \bigoplus_{n \geq 0} \left(\bigoplus_{n_i = 1}^{N-1} \Gamma(\mathbb{P}^m_R, \iota_*(\mathcal{L}^n \otimes \iota^* \mathcal{O}_{\mathbb{P}^m_R(n)})) \right) \\
&= \bigoplus_{n \geq 0} \left(\bigoplus_{n_i = 1}^{N-1} \Gamma(\mathbb{P}^m_R, \iota_* \mathcal{L}^n \otimes \mathcal{O}_{\mathbb{P}^m_R(n)}) \right) \\
&= \bigoplus_{n \geq 0} \left(\Gamma\left(\mathbb{P}^m_R, \iota_* \left(\bigoplus_{n_i = 0}^{N-1} \mathcal{L}^{n_i} \right) \otimes \mathcal{O}_{\mathbb{P}^m_R}(n) \right) \right)
\end{aligned}$$

So $\mathcal{F} = \iota_*(\oplus_{n_i=0}^{N-1} \mathcal{L}^{n_i})$ a coherent sheaf on $\mathbb{P}^m_R$. But for all coherent sheaves on $\mathbb{P}^m_R$ with R noetherian, then for all $k \in \mathbb{Z}$, $\oplus_{n \geq k} \Gamma(\mathbb{P}^m_R, \mathcal{F}(n))$ is a finite $R[T_0, \ldots, T_m]$-module. $\square$

7.28 May 7, 2019

[There is a revision assignment. There will be more details either by email or on the website. You will be able to redo old problems on homeworks and get credit. There is a pseudo-final. It will be a cumulative homework assignment where there will be material from each section from Hartshorne. This will also be an opportunity for extra credit. Again, there will be more details either by email or on the website. This will be done individually.]

Why is algebraic geometry so interesting?

There's a theory of moduli spaces.

One phenomenon that occurs in algebraic geometry that does not appear as much even in nearby fields (eg. in complex analytic geometry) is that moduli spaces of algebrogeometric objects is also an algebrogeometric object. Examples include classifying algebraic curves of genus g (ie. "$\mathcal{M}_g$"), or 1-dimensional subspaces of $\mathbb{R}^n$ (ie. $\mathbb{P}^n_{\mathbb{R}}$), or even finite sets. So the moduli space of compact Riemann surfaces (which are indeed algebraic curves) are nice, whereas allowing for noncompact ones is not so nice.

The question is to classify means of to have some notion of "equivalence" for such objects. How do we know what a "good" equivalence condition is?

Note also that in algebraic geometry, we also want to think of such objects in families, which usually means flat maps $X \to B$ to some base B.

Now the goal is that after picking objects, equivalence relations, and a notion of family, then the set of objects up to isomorphism is something nice (a variety, scheme, stack, etc.)

Example 7.28.1 (Quadruples of points in $\mathbb{P}^1$)**.** We want to classify (p_1, p_2, p_3, p_4) distinct points in $\mathbb{P}^1$. A family 4-tuples of points over a base B is a diagram

$$\begin{tikzcd} B \times \mathbb{P}^1 \arrow[d, "\pi"'] \\ B \arrow[u, bend right, "\sigma_1"'] \arrow[u, bend right=50] \arrow[u, bend right=70] \arrow[u, bend right=90, "\sigma_4"'] \end{tikzcd}$$

where the four sections are disjoint. So the moduli space of 4-tuples is

$$(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \setminus \{\text{diagonals of all } n\text{-tuples in the 4-tuple}\}$$

Then on points, $Q = \{(p_1, p_2, p_3, p_4)\}$. But what what about on families? For example, we have

$$\begin{tikzcd} \{(p_1, p_2, p_3, \lambda p_4) \colon \lambda \in k\} \arrow[d] \\ B = \mathbb{A}^1_k \end{tikzcd}$$

