Contents

Schramm-Loewner Evolution
(SLE)

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Introduction

This is the first expository set of notes on SLE I have written since publishing a book two years ago [**46**]. That book covers material from a year-long class, so I cannot cover everything there. However, these notes are not just a subset of those notes, because there is a slight change of perspective. The main differences are:

- I have defined SLE as a finite measure on paths that is not necessarily a probability measure. This seems more natural from the perspective of limits of lattice systems and seems to be more useful when extending SLE to non-simply connected domains. (However, I do not discuss non-simply connected domains in these notes.)
- I have made more use of the Girsanov theorem in studying corresponding martingales and local martingales.

As in [**46**], I will focus these notes on the continuous process SLE and will not prove any results about convergence of discrete processes to SLE. However, my first lecture will be about discrete processes — it is very hard to appreciate SLE if one does not understand what it is trying to model.

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LECTURE 1

Scaling limits of lattice models

The Schramm-Loewner evolution (SLE) is a measure on continuous curves that is a candidate for the scaling limit for discrete planar models in statistical physics. Although our lectures will focus on the continuum model, it is hard to understand SLE without knowing some of the discrete models that motivate it. In this lecture, I will introduce some of the discrete models. By assuming some kind of "conformal invariance" in the limit, we will arrive at some properties that we would like the continuum measure to satisfy.

1. Self-avoiding walk (SAW)

A self-avoiding walk (SAW) of length n in the integer lattice $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z}$ is a sequence of lattice points

$$\omega = [\omega_0, \ldots, \omega_n]$$

with $|\omega_j - \omega_{j-1}| = 1, j = 1, \ldots, n$, and $\omega_j \neq \omega_k$ for $j < k$. If J_n denotes the number of SAWs of length n with $\omega_0 = 0$, it is well known that

$$J_n \approx e^{\beta n}, \quad n \to \infty,$$

where e^β is the *connective constant* whose value is not known exactly. Here $\approx$ means that $\log J_n \sim \beta n$ where $f(m) \sim g(m)$ means $f(m)/g(m) \to 1$. In fact, it is believed that there is an exponent, usually denoted γ, such that

$$J_n \asymp n^{\gamma - 1} e^{\beta n}, \quad n \to \infty,$$

where $\asymp$ means that each side is bounded by a constant times the other. The exponent ν is defined roughly by saying that the typical diameter (with respect to the uniform probability measure on SAWs of length n with $\omega_0 = 0$) is of order n^ν. The constant β is special to the square lattice, but the exponents ν and γ are examples of lattice-independent *critical exponents* that should be observable in a "continuum limit". For example, we would expect the fractal dimension of the paths in the continuum limit to be $d = 1/\nu$.

To take a continuum limit we let $\delta > 0$ and

$$\omega^\delta(j\delta^d) = \delta\, \omega(j).$$

We can think of ω^δ as a SAW on the lattice $\delta\mathbb{Z}^2$ parametrized so that it goes a distance of order one in time of order one. We can use linear interpolation to make $\omega^\delta(t)$ a continuous curve. Consider the square in $\mathbb{C}$

$$D = \{x + iy : -1 < x < 1, -1 < y < 1\},$$

and let $z = -1, w = 1$. For each integer N we can consider a finite measure on continuous curves $\gamma : (0, t_\gamma) \to D$ with $\gamma(0+) = z, \gamma(t_\gamma) = w$ obtained as follows. To each SAW ω of length n in $\mathbb{Z}^2$ with $\omega_0 = -N, \omega_n = N$ and $\omega_1, \ldots, \omega_{n-1} \in ND$

we give measure $e^{-\beta n}$. If we identify ω with $\omega^{1/N}$ as above, this gives a measure on curves in D from z to w. The total mass of this measure is

$$Z_N(D;z,w) := \sum_{\omega: Nz \to Nw, \omega \subset ND} e^{-\beta|\omega|}.$$

It is conjectured that there is a b such that as $N \to \infty$,

$$Z_N(D;z,w) \sim C(D;z,w)\, N^{-2b}. \tag{1.1}$$

Moreover, if we multiply by N^{2b} and take a limit, then there is a measure $\mu_D(z,w)$ of total mass $C(D;z,w)$ supported on simple (non self-intersecting) curves from z to w in D. The dimension of these curves will be $d = 1/\nu$.

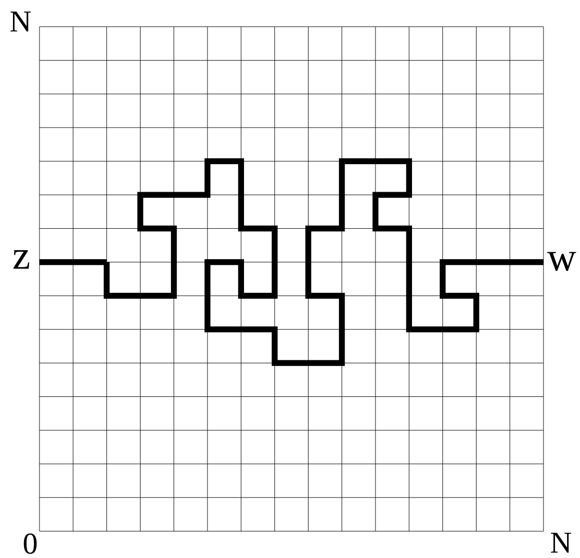

Figure 1. Self-avoiding walk in a domain

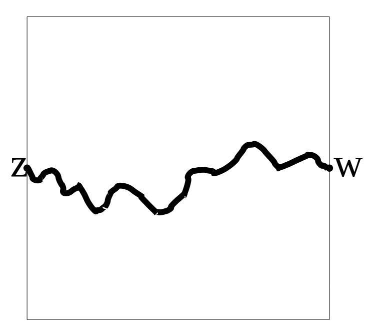

Figure 2. Scaling limit of SAW

Similarly, if D is another domain and $z, w \in \partial D$, we can consider SAWs from z to w in D. If ∂D is smooth at z, w, then (after taking care of the local lattice effects

— we will not worry about this here), we define the measure as above, multiply by N^{2b} and take a limit. We conjecture that we get a measure $\mu_D(z,w)$ on simple curves from z to w in D. We write the measure $\mu_D(z,w)$ as

$$\mu_D(z,w) = C(D;z,w)\,\mu_D^{\#}(z,w),$$

where $\mu_D^{\#}(z,w)$ denotes a probability measure.

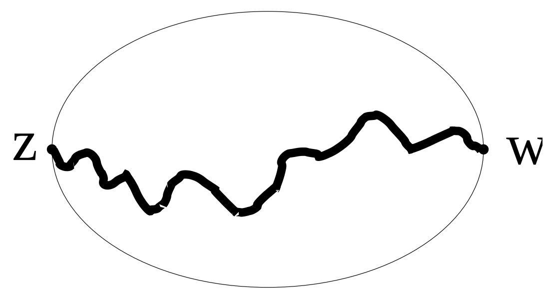

FIGURE 3. Scaling limit of SAW in a different domain

It is believed that the scaling limit satisfies some kind of "conformal invariance". To be more precise we assume the following conformal *covariance* property: if $f : D \to f(D)$ is a conformal transformation and f is differentiable in neighborhoods of $z, w \in \partial D$, then

$$f \circ \mu_D(z,w) = |f'(z)|^b\,|f'(w)|^b\,\mu_{f(D)}(f(z), f(w)).$$

In other words the total mass satisfies the scaling rule

$$C(D;z,w) = |f'(z)|^b\,|f'(w)|^b\,C(f(D); f(z), f(w)),$$

and the corresponding probability measures are conformally *invariant*:

$$f \circ \mu_D^{\#}(z,w) = \mu_{f(D)}^{\#}(f(z), f(w)).$$

Let us be a little more precise about the definition of $f \circ \mu_D^{\#}(z,w)$. Suppose $\gamma : (0, t_\gamma) \to D$ is a curve with $\gamma(0+) = z, \gamma(t_\gamma-) = w$. For ease, let us assume that γ is simple. Then the curve $f \circ \gamma$ is the corresponding curve from $f(z)$ to $f(w)$. At the moment, we have not specified the parametrization of $f \circ \gamma$. We will consider two possibilities:

- **Ignore the parametrization**. We consider two curves equivalent if one is an (increasing) reparametrization of the other. In this case we do not need to specify how we parametrize $f \circ \gamma$.
- **Scaling by the dimension** d**.** If γ has the parametrization as given in the limit, then the amount of time need for $f \circ \gamma$ to traverse $f(\gamma[t_1, t_2])$ is

$$\int_{t_1}^{t_2} |f'(\gamma(s))|^d\,ds.$$

In either case, if we start with the probability measure $\mu_D^{\#}(z,w)$, the transformation $\gamma \mapsto f \circ \gamma$ induces a probability measure which we call $f \circ \mu_D^{\#}(z,w)$.

Remark 1.1.

- Since $\mu^{\#}_D(z,w)$ is a conformal *invariant*, we can define $f \circ \mu^{\#}_D(z,w)$ even if ∂D is not smooth at z, w. (For really bad conformal tranformations, one needs to worry about the continuity of $f \circ \gamma$ at $f(z)$ and $f(w)$, but we do not need to consider transformations that are *that* bad!)
- The Riemann mapping theorem tells us that if D, D_1 are simply connected domains; z, w distinct points in ∂D; z_1, w_1 distinct points in ∂D_1, then there is a one-parameter family of conformal transformations $f : D \to D_1$ with $f(z) = z_1, f(w) = w_1$. In particular, if we know the measure for one simply connected domain D, we know it for all simply connected domains.
- In particular, if we know $\mu^{\#}_{\mathbb{H}}(0,\infty)$, then we know $\mu_D(z,w)$ for all simply connected domains. Here $\mathbb{H}$ denotes the upper half plane.
- The measure $\mu^{\#}_{\mathbb{H}}(0,\infty)$ must be invariant under the dilation $z \mapsto rz$ ($r > 0$).
- Although the map $f : D \to D_1$ with $f(z) = z_1, f(w) = w_1$ is not uniquely defined the quantity $f'(z)\, f'(w)$ is independent of the choice.
- One can choose a unique such f with $|f'(w)| = 1$ in which case the conformal covariance condition becomes

  $$f \circ \mu_D(z,w) = |f'(z)|^b \, \mu_{f(D)}(f(z), f(w)).$$

  When D is a subdomain of $\mathbb{H}$ with $\mathbb{H} \setminus D$ bounded and $w = f(w) = \infty$, then the condition $|f'(w)| = 1$ translates to $f(w') \sim w'$ as $w' \to \infty$.

There are two more properties that we would expect the family of measures $\mu_D(z,w)$ to have. The first of these will be shared by all the examples in this section while the second will not. We just state the properties, and leave it to the reader to see why one would expect them in the limit.

- **Domain Markov property**. Consider the measure $\mu^{\#}_D(z,w)$ and suppose an initial segment of the curve $\gamma(0,t]$ is observed. Then the conditional distribution of the remainder of the curve given $\gamma(0,t]$ is the same as $\mu^{\#}_{D\setminus\gamma(0,t]}(\gamma(t), w)$.

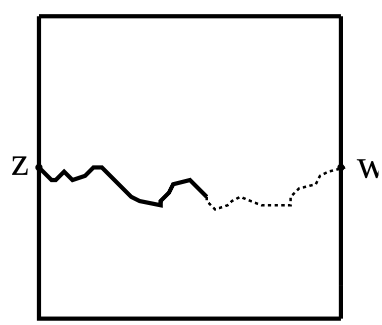

FIGURE 4. Domain Markov property

- **Restriction property**. Suppose $D_1 \subset D$. Then $\mu_{D_1}(z,w)$ is $\mu_D(z,w)$ restricted to paths that lie in D_1. In terms of Radon-Nikodym derivatives, this can be phrased as

  $$\frac{d\mu_{D_1}(z,w)}{d\mu_D(z,w)}(\gamma) = 1\{\gamma(0,t_\gamma) \subset D_1\}.$$

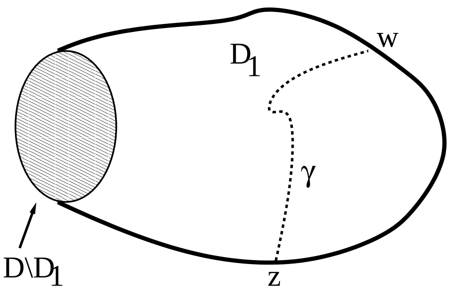

FIGURE 5. Illustrating the restriction property

We have considered the case where $z, w \in \partial D$. We could consider $z \in \partial D, w \in D$. In this case the measure is defined similarly, but (1.1) becomes

$$Z_D(z,w) \sim C(D; z, w)\, N^{-b}\, N^{-\tilde{b}},$$

where $\tilde{b}$ is a different exponent (see Lectures 5 and 6). The limiting measure $\mu_D(z,w)$ would satisfy the conformal covariance rule

$$f \circ \mu_D(z,w) = |f'(z)|^b\, |f'(w)|^{\tilde{b}}\, \mu_{f(D)}(f(z), f(w)).$$

Similarly we could consider $\mu_D(z,w)$ for $z, w \in D$.

2. Loop-erased random walk

We start with simple random walk. Let ω denote a nearest neighbor random walk from z to w in D. We no longer put in a self-avoidance constraint. We give each walk ω measure $4^{-|\omega|}$ which is the probability that the first n steps of an ordinary random walk in $\mathbb{Z}^2$ starting at z are ω. The total mass of this measure is the probability that a simple random walk starting at z immediately goes into the domain and then leaves the domain at w. Using the "gambler's ruin" estimate for one-dimensional random walk, one can show that the total mass of this measure decays like $O(N^{-2})$; in fact (Exercise 1.4)

$$Z_N(D; z, w) \sim C(D; z, w)\, N^{-2}, \quad N \to \infty, \tag{1.2}$$

where $C(D; z, w)$ is the "excursion Poisson kernel", $H_{\partial D}(z, w)$, defined to be the normal derivative of the Poisson kernel $H_D(\cdot, w)$ at z. In the notation of the previous section $b = 1$. For each realization of the walk, we produce a self-avoiding path by erasing the loops in chronological order.

Again we are looking for a continuum limit $\mu_D(z,w)$ with paths of dimension d (not the same d as for SAW). The limit should satisfy

- Conformal covariance
- Domain Markov property

However, we would not expect the limit to satisfy the restriction property. The reason is that the measure given to each self-avoiding walk ω is determined by the number of ordinary random walks which produce ω after loop erasure. If we make the domain smaller, then we lose some random walks that would produce ω and

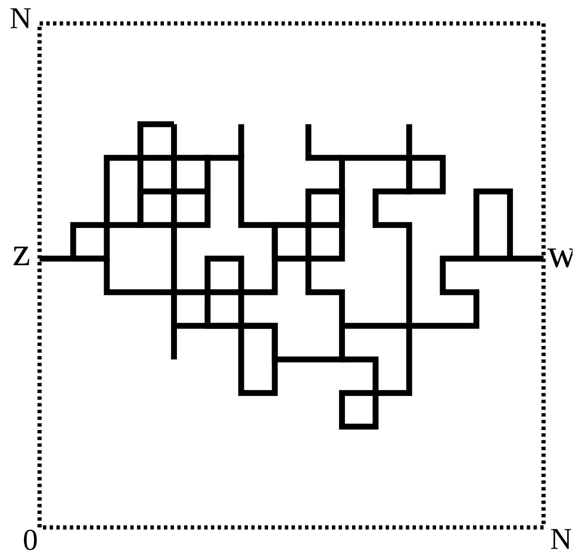

FIGURE 6. Simple random walk in D

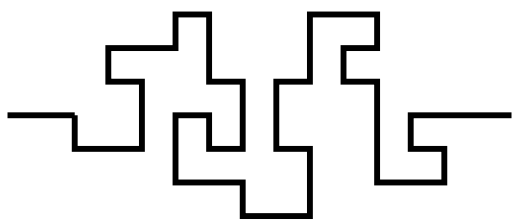

FIGURE 7. The walk obtained from erasing loops chronologically

hence the measure would be smaller. In terms of Radon-Nikodym derivatives, we would expect

$$\frac{d\mu_{D_1}(z,w)}{d\mu_D(z,w)} < 1.$$

3. Percolation

Suppose that every point in the triangular lattice in the upper half plane is colored black or white independently with each color having probability 1/2. A typical realization is illustrated in Figure 8 (if one ignores the bottom row).

We now put a boundary condition on the bottom row as illustrated — all black on one side of the origin and all white on the other side. For any realization of the coloring, there is a unique curve starting at the bottom row that has all white vertices on one side and all black vertices on the other side. This is called the *percolation exploration process*. Similarly we could start with a domain D and two boundary points z, w; give a boundary condition of black on one of the arcs and white on the other arc; put a fine triangular lattice inside D; color vertices black or white independently with probability 1/2 for each; and then consider the path

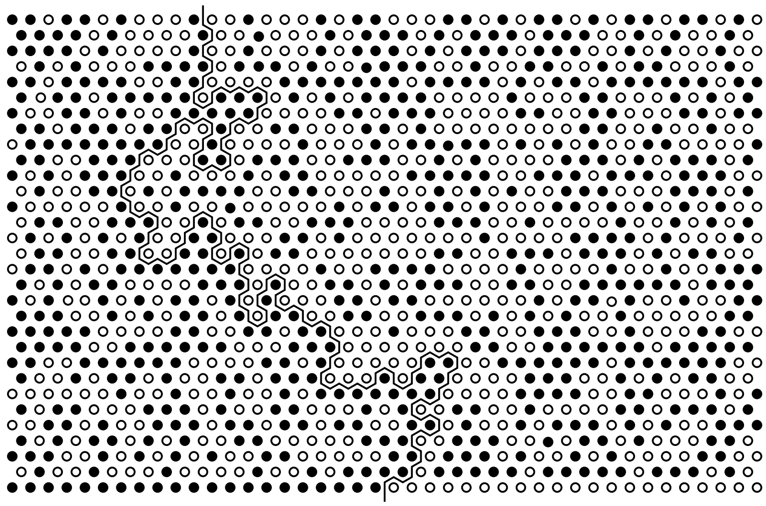

FIGURE 8. The percolation exploration process.

connecting z and w. In the limit, one might hope for a continuous interface. In comparison to the previous examples, the total mass of the lattice measures is one; another way of saying this is that $b = 0$. We suppose that the curve is conformally invariant, and one can check that it should satisfy the domain Markov property.

The scaling limit of percolation satisfies another property called the *locality property*. Suppose $D_1 \subset D$ and $z, w \in \partial D \cap \partial D_1$ as in Figure 5. Suppose that only an initial segment of γ is seen. To determine the measure of the initial segment, one only observes the value of the percolation cluster at vertices adjoining γ. Hence the measure of the path is the same whether it is considered as a curve in D_1 or a curve in D. The locality property is stronger than the restriction property which SAW satisfies. The restriction property is a similar statement that holds for the entire curve γ but not for all initial segments of γ.

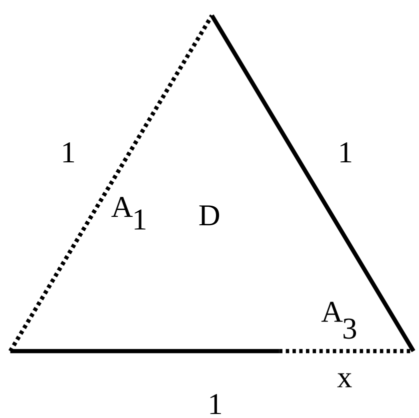

FIGURE 9. Cardy's formula: $P_D(A_1, A_3) = x$.

There is another well known conformal invariant for percolation known as Cardy's formula, named after the physicist who first predicted[1] the formula. Suppose D is a simply connected domain and the boundary is divided into four arcs, A_1, A_2, A_3, A_4 in counterclockwise order. Let $P_D(A_1, A_3)$ be the limit as the lattice spacing goes to zero of the probability that in a percolation cluster as above there is a connected cluster of white vertices connecting A_1 to A_3. This should be a conformal invariant. It turns out that the nicest domain to give the formula is an equilateral triangle as shown in the Figure 9.

4. Ising model

The Ising model is a simple model of a ferromagnet. We will consider the triangular lattice as in the previous section. Again we color the vertices black or white although we now think of the colors as spins. If x is a vertex, we let $\sigma(x) = 1$ if x is colored black and $\sigma(x) = -1$ if x is colored white. The measure on configurations is such that neighboring spins like to have the same sign.

