1 Various preliminaries

We assume that the reader knows the definition of a one-dimensional Brownian motion. We also assume some familiarity with complex analysis although we will develop most of the facts that we need. Those very familiar with complex analysis can go quickly through some sections although it is recommended to read them to see how the results tie in with Brownian motion.

1.1 Notation

We write

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk and $\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$ is the upper half plane.
- More generally, $\mathbb{D}_r = e^{-r}\,\mathbb{D}$ is the disk of radius e^{-r} about the origin and $\mathbb{D}_r(z) = z + \mathbb{D}_r$ is the disk of radius e^{-r} about z.
- $C_r = \partial\mathbb{D}_r, C_r(z) = \partial\mathbb{D}_r(z)$.
- $\mathbb{A}_r = \mathbb{D} \setminus \overline{\mathbb{D}_r} = \{e^{-r} < |z| < 1\}$ is the annulus with boundary $C_0 \cup C_r$.
- We write $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ for the Riemann sphere.

The potential theory of complex Brownian motion makes heavy use of the logarithm function. This affects our choice of notation. There are many times that we will want to write the logarithm of a radius, and by parametrizing our radii exponentially it will make our formulas nicer.

Definition A *(standard) complex Brownian motion* is a process of the form $B_t = B_t^1 + i\,B_t^2$ where B_t^1, B_t^2 are independent one-dimensional Brownian motions. Equivalently, B_t is a standard Brownian motion in $\mathbb{R}^2$ viewed as taking values in $\mathbb{C}$.

When we say complex Brownian motion, we will always mean standard complex Brownian motion. If the context is clear we will say just Brownian motion. If $D \subset \mathbb{C}$ is an open set, then we define

$$\tau_D = \inf\{t \geq 0 : B_t \notin D\},$$

$$\overline{\tau}_D = \inf\{t > 0 : B_t \notin D\}.$$

Note that $\tau_D = \overline{\tau}_D > 0$ if $B_0 \in D$ and $\tau_D = 0$ if $B_0 \in \mathbb{C} \setminus D$. If $B_0 \in \partial D$.

Definition We call a boundary point $z \in D$ *regular* if $\mathbb{P}^z\{\overline{\tau}_D = 0\} = 1$, that is, if Brownian motion starting at z immediately hits the boundary.

Here $\mathbb{P}^w$ means the probability assuming that $B_0 = w$. Note that isolated points of ∂D are not regular points. The next lemma shows that Brownian motion starting near a regular point exits the domain quickly with high probability.

Lemma 1.1. *Suppose D is an open subset of $\mathbb{C}$.*

- *Suppose z is a regular boundary point of ∂D. Then for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|w - z| < \delta$, then*
$$\mathbb{P}^w\{\mathrm{diam}(B[0, \overline{\tau}_D]) \geq \epsilon\} \leq \epsilon.$$
- *If z is an irregular boundary point of ∂D, then $\mathbb{P}^z\{\overline{\tau}_D > 0\} = 1$.*

Proof. Without loss of generality, assume that $z = 0$ and let $\xi_s = \inf\{t \geq 0 : |B_t| = e^{-s}\}$.

Suppose 0 is a regular point. Since $\mathbb{P}\{\overline{\tau}_D = 0\} = 1$, we know that $\mathbb{P}\{B(0, \xi_s] \subset D\} = 0$. Therefore, there exists u such that

$$\mathbb{P}\{B[\xi_u, \xi_s] \subset D\} \leq e^{-s}.$$

We can find $\delta > 0$ such that the distribution of $B(\xi_u)$ given that $|B_0| = \delta$ agrees with that assuming $B_0 = 0$ up to an error of e^{-s} (see comment below). Therefore, for $|w| < \delta$,

$$\mathbb{P}^w\{B[\xi_u, \xi_s] \subset D\} \leq 2e^{-s},$$

and hence

$$\mathbb{P}^w\{\mathrm{diam}[0, \overline{\tau}_D] \geq 2e^{-s}\} \leq 2e^{-s}.$$

Suppose 0 is an irregular point. Then there exists r with

$$\mathbb{P}\{B(0, \xi_r] \subset D\} = \rho > 0.$$

For ease we will assume that $r = 0$ (other r can be handled by scaling, see Exercise 1). For every $\epsilon > 0$ we can find $s \geq 0$ such that

$$\mathbb{P}\{B[\xi_s, \xi_0] \subset D\} \leq \rho + \epsilon.$$

As before, we can find u such that if $|z| \leq e^{-u}$,

$$\mathbb{P}^z\{B[\xi_s, \xi_0] \subset D\} \leq \rho + 2\epsilon.$$

Therefore,

$$\begin{aligned}
\rho &= \mathbb{P}\{B(0, \xi_0] \subset D\} \\
&= \mathbb{P}\{B(0, \xi_u] \subset D\}\, \mathbb{P}\{B(0, \xi_0] \subset D \mid B(0, \xi_u] \subset D\} \\
&\leq \mathbb{P}\{B(0, \xi_u] \subset D\}\, (\rho + 2\epsilon),
\end{aligned}$$

which implies that

$$\mathbb{P}\{B(0, \xi_u] \subset D\} \geq \frac{\rho}{\rho + 2\epsilon}.$$

Hence,

$$\mathbb{P}\{\overline{\tau}_D > 0\} = \lim_{u \to \infty} \mathbb{P}\{B(0, \xi_u] \subset D\} = 1.$$

$\square$

The astute reader will note that we "cheated" a little by not justifying why the hitting probability of C_u starting at w is almost the same as that starting at 0. We could give a proof of this here, but it follows easily from the exact form of the Poisson kernel in the disk which we do below, so we chose not to.

We will call an open set D *regular* if $\mathbb{P}^z\{\tau_D < \infty\} = 1$ for every $z \in D$.

Exercise 1. *Suppose B_t is a complex Brownian motion starting at the origin. Let $\theta \in \mathbb{R}, a > 0$ and*

$$Y_t = e^{i\theta}\, B_t, \quad Z_t = a^{-1}\, B_{a^2 t}.$$

Then Y_t, Z_t are (standard) complex Brownian motions.

Definition

- A *domain* is a connected open subset of $\mathbb{C}$.
- A domain is *simply connected* if $\hat{\mathbb{C}} \setminus D$ is connected.
- If D is an unbounded domain with ∂D compact, we will also consider $D \cup \{\infty\}$ as a domain in $\hat{\mathbb{C}}$.
- Let $\mathcal{D}$ denote the set of all domains $D \subsetneq \mathbb{C}$ such that every $z \in \partial D$ is regular.
- Let $\mathcal{D}^*$ denote the set of all domains $D \subsetneq \mathbb{C}$ such that there exists $z \in \partial D$ that is regular.

Clearly $\mathcal{D} \subset \mathcal{D}^*$. Giving the exact criterion to be in $\mathcal{D}$ is difficult. However, we will now derive a sufficient condition that will suffice for our purposes. We will use the following lemma that makes strong use of the planarity of $\mathbb{C}$.