Observe that Q has a universal family (ie. a tautological family)

$$\begin{tikzcd} Q \times \mathbb{P}^1 \arrow[d] \\ Q \arrow[u, bend right] \arrow[u, bend right=50] \arrow[u, bend right=70] \arrow[u, bend right=90] \end{tikzcd}$$

with $\sigma_i(p_1, \dots, p_4) = p_i$. This is tautological. It is also universal, because for any family

$$
\begin{array}{ccc}
B \times \mathbb{P}^1 & \longrightarrow & Q \times \mathbb{P}^1 \\
\big\downarrow & & \big\downarrow \\
B & \xrightarrow{\varphi} & Q
\end{array}
$$

that is if we have section $x \mapsto (x, \sigma_i(x))$ and $\varphi\colon B' \to B$ then the pullback family along φ is $x \mapsto (x, \sigma_i(\varphi(x)))$, and we have $B \xrightarrow{\varphi} Q$ given by $b \mapsto (\sigma_1(b), \ldots, \sigma_4(b))$. The existence of this universal family is what allows our moduli to "vary in families", any moduli space with such a universal family is a fine moduli space. Another way of characterize is that families $X \to B$ correspond to homomorphisms $\varphi\colon B \to Q$. △

Unfortunately not everything has a fine moduli space. For example, GIT was invented because for many G-invariant problems in algebraic geometry, if we wanted the moduli space to be a scheme, then it is impossible to get a fine moduli space.

Here are first observations about moduli spaces.

Observation 7.28.2. We have that

- An object is just a family over $B = \{*\}$. Hence we have a bijection between objects (up to isomorphism) that we are classifying and our moduli space.
- The universal family is always tautological: if M is a moduli space, then $x \in M$ corresponds to C_x some object we are classifying. If $\iota\colon \{x\} \hookrightarrow M$ is the inclusion and $\pi\colon U \to M$ is some family over M, then $\iota^* U \cong C_x$ for some family $C_x \to \{*\}$. On the other hand, $\iota^* U = \pi^{-1}(x)$, and hence $C_x = \pi^{-1}(x)$ always.

△

Example 7.28.3. We look at 4-tuples in $\mathbb{P}^1$ but up to *projective equivalence*, not isomorphism. Here projective equivalence means that $p \sim q$ whenever there is an automorphism of $\mathbb{P}^1$ mapping p to q (recall that automorphisms of $\mathbb{P}^1$ are Möbius transformations $z \mapsto \frac{az+b}{cz+d}$, and any three given points are mapped to any tree given points by a unique automorphism).

Given a point $p = (p_1, p_2, p_3, p_4)$, we have a quantity $\lambda(p) \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ called the cross-ratio defined to be the image of p_4 under the (unique) automorphism that sends $p_1 \mapsto 0, p_2 \mapsto 1, p_3 \mapsto \infty$. Now $p \sim q$ if and only if $\lambda(p) = \lambda(q)$. Thus, we have a bijection

$$\mathcal{M}_{0,4} := \{(p_1, p_2, p_3, p_4)\}/\sim_{\text{proj}} \longleftrightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

Here $\mathcal{M}_{0,4}$ is the moduli space of genus 0 curves with 4 marked points.

Does there exist a universal form? The answer is yes: we have

$$
\begin{array}{c}
\mathcal{M}_{0,4} \times \mathbb{P}^1 \\
\big\downarrow \uparrow \sigma_1 \cdots \sigma_4 \\
\mathcal{M}_{0,4}
\end{array}
$$

given by $\sigma_1(p) = 0, \sigma_2(p) = 1, \sigma_3(p) = \infty, \sigma_4(p) = p_4$. Note that the fiber over p is a quadruple with cross ratio p (it's trickier to show, but it's true). △

Example 7.28.4. Here are familliar examples of other fine moduli spaces:

- $\mathbb{P}^n$ (more generally Grassmannians)
- $\mathbb{N}$, the moduli space of finite sets up to bijection.

△

However, fine moduli spaces don't always exist.