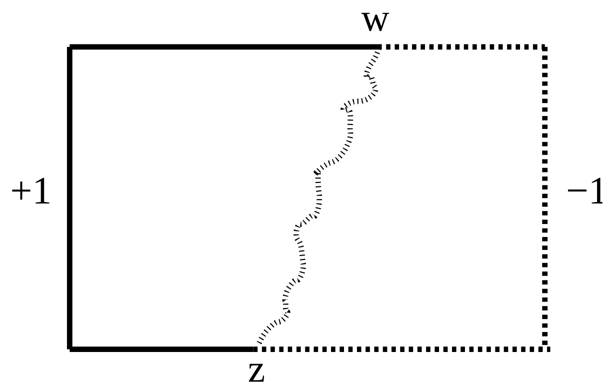

FIGURE 10. Ising interface.

It is easiest to define the measure for a finite collection of spins. Suppose D is a bounded domain in $\mathbb{C}$ with two marked boundary points z, w which give us two boundary arcs. We consider a fine triangular lattice in D; and fix boundary conditions $+1$ and -1 respectively on the two boundary arcs. Each configuration of spins is given energy

$$\mathcal{E} = -\sum_{x\sim y} \sigma(x)\,\sigma(y),$$

where $x \sim y$ means that x, y are nearest neighbors. We then give measure $e^{-\beta\mathcal{E}}$ to a configuration of spins. If β is small, then the correlations are localized and spins separated by a large distance are almost independent. If β is large, there is long-range correlation. There is a critical β_c that separates these two phases.

For each configuration of spins there is a well-defined boundary between $+1$ spins and -1 spins defined in exactly the same way as the percolation exploration

[1]The word "predicted" here means that the formula was found from a nontrivial, but not mathematically rigorous, argument. Much of the work by theoretical physicists using conformal field theory falls into this category. Although nonrigorous, the ideas are deep and involve a number of different areas of mathematics.

process. At the critical β_c it is believed that this gives an interesting fractal curve and that it should satisfy conformal covariance and the domain Markov property.

5. Assumptions on limits

Our goal is to understand the possible continuum limits for these discrete models. We will discuss the boundary to boundary case here but one can also have boundary to interior or interior to interior. (The terms "surface" and "bulk" are often used for boundary and interior.) Such a limit is a measure $\mu_D(z,w)$ on curves from z to w in D which can be written

$$\mu_D(z,w) = C(D;z,w)\,\mu_D^{\#}(z,w),$$

where $\mu_D^{\#}(z,w)$ is a probability measure. The existence of $\mu_D(z,w)$ assumes smoothness of ∂D near z,w, but the probability measure $\mu_D^{\#}(z,w)$ exists even without the smoothness assumption. The two basic assumptions are:

- Conformal covariance of $\mu_D(z,w)$ and conformal invariance of $\mu_D^{\#}(z,w)$.
- Domain Markov property.

The starting point for the Schramm-Loewner evolution is to show that *if we ignore the parametrization of the curves*, then there is only a one-parameter family of probability measures $\mu_D^{\#}(z,w)$ for *simply connected* domains D that satisfy conformal invariance and the domain Markov property. We will construct this family. The parameter is usually denoted $\kappa > 0$. By the Riemann mapping theorem, it suffices to construct the measure for one simply connected domain and the easiest is the upper half plane $\mathbb{H}$ with boundary points 0 and ∞. As we will see, there are a number of ways of parametrizing these curves.

6. Exercises for Lecture 1

Exercise 1.2. let J_n denote the number of SAWs of length n with $\omega_0 = 0$ in $\mathbb{Z}^2$.

(1) Show that there exists a β with $2 \le e^\beta \le 3$ such that

$$\lim_{n\to\infty} \frac{\log J_n}{n} = \beta.$$

(2) Prove that $2 < e^\beta < 3$.

Exercise 1.3. Suppose D is a simply connected domain and $f : D \to D$ is a conformal transformation and $z,w \in \partial D$. Suppose ∂D is smooth near z,w and $f(z) = z, f(w) = w$. Show that $f'(z)\,f'(w) = 1$.

Exercise 1.4. Let

$$V = V_N = \{j + ik \in \mathbb{Z} \times i\mathbb{Z} : 1 \le j \le N-1, 1 \le k \le N-1\}.$$

Let S_n denote a simple random walk starting at $j + ik \in V_N$ and let $\tau = \tau_N = \min\{n : S_n \notin V_N\}$. Let

$$H(j+ik, l+im) = \mathbf{P}^{j+ik}\{S_\tau = l + im\}.$$

We will compute this. Let $\partial V = \{z \in \mathbb{Z}^2 : \mathrm{dist}(z,V) = 1\}$ and $\overline{V} = V \cup \partial V$. Let $\mathcal{A}$ denote the set of real-valued functions f on $\overline{V}$ satisfying:

- f is discrete harmonic in V (i.e., the value of f is the average of f at its nearest neighbors.
- $f \equiv 0$ on $\partial V \setminus \{N + im : m = 1, \ldots, N-1\}$.

Show the following:

(1) Show that there exist $N-1$ linearly independent functions $f_1, \dots, f_{N-1} \in \mathcal{A}$ of the form

$$f_q(j+ik) = \sinh(r_q j)\,\sin(s_q k), \qquad q = 1, \dots, N-1.$$

(2) Show that every $f \in \mathcal{A}$ is a linear combination of $f_1, \dots, f_{N-1}$.

(3) For any $m = 1, \dots, N-1$, find the unique $f \in \mathcal{A}$ for which

$$f(N+im) = 1, \qquad f(N+im') = 0, m' \neq m.$$

Use this to justify (1.2) in Section 2 and show that for non-corner points z,

$$C(D; z, w) = H_{\partial D}(z, w) := \frac{d}{dn} H_D(z, w),$$

where d/dn denotes normal derivative and H_D is the Poisson kernel.

LECTURE 2

Conformal mapping and Loewner equation

1. Important results about conformal maps

Here we summarize the basic facts about conformal maps that one needs in order to use the Loewner equation effectively. A *domain* D will be a connected subset of $\mathbb{C}$. We will call a holomorphic function

$$f : D \longrightarrow D'$$

a *conformal transformation* if it is one-to-one and onto. This implies that $f'(z) \neq 0$ for all $z \in D$. A domain D is *simply connected* if $\mathbb{C} \setminus D$ is connected. We let $\mathbb{D} = \{|z| < 1\}$ denote the unit disk and $\mathbb{H} = \{x + iy : y > 0\}$ the upper half plane. The starting point is the Riemann mapping theorem.

Theorem 2.5 (Riemann mapping theorem)**.** *If D is a proper, simply connected domain in $\mathbb{C}$ and $z \in D$, then there is a unique conformal transformation*

$$f : \mathbb{D} \to D$$

with $f(0) = z, f'(0) > 0$.

In other words, there is a one-to-one correspondence between one-to-one, analytic functions f on $\mathbb{D}$ with $f(0) = 0, f'(0) > 0$ and simply connected proper subdomains of $\mathbb{C}$ containing the origin. The class of such functions with $f'(0) = 1$. is denoted $\mathcal{S}$. Classical function theory devoted much time to the study of $\mathcal{S}$. The high point was the proof by de Branges of the Bieberbach conjecture.

Theorem 2.6 (Bieberbach conjecture, de Branges theorem)**.** *If $f \in \mathcal{S}$ has power series expansion*

$$f(z) = z + \sum_{n=2}^{\infty} a_n\, z^n,$$

then $|a_n| \leq n$ for each n.

Although we do not need such a deep result, we do use some facts that were developed to try to solve the conjecture.

Theorem 2.7 (Koebe (1/4)-theorem)**.** *If $f \in \mathcal{S}$, then $(1/4)\,\mathbb{D} \subset f(\mathbb{D})$.*

Theorem 2.8 (Distortion theorem)**.** *If $f \in \mathcal{S}$, and $|z| \leq r < 1$, then*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}. \tag{2.1}$$

The particular values $1/4$ and those in (2.1) are not as important as the fact that there is some uniform bound over all $f \in \mathcal{S}$. These theorems are important for studying conformal maps even when the domains are not simply connected.

Suppose $f : D_1 \to D_2$ is a conformal transformation with $f(0) = 0$. Let $d_j = \mathrm{dist}(0, \partial D_j)$. If

$$\tilde{f}(z) = \frac{f(d_1 z)}{d_1 \, f'(0)},$$

then $\tilde{f} \in \mathcal{S}$. Therefore, $(1/4)\,\mathbb{D} \subset \tilde{f}(\mathbb{D})$ which implies $d_2 \geq |f'(0)| \, d_1/4$. By interchanging the roles of D_1, D_2, we get the corollary

$$\frac{|f'(0)|}{4} \leq \frac{d_2}{d_1} \leq 4\, |f'(0)|.$$

There is a similar result about harmonic functions that is simple but worth emphasizing. We will state it for the gradient but there are similar results for higher derivatives. These are just corollaries of the Poisson integral formula,

$$u(z) = \int_{\partial \mathbb{D}} u(w)\, H_{\mathbb{D}}(z,w)\, |dw|,$$

where

$$H_{\mathbb{D}}(z,w) = \frac{1}{2\pi} \, \frac{1 - |z|^2}{|z-w|^2}$$

is the Poisson kernel in $\mathbb{D}$ and $|dw|$ means integration with respect to arc length.

Proposition 2.9. *For every $r < 1$, there exists $c_r < \infty$ such that the following holds for all $|z| \leq r$.*

- *(Harnack inequality) If $u : \mathbb{D} \to (0, \infty)$ is harmonic, then*

$$c_r^{-1}\, u(0) \leq u(z) \leq c_r\, u(0).$$

- *(Derivative estimates) If $u : \mathbb{D} \to \mathbb{R}$ is harmonic, then*

$$|\nabla u(z)| \leq c_r \, \|u\|_\infty.$$

In particular, there is a c such that if $u : D \to \mathbb{R}$ is harmonic, then

$$|\nabla u(z)| \leq \frac{c}{\mathrm{dist}(z, \partial D)} \, \|u\|_\infty.$$

The next theorem is a corollary of a stronger theorem known as the Beurling projection theorem. However, the weaker version here is what is used most often in applications (and also has discrete analogues).

Theorem 2.10 (Beurling estimate)**.** *There is a $c < \infty$ such that the following is true. Suppose $\gamma : [0,1] \to \mathbb{C}$ is a continuous curve with $|\gamma(0)| = r < 1 = |\gamma(1)|$. Let B_t be a complex Brownian motion starting at the origin and let $\tau = \tau_{\mathbb{D}} = \inf\{t : |B_t| = 1\}$. Then,*

$$\mathbf{P}\{B[0,\tau] \cap \gamma[0,1] = \emptyset\} \leq c\, r^{1/2}.$$

Remark 2.11. The estimate is sharp when γ is a line segment from r to 1. See Exercise 2.27.

2. Half-plane capacity

If K is a bounded, relatively closed subset of $\mathbb{H}$, let $D = D_K = \mathbb{H} \setminus K$ and

$$\phi_D(z) = \mathrm{Im}(z) - \mathbf{E}^z[\mathrm{Im}(B_{\tau_D})],$$

where B_t is a standard complex Brownian motion and $\tau_D = \inf\{t : B_t \notin D\}$. Then ϕ_D is a positive harmonic function on D that vanishes on ∂D [1] and such that

$$\phi_D(z) = \mathrm{Im}(z) + O(|z|^{-1}), \quad z \to \infty.$$

The *half-plane capacity (from infinity)* of K is defined by

$$\phi_D(z) = \mathrm{Im}\left(z + \frac{\mathrm{hcap}(K)}{z}\right) + o(1), \quad z \to \infty,$$

or, in other words,

$$\mathrm{hcap}(K) = \lim_{z\to\infty} \mathrm{Im}(-1/z)\, \mathbf{E}^z[\mathrm{Im}(B_{\tau_D})].$$

The existence of the limit is included in the following lemma. Let $\mathbb{D}_+ = \mathbb{D} \cap \mathbb{H}$ denote the upper half disk.

Lemma 2.12.

- *If* $r > 0$,

$$\mathrm{hcap}(rK) = r^2\, \mathrm{hcap}(K), \qquad \mathrm{hcap}(r + K) = \mathrm{hcap}(K).$$

- *If* $K \subset \overline{\mathbb{D}_+}$, *then*

$$\mathrm{hcap}(K) = \int_0^\pi \mathbf{E}^{e^{i\theta}}[\mathrm{Im}(B_{\tau_D})] \left(\frac{2}{\pi} \sin\theta\right) d\theta. \tag{2.2}$$

Proof. (sketch) The scaling rule follows immediately from the scaling rule $\phi_D(z) = \phi_{rD}(rz)/r$, and the translation invariance by the rule $\phi_{D+r}(r+z) = \phi_D(z)$. The last equality follows by taking a Brownian motion starting at z and considering the hitting distribution of $\mathbb{R}\cup\overline{\mathbb{D}_+}$, restricted to the unit circle. Then it can be shown (Exercise 2.26) that as $z \to \infty$, the hitting density of the unit circle is given by

$$\frac{2}{\pi}\, \mathrm{Im}(-1/z)\, \sin\theta\, [1 + O(|z|^{-1})], \quad z \to \infty. \tag{2.3}$$

In other words,

$$\mathrm{Im}(-1/z)^{-1}\, \mathbf{E}^z[\mathrm{Im}(B_{\tau_D})] = \left[\int_0^\pi \mathbf{E}^{e^{i\theta}}[\mathrm{Im}(B_{\tau_D})] \left(\frac{2}{\pi}\sin\theta\right) d\theta\right] [1 + O(|z|^{-1})].$$

$\square$

Remark 2.13. From the lemma one can conclude immediately that $\mathrm{hcap}(\overline{\mathbb{D}_+}) = 1$. One can also see this by noting that the function $f(z) = z + z^{-1}$ maps $\mathbb{H} \setminus \overline{\mathbb{D}_+}$ conformally onto $\mathbb{H}$. If $K_1 \subset K$, then $\mathrm{hcap}(K_1) \le \mathrm{hcap}(K)$. In particular, we get the estimate

$$\mathrm{hcap}(K) \le \mathrm{rad}(K)^2,$$

[1] Actually we can only assert that it vanishes at the *regular* points of the boundary. We will not define regular here, but if all the connected components of ∂D are larger than singletons then all points on ∂D are regular.

where $\mathrm{rad}(K) = \sup\{|z| : z \in K\}$. There is no corresponding bound in the opposite direction even for simply connected D. In fact, one can check that there is a c such that for all K

$$\mathrm{hcap}(K) \leq c\,\mathrm{rad}(K)\, \sup\{\mathrm{Im}(z); z \in K\}.$$

Remark 2.14. The proof of (2.2) also gives an error estimate. There is a $c < \infty$ such that for all $K \subset \overline{\mathbb{D}_+}$, and $|z| \geq 2$,

$$\left| \mathrm{Im}(-1/z)^{-1}\, \mathbf{E}^z[\mathrm{Im}(B_{\tau_{D_K}})] - \mathrm{hcap}(K) \right| \leq \frac{c\,\mathrm{hcap}(K)}{|z|}. \tag{2.4}$$

By scaling, this implies that for any K and any $|z| \geq 2\mathrm{rad}(K)$,

$$\left| \mathrm{Im}(-1/z)^{-1}\, \mathbf{E}^z[\mathrm{Im}(B_{\tau_{D_K}})] - \mathrm{hcap}(K) \right| \leq \frac{c\,\mathrm{hcap}(K)\,\mathrm{rad}(K)}{|z|}. \tag{2.5}$$

Note that the error is of order $\mathrm{hcap}(K)\,\mathrm{rad}(K)$ rather than $\mathrm{hcap}(K)^2$. As mentioned above, $\mathrm{rad}(K)$ can be much larger than $\mathrm{hcap}(K)$.

Remark 2.15. In much of the literature on SLE, the half-plane capacity is called just the capacity and denoted cap. However, this can lead to confusion because there are other natural definitions of capacities of sets in $\mathbb{H}$.

If $D = \mathbb{H} \setminus K$ is simply connected, then ϕ_D is the imaginary part of a conformal transformation $g_D : D \to \mathbb{H}$

$$g_D = \mathrm{Re}[g_D] + i\phi_D.$$

We will also write this as g_K. This defines $\mathrm{Re}[g_D]$ up to an additive constant. We define g_D uniquely by specifying that the additive constant should be "0 at infinity", i.e., so that g_D has the expansion

$$g_D(z) = z + \frac{\mathrm{hcap}(K)}{z} + O(|z|^{-2}), \qquad z \to \infty.$$

There is a error estimate similar to (2.5),

$$\left| g_D(z) - \left(z - \frac{\mathrm{hcap}(K)}{z} \right) \right| \leq \frac{c\,\mathrm{hcap}(K)\,\mathrm{rad}(K)}{|z|^2}, \quad |z| \geq 2\mathrm{rad}(K). \tag{2.6}$$

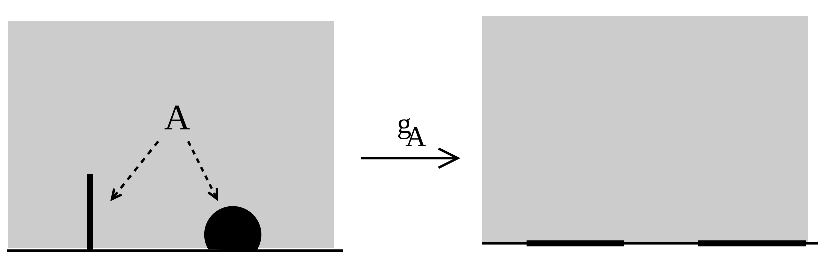

FIGURE 1. The conformal transformation $g_A = g_{\mathbb{H}\setminus A}$.

3. Loewner equation

The last inequality implies a version of the Loewner equation.

Proposition 2.16. *Suppose for each $t > 0$ there is a set K_t as above. Suppose $\dot{a}(0) = \partial_t[\mathrm{hcap}(K_t)]|_{t=0+}$ exists and $r_t := \mathrm{rad}(K_t) \to 0$ as $t \to 0+$. Let $\phi_t = \phi_{K_t}$. Then for fixed $z \in \mathbb{H}$, as $t \to 0+$,*

$$\phi_{K_t}(z) = \mathrm{Im}(z) + \dot{a}(0)\, t\, \mathrm{Im}(1/z) + O(tr_t),$$

i.e.,

$$\partial_t[\phi_t(z)]|_{t=0+} = \dot{a}(0)\,\mathrm{Im}(1/z).$$

If the domains $D_t = \mathbb{H} \setminus K_t$ are simply connected and $g_t = g_{K_t}$,

$$\partial_t[g_t(z)]|_{t=0} = \frac{\dot{a}(0)}{z}.$$

The last proposition is the basis for the following proposition which introduces the *(chordal) Loewner differential equation.* We will not give the details of the proof. One does need to prove that U_t is continuous; the Beurling estimate is a useful tool for this.

Proposition 2.17. *Suppose $\gamma : (0, T] \to \mathbb{H}$ is a simple curve with $\gamma(0+) := U_0 \in \mathbb{R}$. Let $a(t) = \mathrm{hcap}(\gamma(0, t]), g_t = g_{\gamma(0,t]}$, and suppose that a is C^1. Then $g_t(z)$ satisfies*

$$\dot{g}_t(z) = \frac{\dot{a}(t)}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{2.7}$$

where $U_t = g_t(\gamma(t))$. For $z \in \mathbb{H} \setminus \gamma(0, T]$, this is valid for $t \leq T$. For $z = \gamma(s)$, this is valid for $t < s$. The function $t \mapsto U_t$ is continuous.

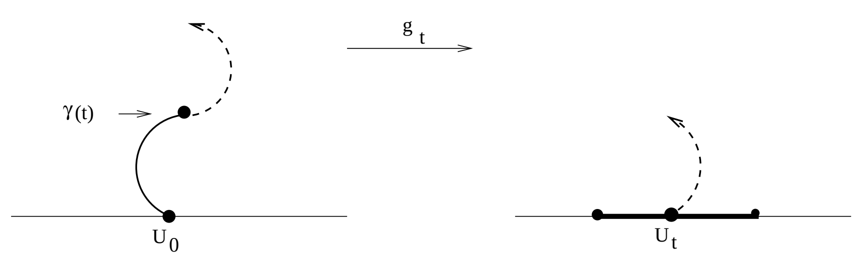

FIGURE 2. The conformal transformation g_t induced by γ .