Lemma 1.2. *There exists $\beta > 0, c < \infty$ such that if B_t is a complex Brownian motion starting at $z \in \mathbb{D}$ and $\tau = \tau_{\mathbb{D}} = \inf\{t : |B_t| = 1\}$, then the probability that the origin lies in the unbounded component of $\mathbb{C} \setminus B[0, \tau]$ is no more than $c\,|z|^\beta$.*

It follows that if D is a domain, $0 \notin D$, and the connected component of $\mathbb{C} \setminus D$ containing 0 also contains at least one point on $\partial\mathbb{D}$, then

$$\mathbb{P}^z\left\{B[0, \tau_D] \not\subset \mathbb{D}\right\} \le c\,|z|^\beta.$$

Proof. Let $p(z)$ be the probability that 0 lies in the unbounded component given $B_0 = z$, and note that $p(z) = p(|z|)$. Let q be the probability that a Brownian motion starting on C_1, the circle of radius e^{-1} about the origin, disconnects C_1 from C_0 before time τ. By constructing a particular event, it is easy to see that $q > 0$ and is independent of the angle of the starting point. Using the strong Markov property and the scaling property of Brownian motion we can see that for $r \ge 1$, $p(e^{-r}) \le (1-q)\, p(e^{-r-1})$. Hence for integer $n \ge 0$, $p(e^{-n}) \le e^{n \log(1-q)}$, and more generally $p(e^{-u}) \le p(e^{-\lfloor u \rfloor}) \le e^{-\log(1-q)}\, e^{u \log(1-q)}$. $\square$

The optimal value of β is called the *disconnection exponent* and is known to be $1/4$. This is significantly harder to show and we will not need it.

Corollary 1.3. *Suppose $D \subsetneq \mathbb{C}$ is a domain such that all the connected components of $\mathbb{C} \setminus D$ are larger than one point. Then $D \in \mathcal{D}$. In particular, all simply connected $D \subsetneq \mathbb{C}$ are in $\mathcal{D}$.*

The proof of Lemma 1.2 establishes the following stronger fact.

Definition If $z \in \mathbb{C}, K \subset \mathbb{C}$, we define

$$\mathrm{rad}_K(z) = \sup\{|w - z| : w \in K\}, \quad \mathrm{rad}_K = \mathrm{rad}_K(0),$$

$$\mathrm{diam}(K) = \sup\{|w - z| : w, z \in K\}.$$

Note that if $z \in K$, then $\mathrm{rad}_K(z) \leq \mathrm{diam}(K) \leq 2\,\mathrm{rad}_K(z)$.

Proposition 1.4. *Suppose D is a domain and w, w' are in the same component of $\mathbb{C} \setminus D$. Then, for all z,*

$$\mathbb{P}^z \left\{ \mathrm{diam}\left(B[0, \tau_D]\right) \geq 2\,|w' - w| \right\} \leq c \left(\frac{|z - w|}{|w' - w|} \right)^\beta ,$$

where c, β are as in Lemma 1.2.

Proposition 1.5. *Suppose D is a domain and z is an irregular boundary point of ∂D. Let h be a strictly positive harmonic function on D. Then there exists a sequence $z_n \in D$ with $z_n \to z$ with* $\liminf_{n \to \infty} h(z_n) > 0$.

Proof. Since z is irregular, we can find a compact $V \subset D$ and $\delta > 0$. such that $\mathbb{P}^z\{\tau_{\mathbb{C} \setminus V} < \overline{\tau}_D\} \geq \delta > 0$. This implies that there exists $z_n \to z$ with $\mathbb{P}^{z_n}\{\tau_{\mathbb{C} \setminus V} < \tau_D\} \geq \delta\}$ and hence $h(z_n) \geq \delta \, \min\{h(w) : w \in V\}$.

□

Proposition 1.6. *If D is a domain and $z \in D$, then with probability one, $B(\tau_D)$ is a regular point of ∂D.*

Proof. Let V be a subset of ∂D and let E be the event $\{B(\tau_D) \in V\}$. Let M_t be the martingale $M_t = \mathbb{P}[E \mid \mathcal{F}_{t \wedge \tau}]$. For $t < \tau_D$, $M_t = h(B_t)$ where h is the positive harmonic function $h(w) = \mathbb{P}^w\{B_{\tau_D} \in V\}$. Assume that V is such that $0 < h < 1$ on D. Note that $M_\tau = 1_E$ and the martingale convergence theorem implies that $M_{\tau-} = M_\tau$ with probability one. In particular, with probability one, if $\zeta := B(\tau_D) \notin V$, then there exists $\gamma : [0, 1) \to D$ with $\gamma(1-) = \zeta$ such that

$$\lim_{t \uparrow 1} h(\gamma(t)) = 0.$$

Since h is a bounded function, one can see using Lemma 1.2 that this last condition implies: if $z_n \to \zeta$, then $h(z_n) \to 0$. This is impossible if ζ is an irregular point by the previous proposition.

□

2 Brownian motion and harmonic functions

2.1 Harmonic functions

We will consider only functions on $\mathbb{R}^2$, or equivalently, $\mathbb{C}$ although much of we state here applies (with appropriate modifications) to functions on $\mathbb{R}^d$. Recall that the Laplacian of a C^2 function is defined by

$$\Delta f(z) = \partial_{xx} f(z) + \partial_{yy} f(z).$$

Definition If D is a domain, then a measurable function $\phi : D \to \mathbb{R}$, is *harmonic (in D)* if it is locally bounded and satisfies the mean-value property. In other words, if $z \in D$ and $\mathrm{dist}(z, \partial D) > \epsilon$, then

$$\phi(z) = MV(\phi, z, \epsilon) := \frac{1}{2\pi} \int_0^{2\pi} \phi(z + \epsilon e^{i\theta})\, d\theta. \tag{1}$$

The next lemma shows that harmonic functions are smooth.

Lemma 2.1. *If ϕ is a harmonic function on a domain, then ϕ is a C^∞ function.*

Proof. It suffices to show that ϕ is C^∞ in a neighborhood of every point and without loss of generality we will assume that $0 \in D$ and we will differentiate in a neighborhood of 0. By shrinking D if necessary, we may assume that

$$\int_D |\phi(z)|\, dA(z) < \infty.$$

Let $\epsilon < \mathrm{dist}(0, \partial D)/2$, and let ψ denote a radially symmetric, nonnegative, C^∞ function on $\mathbb{C}$ that vanishes on $\{|w| \geq \epsilon\}$ and satisfies

$$\int_{\mathbb{C}} \psi(w)\, dA(w) = 1.$$

Since ϕ satisfies the mean value property, we can use polar coordinates to see that that for $|z| < \epsilon$.

$$\phi(z) = \int_{\mathbb{C}} \phi(w)\, \psi(w - z)\, dA(w).$$

Since ϕ is L^1 and ψ is C^∞ with compact support, the right-hand side is infinitely differentiable. □

Proposition 2.2. *A function ϕ is harmonic on D if and only if it is C^2 and satisfies $\Delta\phi(z) = 0$ for every $z \in D$. Moreover, if $D' \subset D$ is a subdomain bounded by a finite disjoint union of C^1 curves in D, then*

$$\int_{\partial D'} \partial_n \phi(z)\, |dz| = 0, \tag{2}$$

where n denotes the unit inward normal.