Example 7.28.5. Consider the moduli spaces of 1-dimensional vector spaces up to isomorphism. We expect the moduli space to be a point. Now suppose we have a family of 1-dimensional spaces, that is to say, a line bundle over a base B. There are two nonisomorphic line bundles on S^1 (namely the trivial one and the Möbius strip), and because $M = \{*\}$ these must induce the same map to M. △

The underlying problem is that the objects of the moduli problem have nontrivial isomorphisms. In this situation, a fine moduli space generally does not exist.

Historically, there are three methods to get a fine moduli space:

1. Modify the moduli problem to impose extra structure which kills the automorphisms (this is commonly done in number theory, where you impose extra symmetries, so you might have an abelian variety and impose that it maps a subgroup to a subgroup).

2. Maybe be okay without a universal family, and the bijection is enough (this is classic, so Riemann did this and calculated eg. the dimension of the space)

3. Look at the moduli stack instead of the space.

Approach 1. Let us try approach 1 above to our problematic Example 7.28.5. For one dimensional vector spaces up to isomorphism, we can fix the datum of $1 \in V$. If an isomorphisms has to send 1 to 1, then there are no nontrivial automorphisms. Families now are no longer line bundles; they are line bundles with a nowhere vanishing section. Then the Möbius strip is no longer a family, for example. In fact, every family over every base B is trivial, and $\{*\}$ is now a fine moduli space for our problem.

Usually, one might be able to fix datum that is natural to the object we are studying.

Approach 2. Now, moduli problems can be described as a contravariant functor

$$\begin{aligned} F\colon \mathbf{Sch} &\to \mathbf{Sets} \\ B &\mapsto F(B) = \text{families over } B/\simeq \\ \varphi &\mapsto F(\varphi) = \text{pullback map on families} \end{aligned}$$

and now a fine moduli space exists if and only if F is representable, so $F \simeq h_M = \operatorname{Hom}(-, M)$. This equivalence is given by Yoneda's lemma: the isomorhpism $U\colon h_M \to F$ can be viewed as an object in $F(M) = \{\text{families over } M\}/\simeq$ (ie. it is the tautological family). Note that for all schemes B,

$$\{\text{families over } B\}/\simeq \longleftrightarrow \{\text{morphisms } B \to M\}.$$

Definition 7.28.6. A coarse moduli space for F is a pair (M, V) where M is a scheme and $V\colon F \to h_M$ is a natural transformation (not necessarily an isomorphism) such that (M, V) is initial among all such pairs and on sets $V_{\{*\}}\colon F(\{*\}) \to \operatorname{Hom}(\{*\}, M)$ is a bijection. △

Thus points of M are in bijection with objects we're trying to classify.

Example 7.28.7. Recall the setup of Example 7.28.5. Now, $\{*\}$ is a coarse moduli space for 1-dimensional vector spaces up to equivalence. △

Approach 3. Harrison talked briefly about moduli stacks. An example of these (just a baby case) are orbifolds, which are like manifolds but with local spaces looking like $\mathbb{R}^n/G$ for a finite group G.

For X a manifold and G a finite group, X/G can be viewed as a groupoid (a category with all morphisms being isomorphisms). Then any orbifold can also be viewed as a groupoid by gluing the local pieces together. Using this groupoid perspective, we can obtain a fine moduli space.

Example 7.28.8. Consider the category $[X/G]$ where the objects correspond to elements of X and morphisms correspond to elements of $X \times G$. Given a scheme B, an orbifold morphism $B \to [X/G]$ is defined such that they're in bijection with pairs (p, ϕ) such that $p\colon P \to B$ is a principal G-bundle, whatever those are, and $\phi\colon P \to X$ is a G-equivariant map. △

Example 7.28.9. If we have $[*/G]$, then orbifold morphisms $B \to [*/G]$ are in bijection to with principal G-bundles over B, so $[*/G]$ is a classifying space (more precisely, a stack) for principal G-bundles. In particular, the moduli stack of 1-dimensional vector spaces over $\mathbb{C}$ is just $[*/\mathbb{C}^*]$, because a line bundle over B is the same as a principal $\mathbb{C}^*$-bundle, where we send $\mathcal{L} \mapsto \mathcal{L} \setminus \{0\text{-section}\}$. $\triangle$