In the last proposition we started with a curve and produced a function U_t. We will reverse the procedure here. Suppose $a : [0, \infty) \to (0, \infty)$ is a strictly increasing C^1 function, and $U : [0, \infty) \to \mathbb{R}$ is a continuous function. We will consider the (chordal) Loewner equation.

$$\dot{g}_t(z) = \frac{\dot{a}(t)}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{2.8}$$

For each $z \in \mathbb{C}\setminus\{0\}$, the solution of the equation above exists up to time $T_z \in (0, \infty]$. Using the continuity of U_t, one can see that for every $\epsilon > 0$ there is a $t > 0$ such that $T_z \geq t$ for $|z - U_0| \geq \epsilon$. Moreover, it can be shown that for fixed t, g_t is the conformal transformation of $H_t := \{z \in \mathbb{H} : T_z > t\}$ onto $\mathbb{H}$ with expansion at infinity

$$g_t(z) = z + \frac{a(t)}{z} + O(|z|^{-2}), \quad z \to \infty.$$

If $f_t(z) = g_t^{-1}(z)$, then f_t is a conformal transformation of $\mathbb{H}$ onto H_t. By differentiating both sides of $f_t(g_t(z)) = z$ with respect to t, we see that f_t satisfies

$$\dot{f}_t(z) = -\frac{\dot{a}(t)\, f_t'(z)}{z - U_t}, \quad f_t(0) = 0.$$

Here, and throughout these lectures, $'$ refers to spatial derivatives.

We have made no assumptions on U_t other than continuity. Since U_t is real-valued, $g_t(\overline{z}) = \overline{g_t(z)}$ and $T_{\overline{z}} = T_z$. Usually we consider the equation only for $z \in \overline{\mathbb{H}}$. If we write

$$g_t(z) = u_t(z) + i v_t(z),$$

then (2.8) becomes

$$\dot{u}_t(z) = \frac{\dot{a}(t)\,(u_t(z) - U_t)}{(u_t(z) - U_t)^2 + v_t(z)^2}, \quad \dot{v}_t(z) = -\frac{\dot{a}(t)\, v_t(z)}{(u_t(z) - U_t)^2 + v_t(z)^2}.$$

For fixed z, we will often write

$$Z_t = Z_t(z) = g_t(z) - U_t = X_t + iY_t,$$

$$X_t = X_t(z) = u_t(z) - U_t, \quad Y_t = Y_t(z) = v_t(z),$$

in which case we can write (2.8) as

$$\dot{u}_t(z) = \frac{\dot{a}(t)\, X_t}{X_t^2 + Y_t^2}, \quad \dot{v}_t(z) = \dot{Y}_t = -\frac{\dot{a}(t)\, Y_t}{X_t^2 + Y_t^2}. \tag{2.9}$$

Differentiating (2.8) with respect to z gives

$$\dot{g}_t'(z) = -\frac{\dot{a}(t)\, g_t'(z)}{Z_t^2}.$$

Since $g_0'(z) = 1$, we can solve this equation

$$g_t'(z) = \exp\left\{-\int_0^t \frac{\dot{a}(s)\, ds}{Z_s^2}\right\}, \tag{2.10}$$

$$|g_t'(z)| = \exp\left\{-\int_0^t \operatorname{Re}\left[\frac{\dot{a}(s)\, ds}{Z_s^2}\right]\right\} = \exp\left\{\int_0^t \frac{\dot{a}(s)\,(Y_s^2 - X_s^2)\, ds}{(X_s^2 + Y_s^2)^2}\right\} \tag{2.11}$$

The last equation can be rewritten as

$$\partial_t |g_t'(z)| = \dot{a}(t)\, |g_t'(z)|\, \frac{Y_t^2 - X_t^2}{(X_t^2 + Y_t^2)^2}. \tag{2.12}$$

4. Maps generated by a curve

Any continuous function U_t and C^1 function $a(t)$ produces the conformal maps g_t and hence the domains H_t. It is not always true that the domains H_t are obtained by slitting $\mathbb{H}$ with a curve γ. By a *curve*, we will mean only a continuous function from an interval in $\mathbb{R}$ into $\mathbb{C}$.

Definition 2.18. Let

$$H_t^{\text{pion}} := \bigcup_{0 \le s \le t} \partial H_s$$

denote the *pioneer points* of H_t. If there is a curve $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ with $\gamma(0) \in \mathbb{R}$ such that

$$H_t^{\text{pion}} = \mathbb{R} \cup \gamma(0,t],$$

we say that g_t is *generated by the curve* γ. (The term pioneer comes from the idea that a pioneer is someone who is on the frontier at some time.)

Note that H_t is the unbounded component of $\mathbb{H} \setminus H_t^{\text{pion}}$. Suppose that g_t is generated by a curve γ. If γ is simple with $\gamma(0,t] \subset \mathbb{H}$, then $H_t^{\text{pion}} = \mathbb{R} \cup \gamma(0,t]$, $H_t = \mathbb{H} \setminus \gamma(0,t]$. If γ is not simple, then it is possible for there to be points in $\mathbb{H} \setminus H_t$ that are not on $\gamma(0,t]$. Since $\partial H_t \subset \mathbb{R} \cup \gamma(0,t]$, we can see that

$$\text{hcap}(\gamma(0,t]) = \text{hcap}(\mathbb{H} \setminus H_t) = a(t).$$

In particular, for every $t, \epsilon > 0$,

$$\gamma(t, t+\epsilon] \cap H_t \neq \emptyset.$$

Also,

$$U_t = g_t(\gamma(t)) = \lim_{\epsilon \to 0+} g_t(\gamma(t+\epsilon)).$$

It is sometimes difficult to tell whether or not the maps are generated by a curve. The next proposition gives a criterion.

Proposition 2.19. *Suppose that $U_t : [0,1] \to \mathbb{R}$ is a continuous function and g_t is the solution to* (2.8) *with $a(t) = t$. Suppose that there exists $v : (0,1] \to (0,1]$ with $v(0+) = 0$ such that for all $0 \le t \le 1$ and all $\epsilon < 1$,*

$$\int_0^\epsilon |f_t'(U_t + iy)|\, dy \le v(\epsilon). \tag{2.13}$$

Then U_t is generated by a curve γ.

Proof. Using (2.13), we can see that the limit

$$\gamma(t) := \lim_{\epsilon \to 0+} f_t(U_t + i\epsilon), \tag{2.14}$$

exists and

$$|\gamma(t) - f_t(U_t + i\epsilon)| \le v(\epsilon), \qquad 0 < t, \epsilon \le 1. \tag{2.15}$$

For fixed $\epsilon > 0$, the function $t \mapsto f_t(U_t + i\epsilon)$ is continuous and hence there is a δ_ϵ such that

$$|f_t(U_t + i\epsilon) - f_s(U_s + i\epsilon)| \le v(\epsilon), \quad |s - t| \le \delta_\epsilon,$$

which implies

$$|\gamma(t) - \gamma(s)| \le 3v(\epsilon), \quad |s - t| \le \delta_\epsilon.$$

This shows that $t \mapsto \gamma(t)$ is a continuous function. The definition of γ shows that $\gamma(t) \in \partial H_t$ and hence H_t is contained in the unbounded component of $\mathbb{H} \setminus \gamma(0,t]$. It is not difficult to show, in fact, that these are equal.

$\square$

Remark 2.20. The uniform bound (2.13) is more than is needed show that the limit (2.14) exists, but it is used to prove the continuity of γ. There exist examples where the limit (2.14) exists for all t but for which γ is not a continuous function of t.

Remark 2.21. The distortion theorem tells us that

$$\int_0^{2^{-n}} |f_t'(U_t + iy)| \asymp \sum_{j=n}^{\infty} 2^{-j}\, |f_t'(U_t + i2^{-j})|.$$

A sufficient condition is to show that $|f_t'(U_t + iy)| \leq \phi(y)$ where ϕ satisfies

$$\sum_{j=1}^{\infty} 2^{-j}\, \phi(2^{-j}) < \infty.$$

5. A flow on conformal maps

The Loewner equation can be considered as a flow on the space of locally real conformal transformations at the origin. Suppose

$$F(z) = \sum_{n=0}^{\infty} \frac{q_n}{n!}\, z^n,$$

is a function analytic in a neighborhood $\mathcal{N} = \mathcal{N}_F$ of the origin with $q_1 > 0$ and $q_n \in \mathbb{R}$ for $n \geq 0$. Assume for ease that $U_0 = 0$. Let $K_t = \mathbb{H} \setminus H_t$. For t sufficiently small, $K_t \subset \mathcal{N}$ and hence we can define $K_t^* = F(K_t)$. Let $g_t^* = g_{K_t^*}$ and set $\psi_t(z) = (g_t^* \circ F \circ g_t^{-1})(z)$. We can write down the differential equation for ψ_t. Assume that $a(t) = t, \dot{a}(t) = 1$. Then the map g_t^* satisfies the equation

$$\dot{g}_t^*(z) = \frac{\psi_t'(U_t)^2}{g_t^*(z) - U_t^*},$$

where $U_t^* = \psi_t(U_t)$. The extra term $\psi_t'(U_t)^2$ arises from the scaling rule $\mathrm{hcap}(rK) = r^2 \mathrm{hcap}(K)$. Since $\psi_t(z) = g_t^* \circ F \circ f_t$, the chain rule gives

$$\dot{\psi}_t(z) = \frac{\psi_t'(U_t)^2}{\psi_t(z) - \psi_t(U_t^*)} - \frac{\psi_t'(z)}{z - U_t}.$$

In particular, since $U_0 = 0$,

$$\dot{\psi}_0(z) = (\Lambda F)(z) := \frac{F'(0)^2}{F(z) - F(0)} - \frac{F'(z)}{z}.$$

Note that ΛF is analytic in $\mathcal{N}$ with $\Lambda F(0) = -3F''(0)/2$. By differentiating this equation, we can find $\dot{\psi}_0^{(k)}(n)$ for all positive integers k.

Lemma 2.22. *If*

$$F(z) = \sum_{n=0}^{\infty} q_n\, x^n,$$

then

$$\Lambda F = -\frac{3q_2}{2} + \left(\frac{q_2^2}{4q_1} - \frac{2q_3}{3}\right) z + \left(\frac{q_2 q_3}{6q_1} - \frac{5q_4}{24} - \frac{q_2^3}{8q_1^2}\right) z^2 + \cdots . \tag{2.16}$$

PROOF. Since $\Lambda(rF + q_0) = r\Lambda F$, it suffices to prove the expansion for $q_0 = 0, q_1 = 1$, for which

$$\Lambda F(z) = \left[\sum_{n=1}^{\infty} \frac{q_n}{n!}\right]^{-1} - \frac{1}{z}\sum_{n=0}^{\infty} \frac{q_{n+1}}{n!} z^n.$$

We expand

$$\left[\sum_{n=1}^{\infty} \frac{q_n}{n!} z^n\right]^{-1} = \frac{1}{z}\left[1 - \left(\sum_{n\geq 2} \frac{q_n}{n!} z^{n-1}\right) + \left(\sum_{n\geq 2} \frac{q_n}{n!} z^{n-1}\right)^2 - \cdots\right],$$

$$\frac{1}{2}\left(\sum_{n\geq 2} \frac{q_n}{n!} z^{n-1}\right) = \frac{q_2}{2} + \frac{q_3}{6} z + \frac{q_4}{24} z^2 + \cdots ,$$

$$\frac{1}{z}\left(\sum_{n\geq 2} \frac{q_n}{n!} z^{n-1}\right)^2 = \frac{q_2^2}{4} z + \frac{q_2 q_3}{6} z^2 + \cdots ,$$

$$\frac{1}{z}\left(\sum_{n\geq 2} \frac{q_n}{n!} z^{n-1}\right)^3 = \frac{q_2^3}{8} z^2 + \cdots ,$$

which gives

$$\frac{1}{F(z)} = \frac{1}{z} - \frac{q_2}{2} + \left(\frac{q_2^2}{4} - \frac{q_3}{6}\right) z + \left(\frac{q_2 q_3}{6} - \frac{q_4}{24} - \frac{q_2^3}{8}\right) z^2 + \cdots \quad .$$

Also,

$$\frac{1}{z}\sum_{n=0}^{\infty} \frac{q_{n+1}}{n!} z^n = \frac{1}{z} + q_2 + \frac{q_3}{2} z + \frac{q_4}{6} z^2 + \cdots ,$$

giving

$$\Lambda F(z) = -\frac{3q_2}{2} + \left(\frac{q_2^2}{4} - \frac{2q_3}{3}\right) z + \left(\frac{q_2 q_3}{6} - \frac{5q_4}{24} - \frac{q_2^3}{8}\right) z^2 + \cdots \quad .$$

□

6. Doubly infinite time

If $U_t : (-\infty, \infty) \to \mathbb{C}$ is a continuous function, we can consider the solution g_t of the Loewner equation

$$\dot{g}_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z. \tag{2.17}$$

Let $\tilde{g}_t(z) = g_{-t}(z), \tilde{U}_t = U_{-t}$. Then $\tilde{g}_t, 0 \leq t < \infty$, satisfies

$$\dot{\tilde{g}}_t(z) = -\frac{a}{\tilde{g}_t(z) - \tilde{U}_t}, \quad \tilde{g}_0(z) = z.$$

Proposition 2.23. *For each $t \geq 0$, $\tilde{g}_t$ is a conformal transformation of $\mathbb{H}$ onto a subdomain $\tilde{H}_t = \tilde{g}_t(\mathbb{H})$ with* $\operatorname{hcap}(\mathbb{H} \setminus H_t) = at$ *satisfying*

$$\tilde{g}_t(z) = z + \frac{at}{z} + O(|z|^{-2}), \quad z \to \infty.$$

PROOF. For fixed $T > 0$, let $V_t = \tilde{U}_{T-t} - U_T$. If $z \in \mathbb{H}$, then $r_t(z) = \tilde{g}_{T-t}(z) - U_T$ satisfies

$$\dot{r}_t(z) = \frac{a}{r_t(z) - V_t}, \qquad r_0(z) = \tilde{g}_T(z) - U_T, \quad r_T(z) = z - U_T.$$

In other words, if we let $g_t(z) = \tilde{g}_{T-t}(\tilde{g}_T^{-1}(z + U_T)) - U_T$, then $g_t(z)$ satisfies

$$\dot{g}_t(z) = \frac{a}{g_t(z) - V_t}, \quad g_0(z) = z.$$

This is the usual Loewner equation. Note that

$$g_T(z) = \tilde{g}_0(\tilde{g}_T^{-1}(z + U_T)) - U_T = \tilde{g}_T^{-1}(z - V_T) + V_T.$$

In particular,

$$\tilde{g}_T'(z - V_T) = (g_T^{-1})'(z). \tag{2.18}$$

□

7. Distance to boundary

Suppose g_t satisfies (2.8), $H_t = \{z : T_z > t\}$, and $z \in \mathbb{H}$. Recall that $H_t^{\text{pion}} := \bigcup_{0 \leq s \leq t} \partial H_s$ denotes the pioneer points of H_t. If γ is generated by a curve γ, then $H_t^{\text{pion}} = \mathbb{R} \cup \gamma(0, t]$. In this section we consider

$$\operatorname{dist}\left[z, H_t^{\text{pion}}\right].$$

If the maps are generated by the curve γ, this is the same as $\operatorname{dist}[z, \gamma(0, t])$. If $t < T_z$, this is also the same as $\operatorname{dist}(z, \partial H_t)$. Therefore,

$$\operatorname{dist}\left[z, H_\infty^{\text{pion}}\right] = \lim_{t \to T_z -} \operatorname{dist}[z, \partial H_t].$$

This exact quantity is not as easy to study as a closely related quantity. For $t < T_z$, we define

$$Y_t = Y_{t,z} = \operatorname{Im}[g_t(z)], \quad \Upsilon_t = \Upsilon_{t,z} = \frac{Y_t}{|g_t'(z)|}. \tag{2.19}$$

Using (2.9) and (2.11) we see that

$$\dot{\Upsilon}_t = -\Upsilon_t \, \frac{2\,\dot{a}(t)\, Y_t^2}{(X_t^2 + Y_t^2)^2}, \tag{2.20}$$

$$\Upsilon_t = \operatorname{Im}[z] \exp\left\{-2 \int_0^t \frac{\dot{a}(s)\, Y_s^2 \, ds}{(X_s^2 + Y_s^2)^2}\right\}.$$

In particular, Υ_t is decreasing in t, so we can define

$$\Upsilon_{\infty,z} = \Upsilon_{T_z -, z} = \exp\left\{-2 \int_0^{T_z} \frac{\dot{a}(s)\, Y_s^2 \, ds}{(X_s^2 + Y_s^2)^2}\right\}, \quad t \geq T_z.$$

The following lemma is an immediate corollary of the Koebe-(1/4) theorem.

Lemma 2.24. *Under the assumptions above, if $t < T_z$,*

$$\frac{\Upsilon_{t,z}}{4} \leq \operatorname{dist}(z, H_t^{\text{pion}}) \leq 4\,\Upsilon_{t,z}. \tag{2.21}$$

Hence,

$$\frac{\Upsilon_{\infty,z}}{4} \leq \operatorname{dist}(z, H_\infty^{\text{pion}}) \leq 4\,\Upsilon_{\infty,z}.$$

The quantity Υ_t is sometimes called the *conformal radius.* Note that (2.20) can be rewritten as

$$\partial_t \Upsilon_t = -2\dot{a}(t)\,\Upsilon_t\,[\pi\, H_{\mathbb{H}}(0, Z_t)]^2,$$

where H denotes the Poisson kernel.

8. Exercises for Lecture 2

Exercise 2.25. Let

$$D = \{x + iy : 0 < x < \infty, 0 < y < \pi\}$$

be a half-infinite rectangle. Use separation of variables, as outlined below, to find $H_D(x + iy, iy')$ for $x > 0, 0 < y, y' < \pi$.

(1) Find all functions of the form

$$\phi(x + iy) = \phi_1(x)\,\phi_2(y)$$

that are harmonic in D and vanish on the horizontal boundaries of D.

(2) Find the linear combination of these functions whose boundary value is the δ-function at iy'.

Exercise 2.26. Use Exercise 2.25 and conformal invariance to justify (2.3) and (2.4).

Exercise 2.27. Let B_t be a standard complex Brownian motion starting at the origin and $\tau = \tau_{\mathbb{D}} = \inf\{t : |B_t| = 1\}$. Find

$$\lim_{\epsilon \to 0} \epsilon^{-1/2}\,\mathbf{P}\{B[0,\tau] \cap [\epsilon, 1] = \emptyset\}.$$

Exercise 2.28. Let D, ϕ_D be as in the beginning of Section 2. Let B_t be a standard complex Brownian motion starting at $z \in D$. For each $R > 0$, let

$$\sigma_R = \inf\{t : B_t \notin D \text{ or } \operatorname{Im}[B_t] = R\}.$$

Show that

$$\lim_{R \to \infty} R\,\mathbf{P}\{\operatorname{Im}[B_{\sigma_R}] = R\} = \phi_D(z).$$

Conclude that if $x \in \mathbb{R}$ and $\operatorname{dist}(x, \mathbb{H} \setminus D) > 0$, then $\partial_y \phi_D(x)$ is the probability that a Brownian motion started at x conditioned to stay in $\mathbb{H}$ forever (i.e., a Brownian excursion) stays in D for all time.

Exercise 2.29. Find g_K for the following sets:

- $K = (0, yi]$
- K is the line segment from 0 to $e^{i\theta}$.

Exercise 2.30. Suppose g_t is the solution to (2.7) with $a(t) = at$. Fix $T > 0$ and let $V_t = U_{T-t} - U_T$. Suppose h_t is the solution to the reverse time Loewner equation

$$\dot{h}_t(z) = \frac{a}{V_t - h_t(z)}, \quad 0 \leq t \leq T.$$

Show that $h_T(z) = g_T^{-1}(z + U_T)$.