Proof. We first claim that if ϕ is C^2 then at each z,

$$\frac{1}{4}\, \Delta\phi(z) = \lim_{\epsilon \downarrow 0} \frac{MV(\phi, z, \epsilon) - \phi(z)}{\epsilon^2}. \tag{3}$$

Without loss of generality we may assume that $z = 0$ and $\phi(z) = 0$ in which case we can write

$$\phi(z) = b_x\, x + b_y y + \frac{1}{2}\,[b_{xx}\, x^2 + b_{yy}\, y^2 + 2\, b_{xy}\, xy] + o(\epsilon^2).$$

Here $b_x, b_y, b_{xx}, b_{yy}, b_{xy}$ are the first and second partial derivatives evaluated at 0. Here we use complex notation $z = x + iy$. Since

$$MV(x^2, 0, \epsilon) + MV(y^2, 0, \epsilon) = MV(x^2 + y^2, 0, \epsilon) = \epsilon^2,$$

we see that $MV(x^2, 0, \epsilon) = MV(y^2; 0, \epsilon) = \epsilon^2/2$, and

$$MV(\phi, 0, \epsilon) = \frac{\Delta\phi(0)}{4}.$$

This gives (3) and it follows immediately that $\Delta\phi \equiv 0$ for harmonic functions.

If ϕ is C^2 and satisfies $\Delta\phi \equiv 0$, then the divergence theorem shows that

$$\int_{\partial D'} \partial_n \phi(z)\, |dz| = -\int_{D'} \Delta\phi(w)\, dA(w) = 0.$$

Applying this to the circles centered at z shows that

$$\frac{dMV(\phi, z, \epsilon)}{d\epsilon} = \frac{1}{2\pi\epsilon} \int_{|z|=\epsilon} \partial_n \phi(z)\, |dz| = \frac{c}{\epsilon},$$

for some $c \in \mathbb{R}$. Since $MV(\phi, z, 0+) = \phi(z)$, we get that $c = 0$ and $\phi(z, \epsilon) = \phi(z)$ for all ϵ.

□

2.2 Optional sampling theorem

There is a close relationship between harmonic function and martingales. Before proceeding we will prove a lemma that is one version of the "optional sampling" or "optional stopping" theorem for martingales. The assumptions we make are significantly stronger than is needed for the result, but it will suffice for our purposes.

Proposition 2.3. *Suppose $M_t, 0 \le t \le T$ is a uniformly bounded continuous martingale, and τ is a stopping time, each with respect to the filtration $\{\mathcal{F}_t\}$. Then $M_{t\wedge\tau}, 0 \le t \le T$ is a continuous martingale with respect to the filtration $\{\mathcal{F}_{\tau\wedge t}\}$.*

We recall that if τ is a stopping time, then $\mathcal{F}_\tau$ is the σ-algebra of all events E such that for all t, $E \cap \{\tau \le t\} \in \mathcal{F}_t$. One thinks of this as all the events that depend on the process only up to the stopping time τ.

Proof. Let $Y_t = M_{t\wedge\tau}$. It is immediate that Y is a continuous process. Let us first assume that τ takes on only a discrete number of values $0 = s_0 < s_1 < \cdots < s_k < \infty$. If $s < t$, then $M_{t\wedge\tau}$ can be written as

$$M_{t\wedge\tau} = \sum_{s_j < t} M_{s_j}\, 1\{\tau = s_j\} + M_t\, 1\{\tau > s_j\}.$$

Using the definition, it is not hard to show that this is a martingale. To illustrate this, we consider the case $s_j = s < t$, in which case

$$\mathbb{E}\left[M_{t\wedge\tau} \mid \mathcal{F}_s\right] = \sum_{k\le j} M_{s_k}\, 1\{\tau = s_k\} + \mathbb{E}\left[M_t\, 1\{\tau > s_j\} \mid \mathcal{F}_{s_j}\right].$$

Since the event $\{\tau > s_j\}$ is the complement of the event $1\{\tau \le s_j\}$ which is $\mathcal{F}_{s_j}$-measurable,

$$\mathbb{E}\left[M_t\, 1\{\tau > s_j\} \mid \mathcal{F}_{s_j}\right] = 1\{\tau > s_j\}\, \mathbb{E}\left[M_t \mid \mathcal{F}_{s_j}\right] = 1\{\tau > s_j\}\, M_{s_j},$$

and hence

$$\begin{aligned} \mathbb{E}\left[M_{t\wedge\tau} \mid \mathcal{F}_{s_j}\right] &= \sum_{k\le j} M_{s_j}\, 1\{\tau = s_j\} + M_{s_j}\, 1\{\tau > s_j\} \\ &= \sum_{k< j-1} M_{s_k}\, 1\{\tau = s_k\} + M_{s_j}\, 1\{\tau \ge s_{j+1}\} = Y_{s_j}. \end{aligned}$$

For more general τ, we approximate τ by discrete stopping times,

$$\tau^j = \tau_{(j+1)/n}, \quad \frac{j}{n} \le \tau < \frac{j+1}{n}.$$

The random variables $M_{t\wedge\tau^j} \to M_t$ with probability one. Since they are bounded, they also converge in L^1.

□

- Let B_t be a standard one-dimensional Brownian motion starting at the origin and suppose that $a, b > 0$. Let $\tau = \inf\{t : B_t = b \text{ or } B_t = -a\}$. Then $B_{t\wedge\tau}$ is a martingale. Therefore, for each t,

  $$0 = \mathbb{E}[B_0] = \mathbb{E}\left[B_{t\wedge\tau}\right].$$

  With probability one $B_{t\wedge\tau} \to B_\tau$. Since $B_{t\wedge\tau}$ is uniformly bounded, we can use the dominated convergence theorem to see that

  $$\mathbb{E}[B_\tau] = \lim_{t\to\infty} \mathbb{E}\left[B_{t\wedge\tau}\right] = 0.$$

  But,

  $$\mathbb{E}[B_\tau] = b\, \mathbb{P}\{B_\tau = b\} - a\, \left[1 - \mathbb{P}\{B_\tau = b\}\right].$$

  Solving, we get

  $$\mathbb{P}\{B_\tau = b\} = \frac{a}{a+b}.$$

  This relation is often referred to as the *gambler's ruin* estimate for one-dimensional Brownian motion. From this we can easily see that one-dimensional Brownian motion is *recurrent*, that is, it keeps returning to the origin.

- One must be careful in using this proposition. If $\mathbb{P}\{\tau < \infty\} = 1$, then with probability one
$$\lim_{t\to\infty} M_{t\wedge\tau} = M_\tau.$$
However, it is not always the case that this limit is in L^1. Indeed, it is possible for
$$\mathbb{E}[M_\tau] \neq \lim_{t\to\infty} \mathbb{E}[M_{t\wedge\tau}].$$
As an example, let $M_t = B_t$ be a standard one-dimensional Brownian motion starting at the origin and let $\tau = \inf\{t : B_t = 1\}$. Recurrence of one-dimensional Brownian motion implies that $\mathbb{P}\{\tau < \infty\} = 1$. However, $\mathbb{E}[B_\tau] \neq \mathbb{E}[B_0]$.