LECTURE 3

Schramm-Loewner evolution (SLE)

We now return to the problem of determining possible candidates for the scaling limit of discrete systems. We will focus on $\mu^{\#}_{D}(z,w)$, and we will not worry about the parametrization. We start by considering $\mu^{\#}_{\mathbb{H}}(0,\infty)$. If we parametrize the curve so that the half-plane capacity grows linearly, then we get conformal maps satisfying

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{3.1}$$

where U_t is now random. Conformal invariance and the domain Markov property translate into conditions on U_t. In fact, they require U_t to be continuous with stationary, independent increments. It is well known that this implies that U_t is a one-dimensional Brownian motion.

Since the process should be invariant under dilations of $\mathbb{H}$, we can see that $h_t(z) := r^{-1}\, g_{r^2 t}(rz)$ should have the same distribution as $g_t(z)$. Note that if g_t satisfies (3.1), then

$$\partial_t\, h_t(z) = \frac{a}{h_t(z) - U^*_t},$$

where $U^*_t = r^{-1}\, U_{r^2 t}$. If U_t is a Brownian motion, then U^*_t has the same distribution as U_t provided that the drift is zero. If the drift is nonzero, they do not have the same distribution.

We can choose the variance of U_t and we can choose the parameter a. A simple time change shows that, in fact, there is only one free parameter. As originally defined, the parameter a was chosen to be 2 and κ was used for the variance of the Brownian motion. Here, we choose the variance of the Brownian motion to be 1 and use a as the free parameter. Choosing $a = 2/\kappa$ gives SLE_κ.

1. Definition

Definition 3.31. The *chordal Schramm-Loewner evolution (from* 0 *to* ∞ *in* $\mathbb{H}$ *parametrized so that* $\mathrm{hcap}(\gamma(0,t]) = at$*)* with parameter $\kappa = 2/a$ is the solution of (3.1) where $U_t = -B_t$ is a standard one-dimensional Brownian motion with $B_0 = 0$. The (random) curve γ that generates the maps $\{g_t\}$ is called the SLE_κ curve.

It is not immediately obvious but has been proved that SLE_κ is generated by a curve.

If $z \in \mathbb{H}$ and we write $Z_t(z) = X_t + iY_t = g_t(z) - U_t$, then (2.9) gives

$$dX_t = \frac{aX_t}{X_t^2 + Y_t^2}\, dt + dB_t, \quad \partial_t Y_t = -\frac{aY_t}{X_t^2 + Y_t^2} = Y_t\, \frac{-aX_t^2 - aY_t^2}{(X_t^2 + Y_t^2)^2}. \tag{3.2}$$

We also let

$$\Upsilon_t = \Upsilon_t(z) = \frac{Y_t}{|g_t'(z)|}, \qquad R_t = R_t(z) = \frac{X_t}{Y_t},$$

$$\Theta_t = \Theta_t(z) = \arg(Z_t), \quad O_t = O_t(z) = (R_t^2+1) = [\sin^2\Theta_t]^{-1},$$

Recall that Υ_t is related to the distance between z and $\gamma(0,t]$. Using (2.12), we see that

$$\partial_t \Upsilon_t = -\Upsilon_t \, \frac{2a\, Y_t^2}{(X_t^2+Y_t^2)^2}.$$

Itô's formula gives

$$d\Theta_t = \frac{(1-2a)\, X_t\, Y_t}{(X_t^2+Y_t^2)^2}\, dt - \frac{Y_t}{X_t^2+Y_t^2}\, dB_t, \tag{3.3}$$

$$dO_t^r = O_t^r \left[\frac{[2r^2+(4a-1)r]\, X_t^2 + r\, Y_t^2}{(X_t^2+Y_t^2)^2}\, dt + \frac{2r\, X_t}{X_t^2+Y_t^2}\, dB_t \right].$$

It is worth remembering that Y_t, Υ_t are differential functions of t and the formulas for them are valid for any driving function U_t. However, X_t, R_t, O_t have non-trivial quadratic variation. If we let

$$N_t = \Upsilon_t^{-u/a}\, Y_t^{\theta/a}\, O_t^r,$$

then the product rule gives

$$dN_t = N_t \left[\frac{[2r^2+(4a-1)r-\theta]\, X_t^2 + [2u+r-\theta]\, Y_t^2}{(X_t^2+Y_t^2)^2}\, dt + \frac{2r\, X_t}{X_t^2+Y_t^2}\, dB_t \right].$$

If the dt term is zero, this is a local martingale. Hence we get the following.

Proposition 3.32. *If $r \in \mathbb{R}$ and*

$$u = u(r) = r^2 + (2a-1)\, r,$$

Then

$$M_t = M_t(z) = \Upsilon_t^{-\frac{u}{a}}\, Y_t^{\frac{2u+r}{a}}\, O_t^r,$$

is a local martingale satisfying

$$dM_t = \frac{2r X_t}{X_t^2+Y_t^2}\, M_t\, dB_t.$$

Example 3.33. $r = \frac{1}{2} - 2a, u = a - \frac{1}{4}$ gives the local martingale

$$M_t = \Upsilon_t^{d-2}\, O_t^{\frac{1}{2}-2a},$$

where

$$d = 1 + \frac{1}{4a} = 1 + \frac{\kappa}{8}.$$

Example 3.34. $r = -2a, u = 2a$ gives the local martingale

$$M_t = |g_t'(z)|^2\, O_t^{-2a}.$$

Example 3.35. Let

$$r = -b = \frac{1-3a}{2}, \quad u = a\tilde{b} = \frac{b\,(1-a)}{2}.$$

This gives the martingale

$$M_t = \Upsilon_t^{-\tilde{b}}\, Y_t^{-b}\, O_t^{-b} = |g_t'(z)|^{\tilde{b}}\, Y_t^{-\tilde{b}}\, [\pi H_{\mathbb{H}}(0, Z_t)]^b. \tag{3.4}$$

Here $H_{\mathbb{H}}$ denotes the Poisson kernel in $\mathbb{H}$,

$$H_{\mathbb{H}}(0, x+iy) = \frac{y}{\pi(x^2+y^2)}.$$

2. Phases

Recall that a curve is simple if it has no self-intersections.

Theorem 3.36. *If $\kappa \leq 4$, SLE_κ paths are simple. If $\kappa > 4$, SLE_κ paths have self-intersections. In fact, if $\kappa > 4$ for every $s < t$ there exist $s < t_1 < t_2 < t$ with $\gamma(t_1) = \gamma(t_2)$.*

Remark 3.37. To be more precise, the theorem states that the facts hold "with probability one". We will feel free to leave this phrase out of statements of theorems.

We will prove a related fact and leave the proof of the theorem as an exercise (Exercise 3.46).

Proposition 3.38. *If $\kappa \leq 4$, $\gamma(0,\infty) \subset \mathbb{H}$. If $\kappa > 4$, $\gamma(0,\infty) \cap \mathbb{R} \neq \emptyset$.*

Proof. Let $x > 0$. Then $\gamma(0,\infty) \cap [x,\infty) \neq \emptyset$ if and only if $T_x < \infty$. We will show that $\mathbf{P}\{T_x < \infty\}$ is 1 if $\kappa > 4$ and equals 0 if $\kappa \leq 4$. Let $Z_t = Z_t(x) = g_t(x) - U_t$. Then the Loewner equations tells us that

$$dZ_t = \frac{a}{Z_t}\, dt + dB_t, \quad Z_0 = x. \tag{3.5}$$

This is the Bessel equation, and it is well known that the probability that Z_t reaches the origin in finite time equals 1 or 0 depending on whether $a < 1/2$ or $a \geq 1/2$. $\square$

Remark 3.39. If W_t is a d-dimensional Brownian motion starting at $z \in \mathbb{R}^d \setminus \{0\}$, then $Z_t = |W_t|$ satisfies (3.5) with $a = \frac{d-1}{2}, x = |z|$, and B_t a standard Brownian motion.

Theorem 3.40. *If $\kappa \geq 8$, then $\gamma[0,\infty) = \overline{\mathbb{H}}$, i.e., γ is plane-filling. If $\kappa < 8$, then for each $z \in \mathbb{H}$, $\mathbf{P}\{z \in \gamma(0,\infty)\} = 0$.*

Proof. Suppose $\kappa = 2/a > 4$ and let $z \in \mathbb{H}$. By using the previous theorem and scaling we can see that $\mathbf{P}\{T_z < \infty\} = 1$. Suppose $\gamma(T_z) \neq z$. Then a straightforward argument (Exercise 3.47) shows that $\Theta_{T_z-} \in \{0,\pi\}$, where $\Theta_t = \Theta_t(z)$. It will be convenient to make the time change

$$\hat{X}_t = X_{\sigma(t)}, \quad \hat{Y}_t = Y_{\sigma(t)},$$

where

$$\partial_t \sigma(t) = X^2_{\sigma(t)} + Y^2_{\sigma(t)}.$$

$$\partial_t \hat{Y}_t = \partial_t Y_{\sigma(t)} = \partial_t \sigma(t)\, Y_{\sigma(t)}\, \frac{-a}{X^2_{\sigma(t)} + Y^2_{\sigma(t)}} = -a\, \hat{Y}_t,$$

i.e., $\hat{Y}_t = e^{-at}$. Under this time change $\log \hat{Y}_t$ is a deterministic linear function. Time T_z in the usual parametrization becomes time ∞ in the time change. Under this time change (3.3) becomes

$$d\hat{\Theta}_t = (\frac{1}{2} - a)\, \sin(2\hat{\Theta}_t)\, dt + \sin\hat{\Theta}_t\, d\hat{B}_t.$$

for a standard Brownian motion $\hat{B}_t$. We have reduced the problem to a question about a one-dimensional diffusion. One can show that if $a \leq 1/4$ it is *not* the case that $\sin\hat{\Theta}_t \to 0$ as $t \to \infty$. We will not prove this but let us sketch why this is true. If we do a time change, this equation looks like

$$d\tilde{\Theta}_t = (\frac{1}{2} - a)\, \frac{\sin(2\tilde{\Theta}_t)}{\sin^2\tilde{\Theta}_t}\, dt + d\tilde{B}_t.$$

For $\tilde{\Theta}_t$ near zero this looks like

$$d\tilde{\Theta}_t = \frac{(1-2a)}{\tilde{\Theta}_t}\, dt + d\tilde{B}_t.$$

by comparison with a Bessel equation, we see that to keep this from reaching the origin we need $1 - 2a \geq 1/2$ or $a \leq 1/4$.

This is not quite enough to prove the statement for $4 < \kappa < 8$. Since this follows from Theorem 3.41 in the next section, we will not bother to give the complete details here. $\square$

If $\kappa < 8$, then $\Theta_{T_z-} = \hat{\Theta}_\infty \in \{0, \pi\}$. Let $\phi(z) = \mathbf{P}\{\Theta_\infty = \pi\}$. Scaling shows that ϕ depends only on the angle. Since $\phi(\hat{\Theta}_t)$ is a martingale, we can use Itô's formula to conclude that $\phi(\theta)$ satisfies

$$2\,(1-2a)\,\phi'(\theta)\,\cos\theta\, + \phi''(\theta)\,\sin\theta = 0, \quad \phi(0) = 0, \phi(\pi) = 1.$$

3. Dimension of the path

In this section we will discuss the dimension of the path γ. We will not give all the details.

Theorem 3.41. *There exists c_* such that if $a > 1/4$, then for all $z \in \mathbb{H}$*

$$\mathbf{P}\{\Upsilon_\infty(z) \leq \delta\} \sim c_*\, G(z)\, \delta^{2-d}, \quad \delta \to 0,$$

where $d = 1 + \frac{1}{4a}$ and G is the "Green's function" for chordal $SLE_{2/a}$ defined by

$$G(y(x+i)) = y^{d-2}\,(x^2+1)^{\frac{1}{2}-2a}.$$

PROOF. By scaling it suffices to prove this result when $z = x + i$. We fix x and we consider the martingale from Example 3.33,

$$M_t = \Upsilon_t^{d-2}\, \sin^{4a-1}\Theta_t = |g_t'(z)|^{2-d}\, G(Z_t),$$

which satisfies

$$dM_t = \frac{(1-4a)\, X_t}{X_t^2 + Y_t^2}\, M_t\, dB_t.$$

One can check that with probability one $M_t \to 0$. Let $\tau_\delta = \inf\{t : \Upsilon_t \leq \delta\}$. For fixed δ,, $M_{t\wedge\tau_\delta}$ is a uniformly bounded martingale, and hence the optional sampling theorem gives

$$G(z) = M_0 = \lim_{t\to\infty} \mathbf{E}[M_{\tau_\delta\wedge t}] = \mathbf{E}[M_{\tau_\delta}; \tau_\delta < \infty] = \delta^{d-2}\, \mathbf{E}[\sin^{4a-1}\Theta_{\tau_\delta}; \tau_\delta < \infty].$$

With the aid of Girsanov (details of this argument are left as Exercise 3.49), we can show that there exists $0 < c_* < \infty$ such that

$$\mathbf{E}[\sin^{4a-1}\Theta_{\tau_\delta}; \tau_\delta < \infty] \sim [1/c_*]\, \mathbf{P}\{\tau_\delta < \infty\}.$$

$\square$

Remark 3.42. In fact it can be shown (but is more difficult) that the Hausdorff dimension of the path $\gamma(0,\infty)$ is $d = 1 + \frac{\kappa}{8}$ if $\kappa \leq 8$.

4. Cardy's formula

In this section, we derive a formula for SLE that corresponds to Cardy's formula for percolation as discussed in Lecture 1, Section 3. Suppose $\gamma(0,\infty)$ is an SLE_κ curve with $\kappa > 4$. Suppose $x, y > 0$. Then with probability one $T_x, T_{-y} < \infty$. We will consider

$$\phi(x,y) = \mathbf{P}\{T_{-y} > T_x\}.$$

Note that scaling implies that $\phi(x,y) = \phi(y/x)$ where $\phi(y) = \phi(1,y)$.

Proposition 3.43 (Cardy's formula)**.** *If $0 < a < 1/2$, then*

$$\phi(y) = \frac{\Gamma(2-4a)}{\Gamma(1-2a)^2} \int_0^{\frac{y}{y+1}} \frac{du}{u^{2a}\,(1-u)^{2a}}. \tag{3.6}$$

Proof. Let $X_t = g_t(1) - U_t, J_t = g_t(-y) - U_t$ and

$$Z_t = \frac{X_t}{X_t - J_t}.$$

Let $\tau = \inf\{t : Z_t \in \{0,1\}\}$ and

$$\psi(r) = \mathbf{P}\{Z_\tau = 0 \mid Z_0 = r\}.$$

Then $\phi(y) = \psi(\frac{1}{y+1})$. Since

$$dX_t = \frac{a}{X_t}\,dt + dB_t, \quad \partial_t[X_t - J_t] = \frac{a}{X_t} - \frac{a}{J_t},$$

we can see that

$$dZ_t = \frac{a}{(X_t - J_t)^2}\left[\frac{1}{Z_t} + \frac{1}{Z_t - 1}\right] dt + \frac{1}{X_t - J_t}\,dB_t.$$

We can do a random time change σ and see that $\tilde{Z}_t := Z_{\sigma(t)}$ satisfies

$$d\tilde{Z}_t = \left[\frac{a}{\tilde{Z}_t} - \frac{a}{1-\tilde{Z}_t}\right] dt + d\tilde{B}_t,$$

where $\tilde{B}_t$ is a standard Brownian motion. However, $\psi(\tilde{Z}_t)$ is a martingale. Using Itô's formula, we can see that this implies

$$\psi''(u) + 2a\left[\frac{1}{u} - \frac{1}{1-u}\right]\psi'(u) = 0.$$

Solving this equation with the boundary conditions $\psi(0) = 1, \psi(1) = 0$ gives (3.6). $\square$

To understand why this corresponds to Cardy's crossing formula for percolation, consider the percolation exploration process as in Lecture 1, Section 3. If we again put the boundary condition of black on the negative real axis and white on the positive real axis then the event $\{T_{-y} > T_1\}$ corresponds to the event that in the (scaled) percolation cluster there is a connected component of black sites connecting $[-y,0]$ to $[1,\infty)$. Hence the "crossing probability" for SLE_κ is given by

$$P_{\mathbb{H}}([-y,0],[1,\infty)) = \phi(y) = \frac{\Gamma(2-4a)}{\Gamma(1-2a)^2}\int_0^{\frac{y}{y+1}} \frac{du}{u^{2a}\,(1-u)^{2a}}.$$

In the case $\kappa = 6$, which we will see in the next lecture corresponds to percolation, this crossing probability becomes

$$P_{\mathbb{H}}([-y,0],[1,\infty)) = \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^{\frac{y}{y+1}} \frac{du}{u^{\frac{2}{3}}\,(1-u)^{\frac{2}{3}}}.$$

This may not appear like a very nice formula, but if we map $\mathbb{H}$ to an equilateral triangle, we get the simple formula in Figure 9.

5. Conformal images of SLE

If D is a simply connected domain and z, w are distinct points in ∂D, then the measure $\mu_D^{\#}(z,w)$ is defined to be the image of $\mu_{\mathbb{H}}^{\#}(0,\infty)$ under a conformal transformation $f : \mathbb{H} \to D$ with $f(0) = z, f(\infty) = w$. Here, and for the rest of these lectures, we are considering the probability measures $\mu_{\mathbb{H}}^{\#}$ as being defined on curves modulo reparametrization. The map f is not unique; however, any other such map f_1 can be written as $f_1(z) = f(rz)$ for some $r > 0$. The invariance of SLE_κ under scaling shows that $\mu_D^{\#}(z,w)$ is well defined.

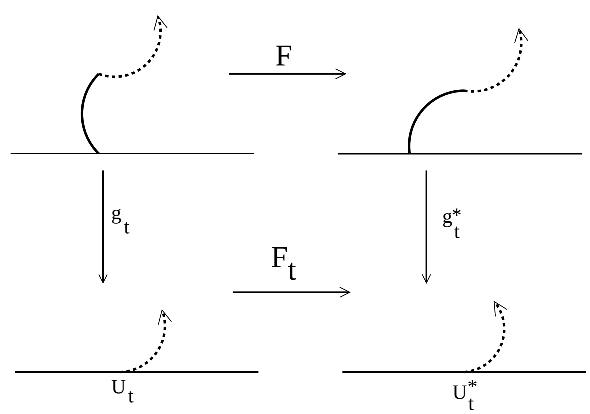

FIGURE 1. The maps g_t^*, F_t. Note that $F_t = g_t^* \circ F \circ g_t^{-1}$.

Here we will assume that

$$F(z) = \sum_{j=0}^{\infty} a_j\, z^j, \quad a_j \in \mathbb{R}, \quad a_1 > 0,$$

is analytic in a neighborhood of the origin. The assumptions on the a_j imply that near 0 F maps $\mathbb{R}$ into $\mathbb{R}$ and for some $\epsilon > 0$, $\epsilon\mathbb{D}_+ = \{z \in \mathbb{H} : |z| < \epsilon\}$ is mapped conformally into $\mathbb{H}$.

Suppose γ is an SLE_κ curve and let $\tau = \inf\{t : \gamma(t) \notin \epsilon\overline{\mathbb{D}_+}\}$. For $t < \tau$, we can define the curve

$$\gamma^*(t) = F(\gamma(t)).$$

We let H_t^* be the unbounded component of $\mathbb{H} \setminus \gamma^*(t)$ and g_t^* the unique conformal transformation of H_t^* onto $\mathbb{H}$ with

$$g_t^*(z) = z + \frac{a^*(t)}{z} + \cdots .$$

We also let $F_t = g_t^* \circ F \circ g_t^{-1}$; in other words, $F_t \circ g_t = g_t^* \circ F$. Using only properties of the Loewner equation, we can see that

$$\partial_t a^*(t) = a\, F_t'(U_t)^2. \tag{3.7}$$

The Loewner equation tells us that

$$\partial_t g_t^*(z) = \frac{a\, F_t'(U_t)^2}{g_t^*(z) - U_t^*},$$

where $U_t^* = g_t^*(\gamma^*(t)) = F_t(U_t)$.

Proposition 3.44. *Under a suitable time change, $\hat{U}_t = U^*_{\sigma(t)}$ satisfies*

$$d\hat{U}_t = b\, \frac{\Phi_t''(\hat{U}_t)}{\Phi_t'(\hat{U}_t)}\, dt + dW_t,$$

where W_t is a standard Brownian motion,

$$b = \frac{3a-1}{2} = \frac{6-\kappa}{2\kappa}$$

is the boundary scaling exponent, and $\Phi_t = F^{-1}_{\sigma(t)}$.

Proof. We use Itô's formula to get

$$dU_t^* = dF_t(U_t) = \left[\dot{F}_t(U_t) + \frac{1}{2} F_t''(U_t)\right] dt + F'(U_t)\, dU_t.$$

The term $\dot{F}_t(U_t)$ can be calculated using the Loewner equation. In fact, it is the first term in (2.16), i.e., $\dot{F}_t(U_t) = -(3a/2)\, F_t''(U_t)$ (the factor a appears because (2.16) assumes $a = 1$). Therefore,

$$dU_t^* = -b\, F_t''(U_t)\, dt + F_t'(U_t)\, dU_t.$$

With an appropriate time change we can write this as

$$d\hat{U}_t = -b\, \frac{F''_{\sigma(t)}(U_{\sigma(t)})}{F'_{\sigma(t)}(U_{\sigma(t)})^2}\, dt + dW_t.$$

A simple calculation shows that

$$-\frac{F''_{\sigma(t)}(U_{\sigma(t)})}{F'_{\sigma(t)}(U_{\sigma(t)})^2} = \frac{\Phi_t''(\hat{U}_t)}{\Phi_t'(\hat{U}_t)}.$$

□

Remark 3.45. The half-plane capacity is not preserved by F. The time change in the proof is exactly the time change needed so that $\text{hcap}(\gamma^*(0, \sigma(t)]) = at$.

6. Exercises for Lecture 3

Exercise 3.46. Show why Proposition 3.38 implies Theorem 3.36

Exercise 3.47. Verify the step in the proof of Theorem 3.40.

Exercise 3.48. Justify (3.7).

Exercise 3.49. The goal is to fill in the details to Theorem 3.41. Let M_t, τ_δ be as in the paragraph following the theorem and assume $z = x + i, \theta_0 = \arg(z)$.

- Show that if we weight by the local martingale M_t then
$$d\Theta_t = \frac{2a\, X_t\, Y_t}{(X_t^2+Y_t^2)^2}\, dt - \frac{Y_t}{X_t^2+Y_t^2}\, dW_t,$$
where W_t is a standard Brownian motion in the weighted measure.
- Show that if we do a random time change $\sigma(t)$ so that $\hat{\Upsilon}_t := \Upsilon_{\sigma(t)} = e^{-2at}$, then $\hat{\Theta}_t = \Theta_{\sigma(t)}$ satisfies

(3.8)
$$d\hat{\Theta}_t = 2a\, \cot \hat{\Theta}_t\, dt + d\hat{W}_t,$$

where $\hat{W}_t$ is a standard Brownian motion.
- Let
$$e(\theta, t) = \mathbf{E}[\sin^{4a-1}\, \hat{\Theta}_t \mid \hat{\Theta}_0 = \theta],$$
where $\hat{\Theta}_t$ satisfies (3.8). Use the Girsanov theorem to prove that
$$\mathbf{P}\{\Upsilon_\infty \le e^{-2at}\} = e(t, \theta_0)\, G(z)\, e^{-2at(2-d)}.$$
- Find a stationary density for (3.8) and use it to show that
$$\lim_{t\to\infty} e(t, \theta_0) = c_*$$
for some $c_* \in (0, \infty)$. Determine c_*.

LECTURE 4

SLE_κ in a simply connected domain D

Let $\mathcal{D}$ denote the set of simply connected subdomains D of $\mathbb{H}$ with $\mathbb{H} \setminus D$ bounded and $\mathrm{dist}(0, \mathbb{H} \setminus D) > 0$. We will write Φ_D for the unique conformal transformation of D onto $\mathbb{H}$ with $\Phi_D(z) = z + o(1)$ as $z \to \infty$ (we denoted this by g_D earlier, but it will be convenient to use a new notation). We will consider SLE_κ from 0 to ∞ in D and show that there are a number of equivalent ways of defining this process. We already have one definition: the image of SLE_κ in $\mathbb{H}$ under the conformal transformation $F_D := \Phi_D^{-1}$. We note that (see Exercise 2.28) $\Phi_D'(0) \in (0, 1]$; in fact $\Phi_D'(0)$ is the probability that a Brownian "excursion" in $\mathbb{H}$ stays in D.

1. Drift and locality

Proposition 3.44 can be restated in the following way. Suppose γ is a curve and let g_t be the corresponding conformal maps. Let

$$\tau = \tau_D = \inf\{t : \mathrm{dist}(\gamma(t), \mathbb{H} \setminus D) = 0\}.$$

Proposition 4.50. *Suppose for $t < \tau$, g_t is the solution to the Loewner equation* (3.1) *where U_t satisfies the stochastic differential equation*

$$dU_t = b\,[\log \Phi_t'(U_t)]'\,dt - dB_t.$$

Here $\Phi_t = \Phi_{D_t}$ where $D_t = g_t(D)$. Then g_t (and the corresponding curves γ) have the distribution of SLE_κ in D stopped at time τ.

In other words, the proposition gives an equivalent definition of SLE_κ in D, at least up to time τ_D. The drift is nontrivial unless $b = 0$ which holds if and only if $a = 1/3, \kappa = 6$.

Theorem 4.51 (Locality)**.** *Suppose γ is a curve with the distribution of SLE_6 in the domain $D \in \mathcal{D}$. The distribution of γ up to time τ_D is the same as that of SLE_κ in $\mathbb{H}$ up to time τ_D.*

If we consider our discrete models, we can see that the only one for which we would definitely expect the locality property is the percolation exploration process. As we saw in the last lecture, the crossing probability for SLE_6 is the same as that predicted by Cardy for percolation.

For other values of κ, we see that SLE_κ in D is obtained (at least for small time) by putting a drift in the Brownian motion. The Girsanov theorem tells us that Brownian motion with drift is absolutely continuous with respect to Brownian motion without drift, at least for small times.

2. Girsanov

The Girsanov theorem is a very useful tool for studying the (local) martingales arising in SLE_κ so we will discuss what we need here. Suppose that M_t is a nonnegative continuous local martingale satisfying

$$dM_t = J_t\, M_t\, dB_t, \qquad M_0 = 1. \tag{4.1}$$

Although M_t may not be a martingale, we can approximate M_t by a uniformly bounded martingale. Indeed, if $\tau_n = \inf\{t : M_t \geq n\}$ and $M_{t,n} = M_{t\wedge\tau_n}$, then for fixed n, $M_{t,n}$ is a uniformly bounded martingale satisfying

$$dM_{t,n} = J_t\, 1\{\tau_n > t\}\, M_{t,n}\, dB_t,$$

and for fixed t, $M_{t,n} \to M_t$ as $n \to \infty$. For any nonnegative martingale N_t satisfying

$$dN_t = J_t\, N_t\, dB_t,$$

there is a measure Q defined by

$$Q(V) = \mathbf{E}[1_V\, N_t],$$

if V if $\mathcal{F}_t$-measurable. The Girsanov theorem states that under the measure Q,

$$W_t = B_t - \int_0^t J_s\, ds,$$

is a standard Brownian motion. In other words, B_t satisfies

$$dB_t = J_t\, dt + dW_t,$$

where W_t is a Brownian motion with respect to Q.

The Girsanov theorem requires that N_t be a martingale. If we only know that M_t is a local martingale satisfying (4.1), then we can still use the Girsanov theorem as long as we run the paths up to a stopping time τ_n. In this case we get the equation

$$dM_t = J_t\, M_t\, dB_t, \quad t < \tau_n.$$

If we weight by the (local) martingale M_t, we can say that B_t satisfies

$$dB_t = J_t\, dt + dW_t, \quad t < \tau_n.$$

One can often use this equation to determine whether or not M_t is actually a martingale. Essentially what keeps the local martingale from being a martingale is the fact that some mass "goes to infinity in finite time". One can check that this happens by time t if and only if

$$\lim_{n\to\infty} \mathbf{E}[M_{t,n}; \tau_n < t] > 0,$$

i.e., if and only if in the *weighted* measure, there is a positive probability of explosion of M_t in finite time.

Remark 4.52. Although we will not prove the Girsanov theorem here, let us give a heuristic reason why it is true. Imagine that $\Delta B_t := B_{t+\Delta t} - B_t$ were equal to $\sqrt{\Delta t}$ with probability $1/2$ and $-\sqrt{\Delta t}$ with probability $1/2$. Then $M_{t+\Delta t}$ is about $(1+J_t\sqrt{\Delta t})M_t$ with probability $1/2$ and $(1-J_t\sqrt{\Delta t})M_t$ with probability $1/2$. If we weight by $M_{t+\Delta t}$, then in the weighted measure ΔB_t equals $\sqrt{\Delta t}$ with probability $(1 + J_t\sqrt{\Delta t})/2$ and equals $-\sqrt{\Delta t}$ with probability $(1 - J_t\sqrt{\Delta t})/2$. Under this measure

$$\mathbf{E}[\Delta B_t] = J_t\, \Delta t.$$

In other words, by weighting by the martingale we obtain a drift of J_t.

3. The restriction martingale

Let D, Φ, Φ_t be as above. We will consider $\Phi_t'(U_t)^b$. In the next proposition, we use the fact that

$$\dot{\Phi}_t'(U_t) = a \left[\frac{\Phi_t''(U_t)^2}{4\,\Phi_t'(U_t)} - \frac{2\,\Phi_t'''(U_t)}{3}\right].$$

This is a property of the Loewner equation and can be seen as the second term in (2.16). With this, the following proposition is a straightforward Itô's formula calculation. The *Schwarzian derivative* of a function f, Sf, is defined by

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3\,f''(z)^2}{2\,f'(z)^2},$$

and the *central charge* $\mathbf{c}$ is defined by

$$\mathbf{c} = \frac{2b(3-4a)}{a} = \frac{(3\kappa-8)\,(6-\kappa)}{2\kappa}.$$

Proposition 4.53. *If U_t is $SLE_{2/a}$ and $D \in \mathcal{D}$, then for $t < \tau_D$,*

$$d[\Phi_t'(U_t)^b] = \Phi_t'(U_t)^b \left[\frac{a\mathbf{c}}{12}\, S\Phi_t(U_t)\, dt - b\,\frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\, dB_t\right].$$

If we let

$$M_t = \exp\left\{-\mathbf{c}\int_0^t \frac{aS\Phi_t(U_t)}{12}\, dt\right\}\, \Phi_t'(U_t)^b,$$

then M_t is a local martingale satisfying

$$dM_t = -b\,\frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\, M_t\, dB_t.$$

In particular, if $\kappa = 8/3$,

$$M_t = \Phi_t'(U_t)^{5/8},$$

is a martingale.

Then under the weighted measure B_t satisfies

$$dB_t = -b\,\frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\, dt + dW_t,$$

where W_t is a Brownian motion in the new measure, i.e., U_t satisfies

$$dU_t = b\,\frac{\Phi_t''(U_t)}{\Phi_t'(U_t)}\, dt - dW_t,$$

Fact. *SLE_κ weighted by $\Phi_t'(U_t)^b$ gives SLE_κ in the smaller domain.*

This fact is valid for all κ provided that t is sufficiently small. For the rest of this lecture we will consider $\kappa \leq 4$ for which the curves are simple. In Proposition 4.57 we will see that

$$S\Phi_t(U_t) \leq 0,$$

so $M_t \leq 1$ for $\kappa \leq 8/3$. This implies that M_t is a martingale. In fact, for all $\kappa \leq 4$, this is a martingale.

Theorem 4.54 (Restriction). *If $\kappa = 8/3$ and $D \in \mathcal{D}$, then*

$$\mathbf{P}\{\gamma(0,\infty) \subset D\} = \Phi_D'(0)^{5/8}.$$

Moreover,

$$\frac{d\mu_D(0,\infty)}{d\mu_{\mathbb{H}}(0,\infty)} = 1\{\gamma(0,\infty) \subset D\}.$$

Proof. We first need to observe (we omit the argument) that if γ is a fixed curve with $\gamma(t) \to \infty$ and $\gamma(0,\infty) \subset D$, then $\Phi_t'(U_t) \to 1$. For each $\epsilon > 0$, let $\rho_\epsilon = \inf\{t : \operatorname{dist}(\gamma(t), \mathbb{H} \setminus D) \le \epsilon\}$.

Since $M_{t,\epsilon} = \Phi_{t\wedge\rho_\epsilon}(U_{t\wedge\rho_\epsilon})^{5/8}$ is a uniformly bounded martingale,

$$\Phi_D'(0)^{5/8} = \mathbf{E}[M_0] = \mathbf{E}[M_{\infty,\epsilon}] = \mathbf{E}[M_\infty; \rho_\epsilon = \infty] + \mathbf{E}[M_{\rho_\epsilon}; \rho_\epsilon < \infty].$$

But we know that *if we weight by the martingale M_t*, the weighted paths have the distribution of $SLE_{8/3}$ in D which is the same as the conformal image of $SLE_{8/3}$ in $\mathbb{H}$ under Φ_D^{-1}. Since $\kappa \le 4$, we know that these paths stay in D and hence

$$\lim_{\epsilon\to 0+} \mathbf{E}[M_{\rho_\epsilon}; \rho_\epsilon < \infty] = 0,$$

and

$$\Phi_D'(0)^{5/8} = \lim_{\epsilon\to 0+} \mathbf{E}[M_\infty; \rho_\epsilon = \infty] = \mathbf{E}[1\{\gamma(0,\infty) \subset D\}] = \mathbf{P}\{\gamma(0,\infty) \subset D\}.$$

□

This generalizes to all $\kappa \le 4$.

Theorem 4.55 (Restriction). *If $\kappa = 2/a \le 4$ and $D \in \mathcal{D}$, then*

$$\frac{d\mu_D(0,\infty)}{d\mu_{\mathbb{H}}(0,\infty)} = M_\infty = 1\{\gamma(0,\infty) \subset D\} \exp\left\{-\mathbf{c}\int_0^\infty \frac{aS\Phi_t(U_t)}{12}\,dt\right\}.$$

For $\kappa \le 8/3$ ($\mathbf{c} \le 0$), the martingale M_t is uniformly bounded and the proof proceeds as the previous proof. For $8/3 < \kappa \le 4$, the martingale is not uniformly bounded. In the next section, we study M_∞.

4. (Brownian) boundary bubbles

The Brownian bubble measure $\nu_{\mathbb{H}}(x,x)$ is an infinite measure on curves $\gamma : (0,t_\gamma) \to \mathbb{H}$ with $\gamma(0+) = \gamma(t_\gamma-) = x$. There are a number of ways of deriving it. One way is to consider Brownian motion starting at $x + \epsilon i$ and conditioned to leave $\mathbb{H}$ at x. Instead of thinking of this as a probability measure, we consider it as a finite measure with total mass $\pi\, H_{\mathbb{H}}(x+\epsilon i, x) = 1/\epsilon$. Here $H_{\mathbb{H}}$ denotes the Poisson kernel. (Recall that $H_{\mathbb{H}}(x+\epsilon i, x) = (\pi\epsilon)^{-1}$.) We define the Brownian bubble measure at x by

$$\nu_{\mathbb{H}}(x) = \lim_{\epsilon\to 0+} \pi\,\nu(x+\epsilon i, x) = \lim_{\epsilon\to 0+} \frac{1}{\epsilon}\,\nu^\#(x+\epsilon i, x),$$

where $\nu^\#(x+\epsilon i, x)$ is the probability measure corresponding to Brownian motion starting at $x + \epsilon i$ conditioned to leave $\mathbb{H}$ at x. If D is a subdomain of $\mathbb{H}$, then we let $\Gamma(x,D)$ be the $\nu_{\mathbb{H}}(x,x)$ measure of all γ with $\gamma(0,t_\gamma) \notin D$. If $\operatorname{dist}(x, \mathbb{H}\setminus D) > 0$, then $\Gamma(x,D)$ is finite. We let $\Gamma(D) = \Gamma(0,D)$.

The quantity $\Gamma(x,D)$ is expressed nicely in terms of excursion measure. If D is a domain and z, w are distinct points in ∂D at which the boundary is smooth, the *excursion Poisson kernel* (sometimes called the Dirichlet to Neumann map) is defined by

$$H_{\partial D}(z,w) = \lim_{\epsilon \to 0+} \epsilon^{-1}\, H_D(z+\epsilon \mathbf{n}, w),$$

where $\mathbf{n}$ denotes the inward unit normal at z. If $f : D \to D'$ is a conformal transformation and the boundary is sufficiently smooth, we have the scaling rule

$$H_{\partial D}(z,w) = |f'(z)|\,|f'(w)|\, H_{\partial D'}(f(z), f(w)).$$

Note that this is the scaling rule from Lecture 1 with $b = 1$. If $D_1 \subset D$ and the boundaries are nice, we define

$$\Gamma_D(z, D_1) = \int_{\partial D_1} H_{\partial D_1}(z,w)\, H_D(w,z)\, |dw|.$$

We could also write this as the integral over $D \cap \partial D_1$. In this notation $\Gamma(x,D) = \Gamma_{\mathbb{H}}(x,D)$. The definition of $\Gamma_D(z,D_1)$ does not need smoothness of $D \cap \partial D_1$ — the same integral can be expressed in terms of Brownian excursion measure. Using the scaling rule for the (regular and excursion) Poisson kernel, we get the following conformal covariance rule

$$\Gamma_D(z, D_1) = |f'(z)|^2\, \Gamma_{f(D)}(f(z), f(D_1)).$$

Example 4.56. Suppose $D = \mathbb{D}_+ = \{z \in \mathbb{H} : |z| < 1\}$. As $\epsilon \to 0$,

$$H_D(\epsilon i, e^{i\theta}) \sim \frac{2\epsilon}{\pi}\, \sin\theta,$$

and hence

$$\Gamma(D) = \int_0^\pi H_{\partial \mathbb{D}_+}(0, e^{i\theta})\, H_{\mathbb{H}}(e^{i\theta}, 0) = \int_0^\pi \frac{2}{\pi}\, \sin^2\theta\, d\theta = 1.$$

The normalization of the bubble measure is chosen so that $\Gamma(\mathbb{D}_+) = 1$.

The next proposition (Exercise 4.68) relates Γ to the Schwarzian derivative.

Proposition 4.57. *Suppose $D \subset \mathbb{H}$ is a simply connected domain with* $\mathrm{dist}(0, \mathbb{H} \setminus D) > 0$. *Suppose $f : D \to \mathbb{H}$ is a conformal transformation. Then,*

$$\Gamma_{\mathbb{H}}(0, D) = -\frac{1}{6}\, Sf(0),$$

where S denotes Schwarzian derivative.

We can write the local martingale M_t from the previous section as

$$M_t = \left[\exp\left\{-\int_0^t a\, \Gamma(U_s, g_s(D))\, ds\right\}\right]^{-\mathbf{c}/2} \Phi_t'(U_t)^b,$$

and if $\kappa \le 4$,

$$M_\infty = 1\{\gamma(0,\infty) \subset D\}\, [e^{-\Theta}]^{-\mathbf{c}/2},$$

where

$$\Theta = \Theta(\gamma, D) = a \int_0^\infty \Gamma(U_t, g_t(D))\, dt.$$

Note that Θ is a deterministic function of γ and D. The factor a comes from the fact that γ has been parametrized so that $\mathrm{hcap}[\gamma(0,t] = at]$. The value of Θ does not depend on the paramertrization; we can write

$$\Theta = \Theta(\gamma, D) = \int_0^\infty \Gamma(U_t, g_t(D))\; d\mathrm{hcap}(\gamma(0,t]).$$

The very nice feature of Θ is conformal invariance. More generally if γ is a curve connecting boundary points in D and $D_1 \subset D$ we can define $\Theta_D(\gamma, D_1)$.

Proposition 4.58. *If $D_1 \subset D$; z, w distinct points on ∂D; and γ a simple curve from z, w in D, then*

$$\Theta_D(\gamma; D_1) = \Theta_{f(D)}(f \circ \gamma; f(D_1)).$$

Remark 4.59. Because Θ is a conformal *invariant*, we can see that we no longer need to assume that ∂D is smooth at z.