2.3 Itô's formula calculation

Suppose D is a domain, $h : D \to \mathbb{R}$ is a harmonic function and $B_t = B_t^1 + iB_t^2$ is a complex Brownian motion starting at $z \in D$. Let $\tau = \tau_D = \inf\{t : B_t \notin D\}$. Then, for $t < \tau$, Itô's formula implies that
$$dh(B_t) = h_x(B_t)\, dB_t^1 + h_y(B_t)\, dB_t^2.$$
Suppose $K \subset D$ is a compact set with $z \in \mathrm{int}(K)$, and let $\tau' = \inf\{t : B_t \in \partial K\}$. Then $h(B_{t\wedge\tau'})$ is a bounded martingale. It follows that
$$\mathbb{E}^z\left[h(B_{t\wedge\tau'})\right] = \mathbb{E}^z[h(B_0)] = h(z).$$
The left-hand side is the same as $MV(z; f, \partial K)$.

Proposition 2.4 (Dirichlet problem)**.** *Suppose $D \in \mathcal{D}$, and h is a bounded continuous function on ∂D. Then there exists a unique bounded continuous function $h : \overline{D} \to \mathbb{R}$ that extends h and is harmonic in D. In fact, for every $z \in D$,*
$$h(z) = \mathbb{E}^z\left[h(\tau_D)\right]. \tag{4}$$

Proof. If h is defined by (4), then h is locally integrable and satisfies the mean value property. Hence, h is harmonic. Conversely if h is harmonic in D and continuous on ∂D, then $M_t = h_{t\wedge\tau_D}$ is a continuous martingale, and (4) satisfies the mean value property. We need to show that h defined as in (4) is continuous on ∂D, and this uses the fact that every point in ∂D is a regular point. □

The assumption that h is bounded is necessary for uniqueness. For example if $D = (0, \infty)$ and $h(0) = 0$, there are an infinite number of harmonic extensions to D given by $h(x) = cx$.

As we will see below, if a Brownian motion starts at $z \in \mathbb{D}$, then,
$$\mathbb{P}^z\{B_{\tau_\mathbb{D}} \in V\} = \frac{1}{2\pi}\int_{|\zeta|=1} \frac{1-|z|^2}{|\zeta - z|^2}\, 1_V(\zeta)\, d|\zeta|,$$
and hence
$$\mathbb{E}^z\left[F(B_{\tau_\mathbb{D}})\right] = \frac{1}{2\pi}\int_{|\zeta|=1} \frac{1-|z|^2}{|\zeta - z|^2}\, F(\zeta)\, d|\zeta|. \tag{5}$$
One can verify the last equality using the previous proposition and checking that the right-hand side is harmonic in $\mathbb{D}$ and obtains the correct boundary value. Using this we can derive the fundamental facts about harmonic functions.

Proposition 2.5 (Derivative estimates)**.** *For every positive integer k, there exists $c_k < \infty$ such that if D is a domain, $h : D \to \mathbb{R}$ is harmonic, and $z \in D$ with* $\mathrm{dist}(z, D) \geq \epsilon$*, then for* $0 \leq j \leq k$*,*

$$|\partial_x^j \partial_y^{k-j} h(z)| \leq c_k \, \epsilon^{-k} \sup\{|h(w)| : |w - z| < \epsilon\}.$$

Proof. By considering $\tilde{h}(w) = h(z + \epsilon w)$, we can see that it suffices to prove the result for $D = \mathbb{D}, z = 0$. In this case we can differentiate under the integral in (5). □

Proposition 2.6 (Harnack principle)**.** *If D is a domain, then for every compact $K \subset D$ there exists $c = c(K, D) < \infty$ such that if $h : D \to (0, \infty)$ is harmonic, then*

$$h(z) \leq c \, h(w), \quad z, w \in K.$$

Proof. Since D is connected, by choosing K larger if necessary, we may assume that K is connected. If $D \supset \mathbb{D}$ and $K = \{|w| \leq 1/2\}$, then the result follows from the explicit form of h in (5). More generally, we can cover K by a finite collection of open balls $\mathbb{D}_r(\zeta_j), j = 1, \ldots, N$ with $\mathbb{D}_{r - \log 2}(\zeta) \subset D$. We write $w \sim z$, if one of these balls contains both w, z. We say that w and z are connected if there exists a sequence $z = z_0, z_1, \ldots, z_k = w$ with $z_{j-1} \sim z_j$ for each j. We claim that all points in K are connected. Indeed, let U_1 be the union of all the disks $\mathbb{D}_r(\zeta_j), j = 1, \ldots, N$ for which z is connected to ζ_j and let U_2 be the union of the other disks. If $U_2 \neq \emptyset$, then U_1, U_2 disconnect K. Hence, for $z, w \in K$ we can find a sequence $z = z_0, z_1, \ldots, z_k = w$ with $z_{j-1} \sim z_j$ for all j. By "erasing loops" if necessary, we can guarantee that $k \leq N$, and hence, and $f(z) \leq c_0^N \, f(w)$. □

It will be useful to have the following convention.

Convention. Suppose D is a domain and $h : D \to \mathbb{R}$ is a harmonic function. We say $h : \overline{D} \to \mathbb{R}$ is an extension of the harmonic function to the boundary, if $\overline{h}$ is continuous at all regular points of ∂D.

While we have stated the derivative estimates and Harnack principle for harmonic functions in $\mathbb{R}^2$, the analogous results hold for harmonic functions in $\mathbb{R}^d$.

Proposition 2.7 (Schwarz reflection, harmonic functions)**.** *Suppose h is a harmonic function defined on $\mathbb{H} \cap \mathbb{D}$ with boundary value* 0 *on* $(-1, 1)$*. If h is extended to $\mathbb{D}$ by $h(x - iy) = -h(x + iy)$ then h is harmonic on $\mathbb{D}$.*

Proof. We show that h has the mean-value property. In other words, we need to show that if $z \in \mathbb{D}$, $r < \mathrm{dist}(z, \partial \mathbb{D})$, $\mathcal{B} = \{w : |z - w| < r\}$, and $\tau = \inf\{t : B_t \in \partial \mathcal{B}\}$, then

$$h(z) = \mathbb{E}^z[h(B_\tau)].$$

If $z \in \mathbb{R}$ this is immediate by symmetry and the case $\mathrm{Im}(z) < 0$ is identical to $\mathrm{Im}(z) > 0$ so we will assume $\mathrm{Im}(z) > 0$. We also assume that $r > \mathrm{Im}(z)$, for otherwise this follows immediately from the harmonicity of h on $\mathbb{H} \cap \mathbb{D}$. Let $V_- = \partial\mathcal{B} \cap \{\mathrm{Im}(\zeta) < 0\}$, $V_+ = \{w : \overline{w} \in V_-\}$ and $V = V_- \cup V_+$. Note that $V_+ \subset \mathcal{B}$. Let $\partial^* = \partial[\mathcal{B} \cap \mathbb{H}]$. We define a sequence of stopping times. Let $\rho_0 = 0$ and

$$\sigma_j = \tau \wedge \inf\{t \geq \rho_{j-1} : B_t \in \partial^*\},$$

$$\rho_j = \tau \wedge \inf\{t \geq \sigma_j : B_t \in V\},$$

Harmonicity in $\mathbb{H} \cap \mathbb{D}$ shows that for each j, $\mathbb{E}^z[h(B_{\sigma_{j+1}})] = \mathbb{E}^z[h(B_{\rho_j})]$ and symmetry shows that $\mathbb{E}^z[h(B_{\rho_j})] = \mathbb{E}^z[h(B_{\sigma_j})]$. Therefore,

$$\mathbb{E}[h(B_\tau)] = \lim_{j \to \infty} \mathbb{E}^z[h(B_{\sigma_j})] = h(z).$$