Remark 4.60. We do not need D or D_1 to be simply connected in order to define $\Gamma_D(z, D_1)$ or $\Theta_D(\gamma; D_1)$. However, Proposition 4.57 which relates the bubble measure to the Schwarzian derivative does assume that the domain is simply connected.

5. Brownian loop measure

A *(rooted) loop* in $\mathbb{C}$ is a continuous function $\gamma : [0, t_\gamma] \to \mathbb{C}$ with $\gamma(0) = \gamma(t_\gamma)$. There is a one-to-one correspondence between loops and ordered triples

$$(z, s, \eta),$$

where $z \in \mathbb{C}, s > 0$, and η is a loop with $t_\eta = 1$ and $\eta(0) = \eta(1) = 0$. The correspondence is by translation and Brownian scaling,

$$t_\gamma = s, \qquad \gamma(t) = z + s^{1/2}\, \eta(t/s), \quad 0 \leq s \leq t_\gamma.$$

An *unrooted loop* is an equivalence class of rooted loops under the equivalence $\tilde{\gamma} \sim \gamma$ if $\tilde{\gamma}(t) = \gamma(t + r)$ for some $r \in \mathbb{R}$ (here addition is modulo t_γ). In other words, an unrooted loop is a loop that forgets where its starting point is.

Definition 4.61.

- The *(rooted) Brownian loop measure* is the measure on loops given by putting the following measure on (z, s, η):

$$\text{area} \times \frac{dt}{2\pi t^2} \times \text{ Brownian bridge},$$

  where Brownian bridge refers to the probability measure on Brownian paths in $\mathbb{R}^2$ conditioned to return to the origin at time 1.
- The *(unrooted) Brownian loop measure* is the measure obtained from the rooted loop measure by forgetting the roots.

Remark 4.62. Recall that the density for a two-dimensional Brownian motion at time t is $(2\pi t)^{-1}\, e^{-|z|^2/2t}$. Roughly speaking, we can think of $(2\pi t)^{-1}$ as the probability that a Brownian motion starting at the origin is at the origin at time t. We think of the unrooted loop measure as giving measure about $(2\pi t)^{-1}$ to each unrooted loop of length t. The rooted loop measure then chooses a root for the loop by using the uniform distribution on $[0, t]$; this gives the extra factor of t^{-1} in the definition. These heuristics can be made precise by considering random walk approximations to the loop measure.

The unrooted loop measure turns out to be conformally invariant. In fact, $\Theta_D(\gamma, D_1)$ is the measure of unrooted loops in D that intersect both γ and D_1.

6. The measure $\mu_D(z,w)$ for $\kappa \leq 4$

The ideas in this lecture can be used to define

$$\mu_D(z,w) = C(D;z,w)\,\mu_D^{\#}(z,w),$$

for all simply connected domains D and distinct boundary points z,w at which ∂D is locally analytic. We allow $w=\infty$. In this case we say ∂D is locally analytic at ∞ if $\partial[f(D)]$ is locally analytic at 0 where $f(z')=1/z'$. Given any two such triples (D,z,w), (D_1,z_1,w_1) there is a unique conformal transformation, which we call the canonical transformation, $f:D\to D'$ with $f(z)=z_1, f(w)=w_1$ and $|f'(w)|=1$ with the appropriate interpretation of this if $w=\infty$ or $w_1=\infty$. (For example, if $w=\infty, w_1\neq\infty$, then as $z'\to\infty$, $|f(z')-w_1|\sim|z'|^{-1}$.) We define $C_{\mathbb{H}}(0,\infty)=1$ and $\mu_{\mathbb{H}}(0,\infty)=\mu_{\mathbb{H}}^{\#}(0,\infty)$ to be the probability measure given by SLE_κ. We then define

$$\mu_D(z,w) = |f'(0)|^{-b}\,[f\circ\mu_{\mathbb{H}}(0,\infty)],$$

where $f:\mathbb{H}\to D$ is the canonical transformation with $f(0)=z, f(\infty)=w$. In other words,

$$C(D;z,w)=|f'(0)|^{-b},\quad \mu_{\mathbb{H}}^{\#}(0,\infty)=f\circ\mu_{\mathbb{H}}^{\#}(0,\infty).$$

Example 4.63. The map $f(z)=z/(1-z)$ is the canonical transformation from $\mathbb{H}$ to $\mathbb{H}$ with $f(0)=0, f(1)=\infty$. Using this we see that $C(\mathbb{H};0,1)=1$. More generally,

$$C(\mathbb{H};x_1,x_2)=|x_2-x_1|^{-2b}.$$

Example 4.64. If $D\in\mathcal{D}$, then Φ_D^{-1} is a canonical transformation from $\mathbb{H}$ to D. Therefore,

$$C(D;x,\infty)=\Phi_D'(x)^b.$$

We get the following properties.

- **Conformal covariance**. If $f:D\to f(D)$ is a conformal transformation, then

$$f\circ\mu_D(z,w)=|f'(z)|^b\,|f'(w)|^b\,\mu_{f(D)}(f(z),f(w)) \tag{4.2}$$

(assuming sufficient smoothness at the boundary points).
- **Boundary perturbation**. If $D_1\subset D$, $\partial D,\partial D_1$ agree near boundary points z,w, then

$$\frac{d\mu_{D_1}(z,w)}{d\mu_D(z,w)}(\gamma)=1\{\gamma\subset D_1\}\,e^{-\mathbf{c}\Theta/2}, \tag{4.3}$$

where Θ is the measure of the set of Brownian loops in D that intersect both γ and $D\setminus D_1$.
- In particular, if $D_1\subset D$, $\partial D,\partial D_1$ agree near boundary points z,w, and $f:D\to f(D)$ is a conformal transformation, then

$$\frac{d\mu_{D_1}(z,w)}{d\mu_D(z,w)}=\frac{d\mu_{f(D_1)}(f(z),f(w))}{d\mu_{f(D)}(f(z),f(w))}.$$

- If $D_1\subset D,\partial D,\partial D_1$ agree near boundary points z,w, then the probability measure $\mu_{D_1}^{\#}(z,w)$ can be obtained from $\mu_D^{\#}(z,w)$ by "weighting paths locally" by $C(D_1;z,w)$, or equivalently by "weighting paths locally" by

$$\frac{d\mu_{D_1}(z,w)}{d\mu_D(z,w)}.$$

Remark 4.65. The last property is stated informally. A precise statement for $D = \mathbb{H}, z = 0, w = \infty$ was given in Section 3. For other D_1, D, we can first map D to $\mathbb{H}$ and use conformal invariance.

Remark 4.66. Because the quantity

$$\frac{d\mu_{D_1}(z,w)}{d\mu_D(z,w)}$$

is a conformal *invariant*, it is well defined even if ∂D is not smooth near z, w (but still assuming that $\partial D, \partial D_1$ agree in neighborhoods of z, w).

Remark 4.67. If $D_1, D_2 \in \mathcal{D}$ and $f : D_1 \to D_2$ is a conformal transformation with $f(\infty) = \infty$, we write $f'(\infty) = r$ if $f(z) \sim z/r$ as $z \to \infty$. In particular, if $f(z) = rz$, then $f'(\infty) = 1/r$. Using this convention, (4.2) holds for such transformations.

7. Exercises for Lecture 4

Exercise 4.68. Prove Proposition 4.57.

Exercise 4.69. Suppose $\gamma : (0, \infty) \to \mathbb{H}$ is a simple curve with $\gamma(0+) = 0, \gamma(z) \to \infty$ as $z \to \infty$. Let g_t be the corresponding conformal maps. Fix $\kappa \leq 4$. Show that if $0 < x < y$,

$$C(\mathbb{H} \setminus \gamma(0,t]; x, y) = \frac{g_t'(x)^b \, g_t'(y)^b}{[g_t(y) - g_t(x)]^{2b}},$$

and

$$C(\mathbb{H} \setminus \gamma(0,\infty); x, y) = \lim_{t \to \infty} \frac{g_t'(x)^b \, g_t'(y)^b}{[g_t(y) - g_t(x)]^{2b}}.$$

Exercise 4.70. Suppose $\kappa \leq 4$. Suppose D is a bounded, simply connected domain, and A_1, A_2 are disjoint, closed, analytic, subarcs of D larger than a single point. Define the measure

$$\mu_D(A_1, A_2) = \int_{A_1} \int_{A_2} \mu_D(z,w) \, |dw| \, |dz|,$$

where $\mu_D(z,w)$ denotes the SLE_κ measure and $|d\cdot|$ denotes integration with respect to arc length. We can write

$$\mu_D(A_1, A_2) = C_D(A_1, A_2) \, \mu_D^{\#}(A_1, A_2),$$

where

$$C_D(A_1, A_2) = \int_{A_1} \int_{A_2} C(D; z, w) \, |dw| \, |dz|,$$

is the total mass and $\mu_D^{\#}(A_1, A_2)$ is a probability measure. Suppose $f : D \to D_1$ is a conformal transformation with $f(A_1), f(A_2)$ being analytic arcs.

- Convince yourself that the integral makes sense (i.e., there is no trouble integrating a "measure-valued" function).
- Show that $0 < C_D(A_1, A_2) < \infty$.
- In the cases of self-avoiding walk, loop-erased walk, and Ising interface, describe what $\mu_D(A_1, A_2)$ represents in terms of limits of discrete models.

- Show that if $\kappa = 2$, then
$$f \circ \mu_D(A_1, A_2) = \mu_{D_1}(f(A_1), f(A_2)).$$
In particular,
$$f \circ \mu_D^{\#}(A_1, A_2) = \mu_{D_1}^{\#}(f(A_1), f(A_2)). \tag{4.4}$$
- Show that (4.4) does not necessarily hold if $\kappa \neq 2$.

LECTURE 5

Radial and two-sided radial SLE_κ

1. Example: SAW II

We return to the example of the self-avoiding walk (SAW) from Lecture 1. We let z be a boundary point and w be an *interior* point and consider the set of SAWs from z to w. We again give the measure $e^{-\beta|\omega|}$ to each walk where β is the critical value and we let $Z_N(D;z,w)$ be the partition function as before. It is conjectured that

$$Z_N(D;z,w) \sim C(D;z,w)\, N^{-b}\, N^{-\tilde{b}},$$

where now we have a *(one-sided) interior scaling exponent* $\tilde{b}$. If we multiply by $N^{b+\tilde{b}}$ and take a limit, we expect to have a measure on simple paths from z to w (or w to z). The model for the continuum limit of this is called *(one-sided) radial SLE*.

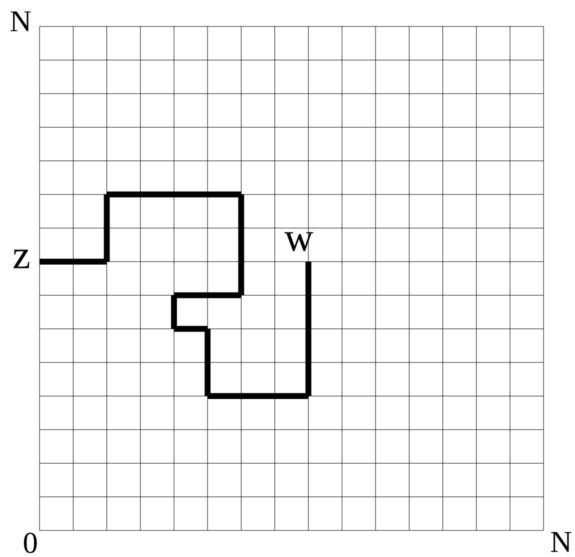

Figure 1. Discrete approximation of one-sided radial

We can also consider a case with two boundary points, z, y and one interior point w. We look at the set of all SAWs ω from z to y that go through w. Equivalently, we can look at the set of pairs of SAWs (ω, ω') where ω goes from w to z; ω' gives from w to y; and $\omega \cap \omega' = \{w\}$. In this case, we expect the partition function to scale like

$$Z_N(D;z,y;w) \sim C(D;z,y;w)\, N^{-2b}\, N^{-\hat{b}},$$

where $\hat{b}$ is a two-sided interior scaling exponent. By comparison to $Z_N(D; z, y)$, we can see that $N^{-\hat{b}}$ should be comparable to the probability that a SAW from z to y goes through w which we can conjecture to be about N^d/N^2 where d is the fractal dimension of the paths. We therefore get

$$\hat{b} = 2 - d.$$

Since this relation holds, we do not adopt the notation $\hat{b}$ but rather just refer to the exponent as $2 - d$.

Consider the marginal measure on ω. Then for any ω from w to z the measure is

$$Z_N(D \setminus \omega; y, w).$$

In other words we can first choose ω using the one-sided measure but then we weight this distribution by the measure of walks ω' from w to y that avoid ω. We will be able to look at the scaling limit of the measure on ω in two different ways: chordal SLE from z to y conditioned to go through w or radial SLE from z to w weighted by the total mass of paths from w to y in $D \setminus \omega$. (Both of these interpretations must be considered in some kind of limit.) If we fix ω and consider the probability measure on ω' obtained by conditioning, then this will be the same as the probability measure given by the normalized measure on $D \setminus \omega$. In the scaling limit, the probability measure on ω' given ω will be $\mu^{\#}_{D\setminus\omega}(w, y)$ (note that w is a boundary point of $D \setminus \omega$).

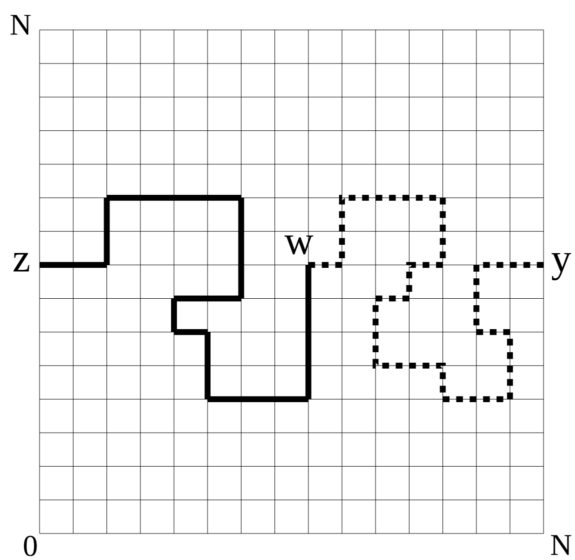

FIGURE 2. Discrete approximation of two-sided radial

Note that

$$\frac{Z_N(D; z, y; w)}{Z_N(D; z, w)\, Z_N(D; y, w)} \asymp N^{-(\hat{b} - 2\tilde{b})}.$$

The left-hand side represents the probability that two SAWs starting at the origin do not intersect. If we recall from Lecture 1 that we expect this probability to decay like $|\omega|^{-(\gamma-1)}$, then we see that we would expect this quantity to decay like $N^{-d(\gamma-1)}$. This gives the following relation between the exponents

$$d(\gamma-1) = \hat{b} - 2\tilde{b} = 2 - d - 2\tilde{b}, \qquad \gamma = 2\nu(1-\tilde{b}).$$

One can also consider the case for two interior points w, w' for which we would expect the scaling

$$Z_N(D; w, w') \sim C(D; w, w')\, N^{-2\tilde{b}}.$$

The corresponding probability measures are called *whole plane SLE*. We will not consider this case in these lectures.

2. Radial SLE_κ

Radial SLE_κ is a measure

$$\mu_D(z,w) = C_D(z,w)\, \mu_D^{\#}(z,w), \quad z \in \partial D, w \in D$$

on paths $\gamma : (0, t_\gamma] \to D$ with $\gamma(0+) = z$ and $\gamma(t_\gamma) = w$. We will consider paths modulo reparametrization, but we could also consider the paths with parametrization. It satisfies the following properties:

- **Domain Markov property (for $\mu_D^{\#}(z,w)$).** Given $\gamma(0,t]$, the distribution of the remainder of the path is the same as $\mu^{\#}_{D\setminus\gamma(0,t]}(\gamma(t), w)$
- **Conformal covariance**. If $f : D \to f(D)$ is a conformal transformation, then

$$C(D; z, w) = |f'(z)|^b\, |f'(w)|^{\tilde{b}}\, C(f(D); f(z), f(w)), \tag{5.1}$$

$$f \circ \mu_D^{\#}(z,w) = \mu^{\#}_{f(D)}(f(z), f(w)).$$

- **Conformal invariance for boundary perturbations (for $\kappa \le 4$).** If $w \in D_1 \subset D$; $z \in \partial D$; and $\partial D_1, \partial D$ agree near z; then $\mu_{D_1}(z,w)$ is absolutely continuous with respect to $\mu_D(z,w)$ and the Radon-Nikodym derivative

$$\frac{d\mu_{D_1}(z,w)}{d\mu_D(z,w)} \tag{5.2}$$

  is a conformal invariant.

Remark 5.71. Here we have introduced the interior scaling exponent

$$\tilde{b} = \frac{\kappa-2}{4}\, b.$$

We discuss below why this value is chosen — essentially, it is the value that makes the quantity on the right hand side of (5.3) a local martingale. The total mass $C(D; z, w)$ is nonzero and finite provided that ∂D is locally analytic at z. One does not need smoothness of the boundary at z to define the probability measures $\mu_D^{\#}(z,w)$ or the Radon-Nikodym derivative (5.2).

Remark 5.72. We will normalize so that $C(\mathbb{H}; 0, i) = 1$. Once we have done this, the scaling rule (5.1) determines $C(D; z, w)$ for all simply connected domains (assuming we have determined $\tilde{b}$). The domain Markov property and conformal invariance will determine $\mu^{\#}_{\mathbb{H}}(0,i)$ and from this we get $\mu_D^{\#}(z,w)$ for all simply connected D.

Example 5.73. We compute $C(\mathbb{H};0,z)$. Let $x>0$ and

$$f(z)=\frac{z(1+x^2)}{x(x-z)},\quad f'(z)=-\frac{1+x^2}{(z-x)^2},\quad f(i)=-\frac{1}{x}+i,$$

which is a conformal transformation of $\mathbb{H}$ onto $\mathbb{H}$ with $f(0)=0$. Note that

$$|f'(i)|=1,\quad |f'(0)|^{-1}=\frac{x^2}{x^2+1}=\pi\, H_{\mathbb{H}}\left(-\frac{1}{x}+i,0\right),$$

where $H_{\mathbb{H}}$ denotes the Poisson kernel. Hence we get

$$C(\mathbb{H};0,y(x+i))=y^{-\tilde b}\,y^{-b}\,\left[\pi\, H_{\mathbb{H}}\,(x+i,0)\right]^b=y^{-\tilde b}\,\left[\pi H_{\mathbb{H}}(y(x+i),0)\right]^b.$$

Example 5.74. We choose (somewhat arbitarily)

$$C(\mathbb{H};\infty,i)=1.$$

The map $F(z)=rz$ satisfies $F'(\infty)=1/r$, so we get

$$C(\mathbb{H};\infty,x+iy)=C(\mathbb{H};\infty,iy)=y^{b-\tilde b},$$

If $w\in D\in\mathcal{D}$, then $\Phi_D'(\infty)=1$ and

$$C(D;\infty,w)=|\Phi_D'(w)|^{\tilde b}\,C(\mathbb{H};\infty,\Phi_D(w)).$$

In particular, if γ is a path and $t<T_w$, then

$$C(\mathbb{H}\setminus\gamma(0,t];\infty,w)=|g_t'(w)|^{\tilde b}\,Y_t^{b-\tilde b}=\Upsilon_t^{-\tilde b}\,Y_t^b.$$

If g_t is chordal SLE_κ from 0 to ∞, then (see (3.4)),

$$M_t=|g_t'(z)|^{\tilde b}\,C(\mathbb{H};U_t,g_t(z))=|g_t'(z)|^{\tilde b}\,C(\mathbb{H};0,Z_t)=\Upsilon_t^{-\tilde b}\,\left[\pi H_{\mathbb{H}}(Z_t,0)\right]^b, \tag{5.3}$$

is a local martingale for $t\le T_z$ satisfying

$$dM_t=(1-3a)\,\frac{X_t}{X_t^2+Y_t^2}\,M_t\,dB_t.$$

We can consider this property as the defining property for the exponent $\tilde b$. If we use the Girsanov theorem and weight the paths by M_t then under the weighted measure $U_t=-B_t$ satisfies

$$dU_t=(3a-1)\,\frac{X_t}{X_t^2+Y_t^2}\,dt-dW_t,$$

where W_t is a Brownian motion in the new measure. This leads to one definition for the probability measure $\mu_{\mathbb{H}}(0,w)$.