□

2.4 Harmonic functions and holomorphic functions

We assume the following facts from an undergraduate course in complex variables.

Definition A function $f : u + iv$ on a domain $D \subset \mathbb{C}$ is called *holomorphic* or *analytic* on D if any of the following equivalent facts hold.

- The derivative
$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$
exists.
- (u, v) is a C^2 function satisfying the Cauchy-Riemann equations
$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v.$$
- For each $z \in D$, we can expand f in a power series
$$f(w) = \sum_{j=0}^{\infty} a_j \, (w - z)^j,$$
where the radius of convergence is at least $\mathrm{dist}(z, \partial D)$.

From the Cauchy-Riemann equations, one can see that u, v are harmonic functions. The next proposition, which is a standard result from a first course in complex variables, gives a partial converse to this fact.

Proposition 2.8. *Suppose D is a simply connected domain and u is harmonic function on D. Then there is a harmonic function v on D, which is unique up to an additive constant, such that $f(z) := u(z) + iv(z)$ is holomorphic.*

Sketch of proof. We use the Cauchy-Riemann equations to find v. Let us fix $z_0 \in D$ and arbitrarily choose $v(z_0) = 0$. If $\gamma : [0,1] \to D$ is a smooth curve with $\gamma(0) = z_0, \gamma(1) = z$, then we define

$$v(z) = \int_\gamma \partial_n u \cdot d\gamma := \int_0^1 \left[\partial_y u(\gamma(t))\, \partial_x \gamma(t) - \partial_x u(\gamma(t))\, \partial_y \gamma(t)\right] \, dt.$$

In order to show this is well defined, we need to show that we get the same value for $v(z)$ regardless of the curve γ. Equivalently, we need to show that if $\gamma(1) = z_0$, then

$$\int_\gamma \partial_n u \cdot d\gamma = 0.$$

If D is simply connected, then the region(s) bounded by γ are entirely in D, and this identity follows from Green's theorem and the fact that u is harmonic. By construction, u, v satisfy the Cauchy-Riemann equation and hence $f = u + iv$ is holomorphic. To show uniqueness, suppose that $\tilde{f} = u + i\tilde{v}$ is also holomorphic. Then $f - \tilde{f}$ is holomorphic and only takes on imaginary values. Hence $f - \tilde{f}$ is constant.

□

The function v is often called the *complex conjugate.* We have used simple connectedness to conclude that there exists an complex conjugate v defined on all of D. Such an extension does not necessarily exist if the domain is not simply connected. For example, if $u(z) = \log|z|$ on $D = \{z : 0 < |z| < 1\}$, then u is harmonic, but there is no holomorphic extension to all of D. However, regardless of the topology of D, we can always find conjugates v defined in a neighborhood of z_0. When trying to determine if two complex domains are conformally equivalent, it is often the case that one can determine the real or imaginary part (or, perhaps, the radial part which is the real part of the exponential, or something similar). This determines the function (up to a constant) locally and then the question becomes whether or not one can extend it to the entire domain D.

Proposition 2.9 (Schwarz reflection, holomorphic functions)**.** *Suppose $f = u + iv$ is a holomorphic function on $\mathbb{D}_+$ with $\lim_{y \downarrow 0} v(x + iy) = 0$ for all $-1 < x < 1$. Then f can be extended to a holomorphic function on $\mathbb{D}$ satisfying $f(\bar{z}) = \bar{f(z)}$.*

Proof. By Proposition 2.7, if we extend v to $\mathbb{D}$ by $v(x) = 0$ and $v(\bar{z}) = -v(z)$, then v is harmonic in $\mathbb{D}$. By Proposition 2.8, there exists u^* (unique up to an additive constant) such that $u^* + iv$ is holomorphic. By uniqueness, we can choose the constant so that $u^* \equiv u$ on $\mathbb{D}_+$. Since $\hat{f}(z) = u(\bar{z}) - iv(\bar{z})$ is holomorhphic in $-\mathbb{D}_+$, we see that $u^* \equiv u + c_0$ in $-\mathbb{D}_+$ for some $c_0 \in \mathbb{R}$. Continuity at the real axis shows that $c_0 = 0$.

□

We have stated Schwarz reflection for functions in $\mathbb{D}_+$, but there is an immediate corollary for functions in $\epsilon\mathbb{D}_+$.

The term conjugate is overused in complex variables! I have given lectures where I have used conjugate three different ways in the same lecture — as the conjugate of a number, as the complex conjugate function above, and also in the algebraic sense of a conjugate function $\tilde{f} = g^{-1} \circ f \circ g$.

2.5 Conformal invariance

If f is a holomorphic function with $f(0) = 0, f'(0) \neq 0$, then locally near zero f looks like a dilation by $|f'(0)|$ and a rotation by $\arg f'(0)$. Brownian motion is invariant under rotation and is also invariant under scaling if one changes the parametrization appropriately. This is the basic reason why the following theorem holds.

Theorem 1. *Suppose $D \subset \mathbb{C}$ is a domain and B_t is a complex Brownian motion starting at $z \in D$. Suppose $f : D \to \mathbb{C}$ is a nonconstant holomorphic function. Let*

$$\xi = \int_0^{\tau_D} |f'(B_s)|^2 \, ds \in (0, \infty],$$

and for $t < \xi$, define $\sigma(t) < \tau_D$ by

$$\int_0^{\sigma(t)} |f'(B_s)|^2 \, ds = t.$$

Then $Y_t = f(B_{\sigma(t)}), 0 \leq t < \xi$ is a complex Brownian motion.

We will need a lemma that states in some sense that all stochastic integrals are time changes of standard Brownian motions. Indeed, a stronger fact is true that we will not prove — all continuous martingales are time changes of standard Brownian motions.

Lemma 2.10. *Suppose B_t is a standard one-dimensional Brownian motion with filtration $\{\mathcal{F}_t\}$ and suppose that A_t is a continuous, adapted process such that there exist $0 < c_1 < c_2 < \infty$ with $c_1 \leq |A_t| \leq c_2$. Let*

$$X_t = \int_0^t A_s \, dB_s,$$

and let

$$\sigma(r) = \inf\{t : \langle X \rangle_t = r\},$$

that is

$$\int_0^{\sigma(r)} A_s^2 \, ds = r.$$

Suppose that for all t, $\mathbb{P}\{\sigma(t) < \infty\} = 1$. Then $W_r := X_{\sigma(r)}$ is a standard Brownian motion with respect to $\tilde{\mathcal{F}}_r = \mathcal{F}_{\sigma(r)}$.

Sketch of Proof. To prove this, one shows that conditioned on $\tilde{\mathcal{F}}_s$ the distribution of $W_{r+s} - W_s$ is that of a Brownian motion with variance r^2. We will do this in the case $s = 0$; the general case is similar. If $\lambda \in \mathbb{R}$, let

$$K_t = \exp\{i\lambda X_t\}.$$

(If one does not want to use Itô's formula with with complex valued processes one can write this as $\cos(\lambda X_t) + i \sin(\lambda X_t)$.) Itô's formula shows that

$$dK_t = K_t \left[i\,\lambda\, A_t \, dB_t - \frac{\lambda^2}{2} \, A_t^2 \, dt \right] = K_t \left[i\,\lambda\, A_t \, dB_t - \frac{\lambda^2}{2} \, d\langle X \rangle_t \right].$$

If $M_t = \exp\{\lambda^2 \langle X \rangle_t / 2\} \, K_t$, then M_t is a local martingale satisfying,

$$dM_t = i \, \lambda M_t \, dB_t.$$

Note that $M_{t \wedge \sigma(r)}$ is a bounded martingale, and hence the optional sampling theorem implies that

$$\mathbb{E}\left[M_{\sigma(r)}\right] = \mathbb{E}[M_0] = 1.$$

But, $M_{\sigma(r)} = e^{\lambda^2 r/2} \, \exp\{i\lambda X_{\sigma(r)}\}$, and hence

$$\mathbb{E}\left[e^{i\lambda W_r}\right] = e^{-\lambda^2 r/2}.$$

Since the characteristic function determines the distribution, we see that $W_r \sim N(0, r)$. □

Proof of Theorem 1. We will give a sketch of the proof relying on some facts from stochastic calculus.

Let $U \subset D$ be a subdomain with $\overline{U}$ compact containing none of the zeros of f', and let $\tau = \tau_U < \tau_D$. Let us write $B_t = B_t^1 + iB_t^2$ and let $X_t = u(B_t^1, B_t^2), Y_t = v(B_t^1, B_t^2)$. Using the fact that u, v are harmonic functions, Itô's formula and the Cauchy-Riemann equations give

$$dX_t = u_x(B_t) \, dB_t^1 + u_y(B_t) \, dB_t^2,$$

$$\begin{aligned} dY_t &= v_x(B_t) \, dB_t^1 + v_y(B_t) \, dB_t^2 \\ &= -u_y(B_t) \, dB_t^1 + u_x(B_t) \, dB_t^2. \end{aligned}$$

Note that $\partial_t \sigma(t) = |f'(B_t)|^{-2} = |\nabla u(B_t)|^{-2}$. If we let $\hat{X}_t = X_{\sigma(t)}, \hat{Y}_t = Y_\sigma(t)$, then

$$d\hat{X}_t = \frac{u_x(\hat{B}_t)}{|\nabla u(\hat{B}_t)|} \, dW_t^1 + \frac{u_y(\hat{B}_t)}{|\nabla u(\hat{B}_t)|} \, dW_t^2,$$

$$d\hat{Y}_t = \frac{-u_y(\hat{B}_t)}{\nabla u(\hat{B}_t)} \, dW_t^1 + \frac{u_x(\hat{B}_t)}{|\nabla u(\hat{B}_t)|} \, dW_t^2,$$

where W_t^1, W_t^2 are independent, standard Brownian motions. This means that $(\hat{X}_t, \hat{Y}_t)$ are independent standard Brownian motions, that is,

$$\hat{B}_t = \hat{X}_t + i \, \hat{Y}_t,$$

is a standard complex Brownian motion at least for $t \leq \tau$. Since this holds for every U, and with probability one the Brownian motion avoids the singular points of U, we can conclude that it holds for $t < \tau_D$.

□

The statement of Theorem 1 is a little nicer if f is a conformal transformation. We say that $f : D \to D'$ is a *conformal transformation* if f is holomorphic, one-to-one, and onto.

Theorem 2. *Suppose D is a domain in $\mathbb{C}$ and $f : D \to f(D)$ is a conformal transformation. Suppose B_t is a complex Brownian motion starting at $z \in D$. Let*

$$\xi = \int_0^{\tau_D} |f'(B_s)|^2 \, ds \in (0, \infty],$$

and for $s < \xi$, define $\sigma(s) < \tau_D$ by

$$\int_0^{\sigma(s)} |f'(B_u)|^2 \, du = s.$$

Then $Y_s := f(B_{\sigma(s)}), 0 \leq s < \xi$ is a complex Brownian motion, and $\xi = \tau_{f(D)} = \inf\{t : Y_t \notin f(D)\}$.

Proof. Note that if $\xi < \infty$, we can extend $Y_s, 0 \leq s \leq \xi$ by continuity. If $\xi < \infty$, we claim that $Y_\xi \in \partial f(D)$. Indeed, if $Y_\xi = w \in f(D)$, then $B_{\tau_D -} = f^{-1}(w) \in D$.

□

2.5.1 Example: Recurrence

Here we show that two-dimensional Brownian motion is *neighborhood recurrent.* To be more precise, with probability one, for all $z \in \mathbb{C}, \epsilon > 0, T < \infty$, there exists $t > T$ with $|B_t - z| < \epsilon$. It suffices to prove this for z with rational coordinates, and the proof is essentially the same for all of them, so let us consider $z = 0$. Let B_t be a complex Brownian motion and let $f(z) = e^z, Y_t = f(B_t) = e^{B_t} = e^{B_t^1} e^{iB_t^2}$. Then Y_t is a time change of a Brownian motion. Since $|Y_t| = e^{B_t^1}$, we see from the recurrence of the one-dimensional Brownian motion B^1 that

$$\liminf_{t \to \infty} |Y_t| = 0.$$

We can also see from this that the Brownian motion is not *pointwise recurrent.* Indeed with probability one, a Brownian motion never visits the origin after time zero. This is obvious for Y_t since 0 is not in the range of the exponential function.

An important corollary of the neighborhood recurrence of Brownian motion is the following: if D is a domain with at least one regular boundary point, then for all $z \in D$,

$$\mathbb{P}^z\{\tau_D < \infty\} = 1.$$

Indeed, if w is a regular point, then there exists a δ such that if the Brownian motion is within distance δ of w, then with probability $1/2$ it leaves the domain before it goes distance one from z. Since we keep returning to the δ neighborhood of z we get infinitely many chances to escape D near w and we will eventually succeed. This fact is used implicitly in the following definition.