Definition 5.75. Let $w\in\mathbb{H}$ and suppose g_t is the solution to the Loewner equation (3.1) where U_t satisfies

$$dU_t=(3a-1)\,\frac{X_t}{X_t^2+Y_t^2}\,dt-dW_t,$$

where $X_t=X_t(w),Y_t=Y_t(w)$. Then the corresponding curve $\gamma(t),0<t<T_w$ is *radial* SLE_κ from 0 to w stopped at time T_w.

Note that for radial SLE_κ,

$$dX_t = \frac{(1-2a)\,X_t}{X_t^2+Y_t^2}\,dt + dW_t,$$

$$d\Theta_t = \frac{aX_tY_t}{(X_t^2+Y_t^2)^2}\,dt - \frac{Y_t}{X_t^2+Y_t^2}\,dW_t.$$

Proposition 5.76.

- *If $\kappa \le 4$, then $T_w < \infty$ and $\gamma(T_w-) = w$.*
- *If $4 < \kappa < 8$, then $T_w < \infty$ and $\gamma(T_w-) \neq w$.*
- *If $\kappa \ge 8$, then $T_w = \infty$.*

PROOF. We consider a new parametrization for which the conformal radius Υ_t decreases deterministically. Assume for ease that $w = x+i$ and let $\rho(t)$ be defined by $\Upsilon_{\rho(t)} = e^{-2at}$. In other words,

$$\dot\rho(t) = \frac{(X_{\rho(t)}^2 + Y_{\rho(t)}^2)^2}{Y_{\rho(t)}^2}.$$

This is valid up to the time that $Y_{\rho(t)} = 0$. Recall that $\Upsilon_t \asymp \mathrm{dist}[w, \gamma(0,t] \cup \mathbb{R}]$. Then $\tilde\Theta_t = \Theta_{\rho(t)}$ satisfies

$$d\tilde\Theta_t = a\,\cot\tilde\Theta_t\,dt + d\tilde W_t, \tag{5.4}$$

where $\tilde W_t$ is a standard Brownian motion. By comparison with a Bessel process we see that if $a \ge 1/2$, then $\sin\tilde\Theta_t > 0$ for all t, but if $a < 1/2$, then $\sin\tilde\Theta_t$ reaches zero in finite time. □

Remark 5.77.

- Our definition shows immediately that radial SLE_κ up to time T_w is absolutely continuous with respect to chordal SLE_κ. (To be more accurate, we need to stop the process slightly before T_w).
- If $\kappa = 2/a = 6$, then radial SLE_κ has the *same* distribution as chordal SLE_κ up to time T_w. This is another version of the locality property for SLE_6.
- For $\kappa > 4$, this definition only defines radial SLE_κ up to time T_w. In order to define the measure $\mu^\#_{\mathbb{H}}(0,i)$ we can consider $\hat g_t$, a conformal transformation of $\mathbb{H}\setminus\gamma(0,t]$ onto $\mathbb{H}$ with $\hat g_t(i) = i$. This is the basis of the original defintion which we describe in the next section. There the definition is done for $\mu^\#_{\mathbb{D}}(z,0)$.

3. Another definition

Here we give an alternative definition of radial SLE_κ in simply connected domains. We will define $\mu^\#_{\mathbb{D}}(z,0)$ for $z \in \partial\mathbb{D}$. Given $\gamma(0,t]$, let $\tilde g_t$ denote the unique conformal transformation of $\mathbb{D}\setminus\gamma(0,t]$ onto $\mathbb{D}$ with $\tilde g_t(0) = 0, \tilde g_t'(0) > 0$. If we parametrize the curve so that $\log \tilde g_t'(0) = t$, then $\tilde g_t$ satisfies

$$\partial_t \tilde g_t(z) = \tilde g_t(z)\,\frac{e^{i2U_t} + \tilde g_t(z)}{e^{i2U_t} - \tilde g_t(z)}, \quad \tilde g_0(z) = z,$$

where $\tilde g_t(\gamma(t)) = e^{i2U_t}$ If we choose $2U_t = B_{\kappa t}$ where B_t is a standard Brownian motion, then γ has the distribution of $\mu^\#_{\mathbb{D}}(z,0)$. $\mu^\#_D(z,w)$ for other simply connected domains is defined by conformal invariance. The equivalence of the definitions of radial SLE_κ can be checked in a straightforward way by studying the image of SLE under conformal transformations. In studying the equation above it is often useful to consider the logarithm: suppose $\tilde{g}_t(e^{2i\theta}) = e^{2i\tilde{h}_t(\theta)}$. (Since $\tilde{g}_t$ vanishes at the origin, there are techinical issues in taking the logarithm. If we stay in a simply connected subdomain of $\mathbb{D}$ that avoids the origin, e.g., a neighborhood of a boundary point, there is no problem.) Then h_t satisfies

$$\partial_t \tilde{h}_t(\theta) = \frac{1}{2} \cot\left(h_t(\theta) - \frac{1}{2} B_{\kappa t}\right).$$

If $h_t = \tilde{h}_{4t/\kappa}$, then this becomes

$$\partial_t h_t(\theta) = a \cot\left(\tilde{h}_t(\theta) - \tilde{W}_t\right),$$

where $\tilde{W}_t$ is a standard Brownian motion. In other words $\Psi_t := h_t(\theta) - \tilde{W}_t$ satisfies

$$d\Psi_t = a \cot \Psi_t \, dt + dW_t,$$

where $W_t = -\tilde{W}_t$ is a standard Brownian motion. Note that this is the same equation as (5.4). If we let $g_t = \tilde{g}_{4t/\kappa}$, then

$$|g_t'(e^{i2\theta})| = \exp\left\{-\int_0^t \frac{a\,ds}{\sin^2 \Psi_s}\right\}.$$

in this parametrization $g_t'(0) = e^{4t/\kappa} = e^{2at}$.

4. Radial SLE_κ in a smaller domain

The computation starts getting a little messy, but we can also do radial SLE in a smaller domain.

Proposition 5.78. *Let $D \in \mathcal{D}$ and $w \in D$. Let*

$$K_t = C(D_t; U_t, g_t(z)) = \Phi_t'(U_t)^b \, |\Phi_t'(g_t(z))|^{\tilde{b}} \, C(\mathbb{H}; 0, L_t),$$

where $L_t = \Phi_t(g_t(z)) - \Phi_t(U_t)$. Then,

$$M_t = \exp\left\{-\int_0^t \frac{a\mathbf{c}}{12} \, S\Phi_t(U_s)\, ds\right\} K_t,$$

is a local martingale.

If $\kappa \le 4$, then (4.3) holds for radial SLE_κ. In particular, for $\kappa = 8/3$, radial SLE_κ satisfies the restriction property.

Proposition 5.79. *Suppose $w \in D_1 \subset D$ where D_1, D are simply connected. Suppose $z \in \partial D$; ∂D is smooth near z; and D_1, D agree near z. Then if γ has the distribution of $\mu_D^{\#}(z, w)$, then the probability that $\gamma(0, t_\gamma] \subset D_1$ is $|F'(z)|^{5/8} \, |F'(w)|^{5/48}$ where F is the conformal transformation of D_1 onto D fixing z, w.*

5. Two-sided radial

Here we let $\kappa < 8$ and introduce a process that can be called either *two-sided radial* SLE_κ or *chordal* SLE_κ *conditioned to go through a point.* This is a measure

$$\mu_D(z_1, z_2; w) = C(D; z_1, z_2; w)\, \mu_D^{\#}(z_1, z_2; w),$$

where D is a domain; z_1, z_2 are distinct points in ∂D; and $w \in D$. We define

$$C(\mathbb{H}; 0, \infty; w) = G(w),$$

where $G(y(x+i)) = y^{d-2}\,(x^2+1)^{\frac{1}{2}-2a}$ is the Green's function for chordal SLE_κ as introduced in Lecture 3. For other simply connected domains, we define

$$C(D; z_1, z_2; w) = |f'(0)|^{-b}\,|f'(w')|^{-\hat{b}}\, G(w') = |f'(0)|^{-b}\,|f'(w')|^{d-2}\, G(w').$$

where $f : \mathbb{H} \to D$ is the canonical transformation with $f(0) = z_1, f(\infty) = z_2$ and $w' = f^{-1}(w)$. It satisfies the following conformal covariance relations.

- **Conformal covariance**. If $f : D \to f(D)$ is a conformal transformation, then

$$C(D; z_1, z_2; w) = |f'(z_1)|^b\,|f'(z_2)|^b\,|f'(w)|^{2-d}\, C(f(D); f(z_1), f(z_2); f(w)), \tag{5.5}$$

$$f \circ \mu_D^{\#}(z_1, z_2; w) = \mu_{f(D)}^{\#}(f(z_1), f(z_2); f(w)).$$

- **Conformal invariance for boundary perturbations** (for $\kappa \le 4$). If $w \in D_1 \subset D$; $z \in \partial D$; and $\partial D_1, \partial D$ agree near z_1, z_2; then $\mu_{D_1}(z_1, z_2; w)$ is absolutely continuous with respect to $\mu_D(z_1, z_2; w)$ and the Radon-Nikodym derivative

$$\frac{d\mu_{D_1}(z_1, z_2; w)}{d\mu_D(z_1, z_2; w)} \tag{5.6}$$

  is a conformal invariant.

There are two different ways to think of the probability measure $\mu_D^{\#}(z_1, z_2; w)$. We illustrate this with $D = \mathbb{H}, z_1 = 0, z_2 = \infty$.

5.1. Chordal weighted by G

Recall that if $z \in \mathbb{H}$ and $M_t = M_t(z) = |g_t'(z)|^{2-d}\, G(Z_t)$, then M_t is a local martingale satisfying

$$dM_t = (1-4a)\,\frac{X_t}{X_t^2 + Y_t^2}\, M_t\, dB_t.$$

Hence if we weight the paths by the martingale, then

$$dB_t = (1-4a)\,\frac{X_t}{X_t^2 + Y_t^2}\, dt + d\hat{B}_t,$$

where $\hat{B}_t$ denotes a Brownian motion in the new measure. In other words,

$$dX_t = (1-3a)\,\frac{X_t}{X_t^2 + Y_t^2}\, dt + d\hat{B}_t,$$

$$d\Theta_t = \frac{2aX_tY_t}{(X_t^2 + Y_t^2)^2}\, dt - \frac{Y_t}{X_t^2 + Y_t^2}\, d\hat{B}_t.$$

If we do the time reparametrization as in the previous section, we can write

$$d\tilde{\Theta}_t = 2a \cot \tilde{\Theta}_t\, dt + d\hat{W}_t, \tag{5.7}$$

where $\hat{W}_t$ is a standard Brownian motion (in the weighted measure).

5.2. Radial weighted by radial weights

We will now show that we can consider two-sided radial SLE_κ from 0 to w in $\mathbb{H}$ as (one-sided) radial SLE_κ weighted by $C(\mathbb{H} \setminus \gamma(0,t]; \infty, w)$. For ease, let us assume $w = i$. Let us start with radial SLE_κ parametrized by Υ_t and recall that

$$C(\mathbb{H} \setminus \gamma(0, \rho(t)]; \infty, w) = \Upsilon_{\rho(t)}^{-\tilde{b}} \, Y_{\rho(t)}^{b} = e^{-2a\tilde{b}} \, \tilde{Y}_t^b.$$

Note that

$$\partial_t \tilde{Y}_t = -a \, \csc^2 \, \tilde{\Theta}_t \, \tilde{Y}_t,$$

i.e.,

$$J_t := \tilde{Y}_t^b = \exp\left\{-ab \int_0^t \frac{ds}{\sin^2 \tilde{\Theta}_s}\right\}.$$

Recall that for radial SLE_κ, $\tilde{\Theta}_t$ satisfies (5.4). Let us now assume that there is a function $\phi(\theta)$ such that

$$\phi(\theta) = \lim_{t \to \infty} e^{\lambda t} \, \mathbf{E}[J_t \mid \tilde{\Theta}_0 = \theta].$$

Then $e^{\lambda t} \, \phi(\tilde{\Theta}_t) \, J_t$ is a martingale. Using Itô's formula, we get the equation

$$\phi''(\theta) + 2a \, \phi'(\theta) \, \cot\theta + \left(2\lambda - 2ab \csc^2 \theta\right) \, \phi(\theta) = 0,$$

which gives $\phi(\theta) = e^{3at/2} \, \sin^a \theta$. Therefore, $M_t = e^{3at/2} \, \sin^a \tilde{\Theta}_t$ satisfies

$$dM_t = a \, \cot \tilde{\Theta}_t \, dW_t.$$

If we do Girsanov and weight by M_t, then $\tilde{\Theta}_t$ satisfies

$$d\tilde{\Theta}_t = 2a \, \cot \tilde{\Theta}_t \, dt + d\tilde{W}_t,$$

where $\tilde{W}_t$ is a standard Brownian motion in the new measure. Note that this is the same as (5.7).

6. Exercises for Lecture 5

Exercise 5.80. Redo Section 1 for loop-erased walk.

Exercise 5.81. Verify (5.5).

LECTURE 6

Intersection exponents

In this lecture we will compute the intersection exponents for $SLE_\kappa, \kappa \leq 4$. The particular case of $\kappa = 8/3$ gives the Brownian motion intersection exponents. We fix $a = 2/\kappa \geq 1/2$ and constants depend on a.

1. One-sided

We assume that g_t satisfies (3.1) where $U_t = -B_t$ is a standard Brownian motion. If $z \in \mathbb{H}$ we write $Z_t = Z_{t,z} = g_t(z) - U_t$ and we write $Z_t = X_t + iY_t$. Recall that

$$dX_t = \frac{aX_t}{X_t^2 + Y_t^2}\,dt + dB_t, \qquad dY_t = -\frac{aY_t}{X_t^2 + Y_t^2}\,dt.$$

$$d|g_t'(z)| = |g_t'(z)|\,\frac{a(Y_t^2 - X_t^2)}{(X_t^2 + Y_t^2)^2}\,dt. \tag{6.1}$$

We will consider the case $z = x \in \mathbb{R} \setminus \{0\}, t \leq T_x$, we have $g_t'(x) \in (0,1]$,

$$dX_t = \frac{a}{X_t}\,dt + dB_t, \quad \partial_t g_t'(x) = -\frac{a\,g_t'(x)}{X_t^2}, \quad g_t'(x) = \exp\left\{-a\int_0^t \frac{ds}{X_s^2}\right\}. \tag{6.2}$$

Define

$$\lambda_0 = \lambda_0(a) = -\frac{(2a-1)^2}{8a},$$

$$q(\lambda) = q_+(\lambda) = q(\lambda;a) = \frac{(1-2a) + \sqrt{(2a-1)^2 + 8a\lambda}}{2}, \quad \lambda \geq \lambda_0, \tag{6.3}$$

$$q_-(\lambda) = q_-(\lambda;a) = \frac{(1-2a) - \sqrt{(2a-1)^2 + 8a\lambda}}{2}, \quad \lambda \geq \lambda_0.$$

Note that $q = q_\pm(\lambda)$ is a solution to the quadratic equation

$$q^2 + (2a-1)\,q - 2a\lambda = 0, \tag{6.4}$$

and that

$$q(\lambda_1) + q(\lambda_2) = q\left(\lambda_1 + \lambda_2 + \frac{q(\lambda_1)q(\lambda_2)}{a}\right). \tag{6.5}$$

Proposition 6.82. *Suppose $x > 0$, $\lambda \geq \lambda_0$, $q = q(\lambda)$. Then $M_t := X_t^q\, g_t'(x)^\lambda$ is a martingale satisfying*

$$dM_t = M_t\,\frac{q}{X_t}\,dB_t. \tag{6.6}$$

If $q = q_-(\lambda)$, the same holds with "local martingale" replacing "martingale".

Proof. Itô's formula gives

$$dX_t^q = X_t^q \left[\frac{qa + \frac{1}{2}q(q-1)}{X_t^2} \, dt + \frac{q}{X_t} \, dB_t \right],$$

and hence the product rule and (6.2) give

$$dM_t = M_t \left[\frac{qa + \frac{1}{2}q(q-1) - a\lambda}{X_t^2} \, dt + \frac{q}{X_t} \, dB_t \right].$$

By (6.4), we see that M_t is a local martingale. If we weight the paths by M_t, the paths B_t satisfy

$$dB_t = \frac{q}{X_t} \, dt + dW_t,$$

where W is a standard Brownian motion in the new weighting. Hence X_t satisfies

$$dX_t = \frac{a+q}{X_t} \, dt + dW_t. \tag{6.7}$$

If $\lambda \geq \lambda_0$ then $q \geq (1/2) - a$ and hence $a + q \geq 1/2$. Therefore, the weighted paths never reach the origin. Also, since the weighted paths follow a Bessel process, one can use the criterion from Section 2 to see that M_t is actually a martingale. □

If we consider the pair of processes $U_t = -B_t$ and $K_t = g_t(x)$, then the pair of processes (U_t, K_t) satisfy the simple system

$$dU_t = \frac{q}{U_t - K_t} \, dt - dW_t, \qquad \partial_t K_t = \frac{a}{K_t - U_t}. \tag{6.8}$$

This is an example of an $SLE(\kappa, \rho)$ process where $\rho = q\kappa$. (The parameter ρ is the coefficient of the drift of U_t if the process is parametrized so that $\mathrm{hcap}(\gamma(0,t]) = 2t$. This is probably not the best way to parametrize these processes.)

Example 6.83. Let $x > 0$, $\lambda = b, q = q(\lambda) = a$. Note that $C(\mathbb{H} \setminus \gamma(0,t]; x, \infty) = g_t'(x)^b$. Then (6.7) becomes

$$dX_t = \frac{2a}{X_t} \, dt + dW_t,$$

and (6.8) becomes

$$dU_t = \frac{a}{U_t - K_t} \, dt - dW_t, \qquad \partial_t K_t = \frac{a}{K_t - U_t}. \tag{6.9}$$

This is the $SLE(\kappa, \rho)$ process with $\rho = a\kappa = 2$. It is also the measure for *two-sided chordal* SLE_κ from $(0, x)$ to ∞ in $\mathbb{H}$. We can let $x \to 0$ and obtain *two-sided chordal* SLE_κ from 0 to ∞ in $\mathbb{H}$. If we are only interested in the distribution of $X_t = K_t - U_t$ we can replace (6.9) with

$$dU_t = \frac{a}{U_t - K_t} \, dt - \sqrt{r(t)} \, dW_t^1, \qquad dK_t = \frac{a}{K_t - U_t} \, dt - \sqrt{1 - r(t)} \, dW_t^2,$$

where $0 \leq r(t) \leq 1$ and W^1, W^2 are independent standard Brownian motions. The value of $r(t)$ determines the ratio of growth rates of the two paths.

Example 6.84. Consider $\mu_{\mathbb{H}}(0, x) = x^{-2b} \, \mu_{\mathbb{H}}^{\#}(0, x)$. We can obtain the probability measure $\mu_{\mathbb{H}}^{\#}(0, x)$ from SLE_κ from 0 to ∞ by weighting by X_t^{-2b}, i.e., by choosing $q_- = -2b$. Then (6.7) becomes

$$dX_t = \frac{1 - 2a}{X_t} \, dt + dW_t,$$

and (6.8) becomes

$$dU_t = \frac{-2b}{U_t - K_t}\,dt - dW_t, \qquad \partial_t K_t = \frac{a}{K_t - U_t}.$$

This is $SLE(\kappa,\rho)$ with

$$\rho = -2b\kappa = \kappa - 6.$$

In this case the local martingale M_t is not a martingale. This can be see by noting that the amount of time (in hcap from infinity) to go from 0 to x is finite, i.e., there is explosion in finite time.