Definition If $D \in \mathcal{D}^*$ and $z \in D$, then the *harmonic measure* $\mathrm{hm}_D(z, \cdot)$ is defined to be the distribution of $B(\tau_D)$ assuming $B_0 = z$. In other words, the probability that a Brownian motion starting at z exits D at V is $\mathrm{hm}_D(z, V)$. More generally, if f is a function defined on ∂D, we let

$$\mathrm{hm}_D(z, f) = \mathbb{E}^z\left[f(B_\tau)\right] = \int_{\partial D} f(w) \, d\mathrm{hm}_D(z, w).$$

If $f : D \to f(D)$ is a conformal transformation, that extends to a homemomorphism of $\overline{D}$, then

$$\mathrm{hm}_{f(D)}(f(z), f(V)) = \mathrm{hm}_D(z, V). \tag{6}$$

There is a similar formula that holds if f does not extend to a homemorphism, but to explain it requires a discussion of prime ends which we will do later. We For example, if $D = \mathbb{D} \cap \mathbb{H}$, then $f(z) = z^2$ is a conformal transformation of D onto $f(D) = \mathbb{D} \setminus [0, 1)$. The boundary point $1/4 \in \partial f(D)$ corresponds to two equivalences classes, one for curves approaching $1/4$ from above and the other for curves approaching $1/4$ from below. These correspond to curves in D that leave D at $1/2$ and $-1/2$ respectively. Examples like this where there are "two-sided" points are easy to handle, even if we have to be careful about our notation. Once we have defined prime ends, the generalization will be straightforward.

The definition of harmonic measure does not require any smoothness of the boundary. However, if the boundary is nice, then one can write harmonic measure as an integral of a kernel called the *Poisson kernel*. Rotational invariance of Brownian motion shows that harmonic measure on $\mathbb{D}$ centered at zero is the uniform distribution. We define $H_D(z, w)$ to be the Poisson kernel normalized so that

$$H_{\mathbb{D}}(0, e^{i\theta}) = \frac{1}{2}.$$

In other words, if V is sufficiently smooth,

$$\mathrm{hm}_D(z, V) = \frac{1}{\pi} \int_V H_D(z, w) \, |dw|. \tag{7}$$

Here $|dw|$ represents integration with respect to arc length, that is, a traditional line integral from vector calculus rather than a complex integral along a curve. The term "sufficiently smooth" is a little vague. We will only use the Poisson kernel at places where ∂D is locally an analytic curve. If $f : D \to f(D)$ is a conformal transformation, D is locally analytic at w,. and $f(D)$ is locally analytic at $f(w)$, then the Poisson kernel satisfies the "conformal covariance" relation

$$H_D(z, w) = |f'(w)| \, H_{f(D)}(f(z), f(w)).$$

Suppose $f : \mathbb{D} \to D$ is a conformal transformation. Then boundary ∂D can be very rough. For example, there can be points $w \in \partial D$ such that there is no continuous path $\eta : [0, 1) \to D$ with $\eta(1-) = w$. However, the harmonic measure of such points has to be zero. This is immediate from the definition of harmonic measure in terms of Brownian motion.

The Poisson kernel is naturally defined up to a constant. We made a nonstandard choice in (7) which is convenient for later work with Schramm-Loewner evolution. Under this normalization, if $x > 0$,

$$H_{\mathbb{H}}(0, x) = x^{-2}.$$

2.5.2 Example: the annulus

Recall that $\mathbb{A}_r$ is the annulus $\mathbb{A}_r = \mathbb{D} \setminus \overline{\mathbb{D}_r} = \{z : e^{-r} < |z| < 1\}$.

Proposition 2.11. *If $z \in \mathbb{A}_r$ and $\tau = \tau_{\mathbb{A}_r}$, then*

$$\mathbb{P}^z\{|B_\tau| = e^{-r}\} = \frac{-\log|z|}{r}. \tag{8}$$

Proof. Let us give two similar proofs. First, note that $\phi(z) := -\log|z|$ is a bounded harmonic function in $\mathbb{A}_r$ that is continuous on $\overline{\mathbb{A}_r}$. This can be checked by differentiation or by noting that it is the real part of the (locally) analytic function $-\log z$. Therefore, by Proposition 2.4,

$$\phi(z) = \phi(B_0) = \mathbb{E}^z\left[\phi(B_\tau)\right] = r\,\mathbb{P}^z\left\{|\phi(B_\tau)| = e^{-r}\right\}.$$

Alternatively we can let $Y_t = \exp\{B_t\}$. Then the probability is the same as the probability that the one-dimensional Brownian motion B_t^1 starting at $\log|z|$ reaches level $-r$ before reaching level 0 which by the gambler's ruin estimate is $-\log|z|/r$.

□

Proposition 2.12. *Suppose ϕ is a harmonic function on the annulus $\mathbb{A}_r = \{e^{-r} < |z| < 1\}$. Let*

$$M_s = MV(\phi, 0, e^{-s}) = \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{-s+i\theta})\,d\theta,$$

be the average value of ϕ on the circle of radius e^{-s}. Then there exist a, b such that

$$M_s = as + b, \quad 0 < s < r.$$

Moreover, for each s,

$$\int_{C_s} \partial_n\phi(w)\,|dw| = 2\pi a,$$

where n is the inward unit normal.

Proof. Suppose that $0 < p < s < q < r$. Let B_t be a Brownian motion starting uniformly on C_s, the circle of radius e^{-s}, and let T be the first time t that $B_t \in C_p \cup C_q$. Then

$$\mathbb{P}\{B_T \in C_p\} = \frac{q-s}{q-p}.$$

By rotational symmetry, given that $B_T \in C_p$ (or given $B_T \in C_q$), the distribution of the angular part is uniform. Therefore,

$$M_s = \frac{q-s}{q-p}\,M_p + \frac{s-p}{q-p}\,M_q = \frac{M_q - M_p}{q-p}\,s + \frac{qM_p - pM_q}{q-p}.$$

This establishes the result for $p < s < q$ and by letting $p \to 0, q \to r$, we get the first assertion. For the final assertion note that

$$2\pi a = 2\pi\partial_s M_s = \partial_s \int_0^{2\pi} \phi(e^{-s+i\theta})\,d\theta = \partial_s \int_{C_s} \phi(w)\,e^s\,|dw| = \int_{C_s} \partial_n\phi(w)\,|dw|.$$

□

3 Green's function

The Green's function $G_D(z,w)$ is the normalized probability that a Brownian motion starting at z visits w before leaving D. As stated this does not make sense since the probability that the Brownian motion visits w is zero. However, we can make sense of it as a limit,

$$G_D(z,w) = \lim_{\epsilon \downarrow 0} \log[1/\epsilon]\, \mathbb{P}^z\{\mathrm{dist}(w, B[0,\tau_D]) \le \epsilon\}.$$

We will show this limit exists in this section and derive some properties. We will first consider the case $w = 0$ and write just $G_D(z)$ for $G_D(z,0)$. Throughout this section, we let

$$\sigma_s = \inf\{t : |B_t| = e^{-s}\}.$$

We will make no topological assumptions about the domain D. We only require that the boundary contain a regular point, that is, that $\mathbb{P}^z\{\tau_D < \infty\} = 1$.