Proposition 6.85. *If $\lambda \geq \lambda_0$ and $q = q(\lambda)$, there is a constant c such that for all $t, x > 0$,*

$$\mathbf{E}[g_{tx^2}'(x)^\lambda] = \mathbf{E}[g_t'(1)^\lambda] \sim c\, t^{-q/2}.$$

Proof. (Sketch) The equality holds by scaling so we may assume $x = 1$. This is a fact about Bessel processes; we are computing $\mathbf{E}[J_t]$ where

$$J_t = \exp\left\{-a\lambda \int_0^t \frac{ds}{X_s^2}\right\},$$

and X_t is a Bessel process satisfying

$$dX_t = \frac{a}{X_t}\,dt + dB_t.$$

We know that $M_t = J_t\, X_t^q$ is a martingale satisfying (6.6). If we weight the paths by M_t, then the weighted paths satisfy (6.7) where W_t is a standard Brownian motion in the new measure. In particular,

$$\mathbf{E}[J_t] = \mathbf{E}[M_t\, X_t^{-q}] = M_0\, \tilde{\mathbf{E}}[X_t^{-q}],$$

where $\tilde{\mathbf{E}}$ denotes expectations in the new measure $\tilde{\mathbf{P}}$. In the $\tilde{\mathbf{P}}$ measure, X_t satisfies (6.7), and hence has a limiting distribution proportional to $r^{2(a+q)}\, e^{-r^2/2}$. The constant c is the expectation of X^{-q} with respect to this limit measure. □

Remark 6.86. We will interpret q as an "intersection exponent" or "crossing exponent" for SLE_κ. Suppose $L > 0$ is large (we will take asymptotics as $L \to \infty$) and γ is an SLE_κ curve from 0 to ∞. Let us consider (see Exercise 4.69)

$$C(\mathbb{H} \setminus \gamma(0,\infty); 1, L) = \lim_{t\to\infty} \frac{g_t'(1)^b\, g_t'(L)^b}{(g_t(L) - g_t(1))^{2b}}.$$

Since the diameter of $\gamma(0,t]$ is of order $\sqrt{t}$, we can expect that

$$C(\mathbb{H} \setminus \gamma(0,\infty); 1, L) \asymp \frac{g_{L^2}'(1)^b\, g_{L^2}'(L)^b}{(g_{L^2}(L) - g_{L^2}(1))^{2b}} \asymp L^{-2b}\, g_{L^2}'(1)^b,$$

$$\mathbf{E}\left[C(\mathbb{H} \setminus \gamma(0,\infty); 1, L)^r\right] \asymp \mathbf{E}\left[\frac{g_{L^2}'(1)^{br}\, g_{L^2}'(L)^{br}}{(g_{L^2}(L) - g_{L^2}(1))^{2br}}\right] \asymp L^{-2br}\, \mathbf{E}[g_{L^2}'(1)^{br}].$$

Or, in other words,

$$\mathbf{E}\left[C(\mathbb{H} \setminus \gamma(0,\infty); 1, L)^r\right] \asymp L^{-q(br) - 2b(r-1)}. \tag{6.10}$$

2. Two-sided

We will now do a similar result for two processes.

Lemma 6.87. *Suppose* $y < 0 < x$, $X_t = g_t(x) - U_t, \tilde{X}_t = U_t - g_t(y)$. *Let* $\lambda_1, \lambda_2 \geq \lambda_0$ *and let* $q = q(\lambda_1), \tilde{q} = q(\lambda_2), r = q\tilde{q}/a$. *Then*

$$M_t = X_t^q \, \tilde{X}_t^{\tilde{q}} \, (X_t + \tilde{X}_t)^r \, g_t'(x)^{\lambda_1} \, g_t'(y)^{\lambda_2}$$

is a martingale.

Proof. Recalling that

$$dX_t = \frac{a}{X_t}\, dt + dB_t, \quad d\tilde{X}_t = \frac{a}{\tilde{X}_t}\, dt - dB_t,$$

we can use Itô's formula and the product rule to show that

$$dM_t = M_t \left[K_t \, dt + \left(\frac{q}{X_t} - \frac{\tilde{q}}{\tilde{X}_t} \right) dB_t \right],$$

where

$$K_t = \frac{qa + \frac{1}{2}q(q-1) - a\lambda_1}{X_t^2} + \frac{\tilde{q}a + \frac{1}{2}\tilde{q}(\tilde{q}-1) - a\lambda_2}{\tilde{X}_t^2} + \frac{ar - \tilde{q}q}{X_t \, \tilde{X}_t} = 0,$$

for our choice of $q, \tilde{q}, r$. If we weight the paths by M_t, we get

$$dB_t = \left(\frac{q}{X_t} - \frac{\tilde{q}}{\tilde{X}_t} \right) dt + dW_t.$$

Therefore,

$$dX_t = \left(\frac{a+q}{X_t} - \frac{\tilde{q}}{\tilde{X}_t} \right) dt + dW_t,$$

$$d\tilde{X}_t = \left(\frac{a+\tilde{q}}{\tilde{X}_t} - \frac{q}{X_t} \right) dt - dW_t.$$

By examining these coupled Bessel-like processes we can see that M_t is a martingale. $\square$

This lemma and (6.5) go a long way to proving an important estimate. For large t, $X_t \approx t^{1/2}, \tilde{X}_t \approx t^{1/2}$. Since $\mathbf{E}[M_t] = M_0$, we get

$$\mathbf{E}[g_t'(x)^{\lambda_1} \, g_t'(y)^{\lambda_2}] \asymp t^{-q(\lambda_1, \lambda_2)/2},$$

where

$$q(\lambda_1, \lambda_2) = q(\lambda_1) + q(\lambda_2) + \frac{q(\lambda_1)\, q(\lambda_2)}{a}.$$

In fact, one can show that for each $x > 0$ there is a $c_x > 0$ such that

$$\lim_{t \to \infty} t^{q(\lambda_1, \lambda_2)/2} \, \mathbf{E}[g_t'(x)^{\lambda_1} \, g_t'(-1)^{\lambda_2}] = c_x.$$

We will now define the chordal crossing exponents $\tilde{\xi} = \tilde{\xi}_\kappa$

- $\tilde{\xi}(\lambda) = \lambda$.
- $$\tilde{\xi}(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2 + \frac{q(\lambda_1)\, q(\lambda_2)}{a}.$$

  Note that

  $$\tilde{\xi}(\lambda_1, \lambda_2) = q^{-1}\left(q(\tilde{\xi}(\lambda_1)) + q(\tilde{\xi}(\lambda_2)) \right) = q^{-1}\left(q(\lambda_1) + q(\lambda_2) \right).$$

- More generally we define

$$\tilde{\xi}(\lambda_1, \lambda_2, \ldots, \lambda_n) = q^{-1}\left(q(\lambda_1) + \cdots + q(\lambda_n)\right).$$

Clearly $\tilde{\xi}$ is a symmetric function and one can check that it satisfies the "cascade relation"

$$\tilde{\xi}(\lambda_1, \ldots, \lambda_n) = \tilde{\xi}(\tilde{\xi}(\lambda_1, \ldots, \lambda_k), \tilde{\xi}(\lambda_{k+1}, \ldots, \lambda_n)).$$

If $\tilde{\xi}_n = \tilde{\xi}(\underbrace{b, \cdots, b}_{n})$, then

$$\tilde{\xi}_{n+1} = \tilde{\xi}(b, \tilde{\xi}_n) = \tilde{\xi}_n + b + q(\tilde{\xi}_n) = \tilde{\xi}_n + \frac{a + \sqrt{(2a-1)^2 + 8a\tilde{\xi}_n}}{2},$$

which yields by induction

$$\tilde{\xi}_n = \frac{an^2 + (2a-1)\,n}{2}.$$

Example 6.88. The case $\kappa = 8/3$ gives the (chordal or half-plane) Brownian intersection exponents. Here we have

$$q(\lambda) = -\frac{1}{4} + \frac{1}{4}\sqrt{1 + 24\lambda},$$

$$\tilde{\xi}(\beta, \lambda) = \beta + \lambda + \frac{q(\beta)}{3}\left[\sqrt{1 + 24\lambda} - 1\right].$$

In particular, $q(1) = 1, q(2) = 3/2$ and hence

$$\tilde{\xi}(1, \lambda) = 1 + \lambda + \frac{4}{3}\,q(\lambda) = \lambda + \frac{2}{3} + \frac{1}{3}\sqrt{1 + 24\lambda},$$

$$\tilde{\xi}(2, \lambda) = 2 + \lambda + 2q(\lambda) = \lambda + \frac{3}{2} + \frac{1}{2}\sqrt{1 + 24\lambda}.$$

3. Nonintersecting SLE_κ

We will assume $\kappa \le 4$ and k is a positive integer. Suppose D is a simply connected domain and $z_1, \ldots, z_k, w_1, \ldots, w_k$ are distinct points on the boundary of D. We will assume that ∂D is locally analytic near all of these boundary points. We define $SLE_D(z_1, \ldots, z_k; w_1, \ldots, w_k)$ which is a measure on k-tuples of curves $\bar{\gamma} = (\gamma^1, \ldots, \gamma^k)$ with γ^j connecting z_j to w_j. The measure is defined inductively on k.

- If $k = 1$, $\mu_D(z_1, w_1)$ is SLE_κ from z_1 to w_1 in D.
- Suppose $SLE_D(z_1, \ldots, z_{k-1}; w_1, \ldots, w_{k-1})$ has been defined. This is a measure on $(k-1)$-tuples $\hat{\gamma} = (\gamma^1, \ldots, \gamma^{k-1})$. For each realization of $\hat{\gamma}$, let $D_{\hat{\gamma}}$ denote the connected component of $D \setminus \hat{\gamma}$ that contains z_k and w_k on the boundary. Then for the marginal on $\hat{\gamma}$ we define

$$\frac{d\mu_D(z_1, \ldots, z_k; w_1, \ldots, w_k)}{d\mu_D(z_1, \ldots, z_{k-1}; w_1, \ldots, w_{k-1})}\,(\bar{\gamma}) = C(D_{\hat{\gamma}}; z_k, w_k).$$

  If z_k, w_k are not on the boundaries of the same connected component we set $C(D_{\hat{\gamma}}; z_k, w_k) = 0$.
- The conditional probability measure on z_k, w_k given $\hat{\gamma}$ is that of $\mu_{D_{\hat{\gamma}}}(z_k, w_k)$.

While the definition above makes it appear that the definition depends on the order that we specify the pairs $(z_1, w_1), \ldots, (z_k, w_k)$, we can give an equivalent definition that is independent. Let $Y = Y_D(z_1, \ldots, z_k, w_1, \ldots, w_k)$ denote the Radon-Nikodym derivative of $\mu_D(z_1, \ldots, z_k; w_1, \ldots, w_k)$ with respect to the product measure

$$\mu_D(z_1, w_1) \times \cdots \times \mu_D(z_k, w_k).$$

Then if $\bar{\gamma} = (\gamma^1, \ldots, \gamma^k)$,

$$Y(\bar{\gamma}) = 1\{\gamma^j \cap \gamma^n = \emptyset, 1 \leq j < n \leq k\}\, e^{-\Theta \mathbf{c}/2}$$

where Θ denotes the Brownian loop measure of the set of loops that intersect at least two of the curves. We will not give the proof which is essentially an application of the ideas from Lecture 4.

The quantity

$$\frac{C(D; z_1, z_2, \ldots, z_k; w_1, w_2, \ldots, w_k)}{C(D; z_1, w_1)\, C(D; z_2, w_2) \cdots C(D; z_k, w_k)}$$

is a conformal invariant. For $k = 2$, it is given by $\phi(L)$ defined in (6.10).

4. Radial exponent and SAW III

Let us consider radial SLE_κ in $\mathbb{D}$ from 1 to 0 parametrized as in Lecture 5, Section 3 with $g_t'(0) = e^{2at}$. Let $\theta \in (0, \pi)$. We will consider $\mathbf{E}[|g_t'(e^{2i\theta})|^\lambda]$. As seen in that section, this is an estimate about a one-dimesional diffusion. Indeed,

$$|g_t'(e^{2i\theta})|^\lambda = \exp\left\{-a\lambda \int_0^t \frac{dt}{\sin^2 \Psi_t}\, dt\right\},$$

where Ψ_t satisfies

$$d\Psi_t = a \cot \Psi_t\, dt + dW_t. \tag{6.11}$$

Proposition 6.89. *If $a \geq 1/4$ and $\lambda > \lambda_0$, then there exists c such that*

$$\mathbf{E}[|g_t'(e^{2i\theta})|^\lambda] \sim c\, e^{-2a\beta t} \sin^r \theta = c\, g_t'(0)^{-\beta} \sin^q \theta.$$

where

$$q = q(\lambda) = \frac{(1 - 2a) + \sqrt{(2a-1)^2 + 8a\lambda}}{2}, \quad \beta = \beta(\lambda) = \frac{\lambda}{2} + \frac{q}{4a}. \tag{6.12}$$

Proof. (Sketch) If

$$r^2 + r(2a - 1) - 2a\lambda = 0, \quad k = a\lambda + \frac{r}{2}, \tag{6.13}$$

then $M_t = |g_t'(e^{2i\theta})|^\lambda \sin^r \Psi_t\, e^{kt}$, is a local martingale satisfying

$$dM_t = r \cot \Psi_t\, M_t\, dW_t.$$

If r is chosen as in (6.12), then using Girsanov we can show that M_t is actually a martingale and hence

$$\sin^r \theta = e^{kt}\, \mathbf{E}\left[|g_t'(e^{2i\theta})|^\lambda \sin^r \Psi_t\right].$$

We then proceed as in Proposition 6.85. $\square$

Example 6.90. We have already seen an example of this exponent. Let $\lambda = b$. Then $r = a, \beta = 3a/4$. If C denotes the raidal SLE total mass, then

$$C(\mathbb{D} \setminus \gamma(0,t]; e^{2i\theta}, 0) = g_t'(0)^{\tilde{b}} \, |g_t'(e^{2i\theta})|^b.$$

Hence we can interpret this as

$$\mathbf{E}\left[C(\mathbb{D} \setminus \gamma(0,t]; e^{2i\theta}, 0)\right] \sim c \, \sin^b \theta \, g_t'(0)^{\tilde{b} - \frac{3a}{4}}.$$

Example 6.91. (SAW III) Here we will show how to predict a critical exponent for self-avoiding walk. Define the exponent u by saying that the probability that two self-avoiding walks of n steps starting at the origin avoid each other decays like n^{-u}. The typical distance of a SAW of length n is $n^{1/d}$ where $d = 4/3$ is the fractal dimension of the paths. Hence, we can rephrase the definition of u as saying that the probability that two SAWs go distance R without an intersection decays like R^{-ud}. Given the conjectured relation between SAWs and $SLE_{8/3}$, we can guess that if γ^1, γ^2 are independent radial $SLE_{8/3}$ in $\mathbb{D}$ going to 0 starting at $+1, -1$, respectively, then the probability that γ^1 reaches the ball of radius δ about the origin without hitting the path of γ^2 decays like δ^{ud}. The restriction property tells us that for a fixed realization of γ^1 (with corresponding conformaal maps g_t), the probability that γ^2 avoids $\gamma^1(0,t]$ is given by

$$g_t'(0)^{\tilde{b}} \, |g_t'(-1)|^b,$$

with $\tilde{b} = 5/48, b = 5/8$. Also, the Koebe-(1/4) theorem tells us that the distance from the origin to $\gamma^1(0,t]$ is comparable to $g_t'(0)$. Our proposition tells us that

$$\mathbf{E}[|g_t'(-1)|^b] \asymp g_t'(0)^{-\beta}, \quad \beta = \beta(b) = \frac{3a}{4} = \frac{9}{16}.$$

Hence the probability of no intersection is given by

$$\mathbf{E}[g_t'(0)^{\tilde{b}} \, |g_t'(-1)|^b] \sim g_t'(0)^{-11/24}.$$

This gives $ud = 11/24, u = 11/32$ and matches the prediction first given by Nienhuis. The exponent u is the same as $\gamma - 1$ described in Lecture 1, Section 1.

The radial exponent ξ is defined by

$$\xi(b, \lambda) = \tilde{b} + \beta(\lambda) = \tilde{b} + \frac{\lambda}{2} + \frac{q(\lambda)}{4a}.$$

More generally, we define

$$\xi(b, \lambda_1, \ldots, \lambda_k) = \xi(b, \tilde{\xi}(\lambda_1, \ldots, \lambda_k)).$$

Note that the chordal exponent $\tilde{\xi}$ appears in the definition of ξ. The exponent $\xi(\lambda_1, \lambda_2)$ is defined so that the foll0wing holds:

$$\xi(b, \lambda, \lambda_2) = \xi(b, \tilde{\xi}(\lambda, \lambda_2)) = \xi(\tilde{\xi}(b, \lambda_1), \lambda_2).$$

Example 6.92. If $\kappa = 8/3$, then $\tilde{\xi}(5/8, 1/8) = 1, q(1/8) = 1/4, \tilde{\xi}(5/8, 5/8) = 2, q(5/8) = 3/4$,

$$\xi(5/8, \lambda) = \frac{5}{48} + \frac{\lambda}{2} + \frac{1}{3} \, q(\lambda) = \frac{1}{48} + \frac{\lambda}{2} + \frac{1}{12} \sqrt{1 + 24\lambda}.$$

One can check that

$$\xi(1, \lambda) = \frac{1}{8} + \frac{\lambda}{2} + \frac{1}{8} \sqrt{1 + 24\lambda},$$

$$\xi(2,\lambda) = \frac{11}{24} + \frac{\lambda}{2} + \frac{5}{24}\sqrt{1+24\lambda}.$$

These are the one-sided and two-sided Brownian intersection exponents. The values $\xi(1,0) = 1/4, \xi(2,0) = 2/3$ are the one-sided and two-sided disconnection exponents.

5. Exercises for Lecture 6

Exercise 6.93. Find the constant c in Proposition 6.85.

Exercise 6.94. Complete the details in the proof of Proposition 6.89. Note that in the proposition, r was chosen to be a particular root of the quadratic equation in (6.13). Show that if we choose r to be the other root of that equation, then M_t is not a martingale (even though it is a local martingale). Express c in terms of invariant densities for diffusions of the form (6.11).

Exercise 6.95. In Proposition 6.89, note that $\beta > \lambda/2$ if $\lambda > 0$ and $\beta \sim \lambda/2$ as $\lambda \to \infty$. Show how these facts can be deduced from the Beurling estimate.

Exercise 6.96. Fix $\kappa \leq 4$ and positive integer n. Let $0 < y_1 < \cdots < y_n < \pi$ and let $\mathcal{R}_L = [0,L] \times [0,\pi]$. Let $z_j = y_j i, w_j = w_{j,L} = L + y_j i$. Show that as $L \to \infty$,

$$C(\mathcal{R}_L; z_1, \ldots, z_n; w_1, \ldots, w_n) \asymp e^{-L\tilde{\xi}_n},$$

where $\tilde{\xi}_n$ is as defined in Section 2.

Exercise 6.97. For $\kappa = 2$, it can be shown using an idenity of Fomin that

$$C(\mathcal{R}_L; z_1, \ldots, z_n; w_1, \ldots, w_n) = \det\left[H_{\partial\mathcal{R}_L}(z_j, w_k)\right].$$

Use this identity to give another proof that

$$\tilde{\xi}_n = \frac{n^2+n}{2}, \quad \kappa = 2.$$

Tables for reference

TABLE 1. Parameters for SLE

κ	variance in driving function	$\kappa > 0$
a	rate of capacity growth	$\frac{2}{\kappa}$
d	dimension of paths	$1 + \frac{\kappa}{8} = 1 + \frac{4}{a}, \quad \kappa \leq 8$
b	boundary scaling exponent	$\frac{3a-1}{2} = \frac{6-\kappa}{2\kappa}$
$\tilde{b}$	one-sided interior scaling exponent	$\frac{1-a}{2a}\, b = \frac{\kappa-2}{4}\, b$
$\hat{b}$	two-sided interior scaling exponent	$2 - d = 1 - \frac{\kappa}{8}$
$\mathbf{c}$	central charge	$\frac{2b(3-4a)}{a} = \frac{(3-4a)(3a-1)}{a} = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$

TABLE 2. Some discrete models

	κ	a	b	$\tilde{b}$	$\mathbf{c}$	d
loop-erased walk	2	1	1	0	-2	$\frac{5}{4}$
self-avoiding walk	$\frac{8}{3}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{5}{48}$	0	$\frac{4}{3}$
Ising interface	3	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{11}{8}$
harmonic explorer free field interface	4	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	1	$\frac{3}{2}$
percolation interface	6	$\frac{1}{3}$	0	0	0	$\frac{7}{4}$
uniform spanning tree	8	$\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{3}{16}$	-2	2

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