Definition Let $\mathcal{U}_r$ (resp., $\mathcal{U}_r^{\mathrm{s}}$) denote the set of domains (resp., simply connected domains) containing the origin with at least one regular boundary point and $\mathrm{dist}(0,\partial D) \ge r$. If $r = 1$ we write just $\mathcal{U}, \mathcal{U}^{\mathrm{s}}$. There are natural bijections $\mathcal{U} \leftrightarrow \mathcal{U}_r$ and $\mathcal{U}^{\mathrm{s}} \leftrightarrow \mathcal{U}_r^{\mathrm{s}}$ given by $D \leftrightarrow rD$.

Proposition 3.1. *Suppose $D \in \mathcal{U}_r$ and $z \in D \setminus \{0\}$. Then the limit*

$$G_D(z) = \lim_{s \to \infty} s\, \mathbb{P}^z\{\sigma_s < \tau_D\},$$

exists and lies in $(0,\infty)$. Moreover, $G_{rD}(rz) = G_D(z)$.

We note that (8) establishes the result for $D = \mathbb{D}$ for which

$$G_D(z) = -\log|z|.$$

Proof. We write $\tau = \tau_D$. It suffices to prove the result for $D \in \mathcal{U}$, after which we can use conformal invariance of Brownian motion to see that

$$G_{e^u D}(e^u z) = \lim_{s\to\infty} s\, \mathbb{P}^{e^u z}\{\sigma_s < \tau_{e^u D}\} = \lim_{s\to\infty} s\, \mathbb{P}^z\{\sigma_{s-u} < \tau\} = \lim_{s\to\infty} (s-u)\, \mathbb{P}^z\{\sigma_{s-u} < \tau\} = G_D(z).$$

Since $\mathbb{D} \subset D$ and ∂D contains a regular point, the Harnack principle (Proposition 2.6) implies that

$$\inf_{|\zeta|=1} \mathbb{P}^\zeta\{\tau < \sigma_1\} =: \rho = \rho_D > 0. \tag{9}$$

For $s > 1$, let

$$q_s = \sup_{|\zeta|=1} \mathbb{P}^\zeta\{\sigma_s < \tau\}.$$

By the strong Markov property, this is the same as the supremum over $|\zeta| \ge 1$. We claim that

$$q_s \le \frac{1}{s\rho}, \tag{10}$$

To see this, note that if $|\zeta| = 1$ and $s > 1$, then

$$\mathbb{P}^\zeta\{\sigma_s < \tau\} \le \mathbb{P}^\zeta\{\sigma_1 < \tau\} \sup_{|\zeta'|=1/e} \mathbb{P}^{\zeta'}\{\sigma_s < \tau\} \le (1-\rho) \sup_{|\zeta'|=1/e} \mathbb{P}^{\zeta'}\{\sigma_s < \tau\}.$$

If $|\zeta'| = 1/e$, then using (8), we get

$$\begin{aligned}\mathbb{P}^{\zeta'}\{\sigma_s < \tau\} &= \mathbb{P}^{\zeta'}\{\sigma_s < \sigma_0\} + \mathbb{P}^{\zeta'}\{\sigma_s > \sigma_0\}\,\mathbb{P}^{\zeta'}\{\sigma_s < \tau \mid \sigma_s > \sigma_0\} \\ &\leq \frac{1}{s} + \frac{s-1}{s}\, q_s.\end{aligned}$$

By taking the supremum over $|\zeta| = 1$, we see that

$$q_s \leq (1-\rho)\,\left[\frac{1}{s} + \frac{s-1}{s}\, q_s\right] \leq \frac{1}{s} + (1-\rho)\, q_s.$$

which gives (10).

We now fix z and let $f(s) = \mathbb{P}^z\{\sigma_s < \tau\}$. Then, if $|z| > e^{-s}$,

$$\begin{aligned}f(s+1) = \mathbb{P}^z\{\sigma_{s+1} < \tau\} &= \mathbb{P}^z\{\sigma_s < \tau\}\,\mathbb{P}^z\{\sigma_{s+1} < \tau \mid \sigma_s < \tau\} \\ &= f(s)\,\mathbb{P}^z\{\sigma_{s+1} < \tau \mid \sigma_s < \tau\}.\end{aligned}$$

If $|\zeta| = e^{-s}$, then (8) and (10) imply that

$$\begin{aligned}\mathbb{P}^{\zeta}\{\sigma_{s+1} < \tau\} &= \mathbb{P}^{\zeta}\{\sigma_{s+1} < \sigma_0\} + \mathbb{P}^{\zeta}\{\sigma_0 < \sigma_{s+1}\}\,\mathbb{P}^{\zeta}\{\sigma_{s+1} < \tau \mid \sigma_0 < \sigma_{s+1}\} \\ &\leq \frac{s}{s+1} + \frac{1}{s+1}\,\frac{1}{\rho\,(s+1)} = \frac{s}{s+1} + O(s^{-2}),\end{aligned}$$

where here and throughout the remainder of the proof, the $O(\cdot)$ terms can depend on D. Therefore,

$$f(s+1) = f(s)\,\left[1 - \frac{1}{s+1} + O(s^{-2})\right],$$

$$\log f(s+1) = \log f(s) - \frac{1}{s+1} + O(s^{-2}).$$

This equation implies the existence of a constant which we call $G_D(z)$ such that for integer s,

$$\mathbb{P}^z\{\sigma_s < \tau\} = f(s) = \frac{G_D(z)}{s}\,\left[1 + O(s^{-1})\right]. \tag{11}$$

If $0 \leq u \leq 1$, the same argument shows that

$$f(s+u) = f(s)\,\frac{s}{s+u}\,\left[1 + O(s^{-1})\right] = \frac{G_D(z)}{s+u}\,\left[1 + O(s^{-1})\right],$$

and hence (11) holds for all s.

□

We extend $G_D(z)$ to be a function on $\mathbb{C}\setminus\{0\}$ by setting $G_D(z) = 0,\ z \notin D$. If D is open but not connected and $\tilde{D}$ is the connected component of D containing the origin, we define $G_D(z) = G_{\tilde{D}}(z)$. If $\mathbb{C}\setminus D$ is compact, then we can extend G_D to infinity,

$$G_D(\infty) = \lim_{z\to\infty} G_D(z).$$

We state the next proposition for $D \in \mathcal{U}$ but it extends immediately to a result about $D \in \mathcal{U}_r$ by scaling.

Proposition 3.2. *There exists $c < \infty$ such that if $D \in \mathcal{U}$, the following holds.*

1. *G_D is a positive harmonic function on $D \setminus \{0\}$ that vanishes on ∂D.*
2. *There exists $c_D < \infty$ such that for all z,*
$$G_D(z) \leq \log_+(1/|z|) + c_D.$$
3. *G_D is continuous at every regular point of ∂D.*
4. *If h_D is defined by*
$$h_D(z) = G_D(z) + \log|z|, \quad z \in \mathbb{D} \setminus \{0\},$$
$$h_D(0) = \frac{1}{2\pi} \int_0^{2\pi} G_D(e^{r+i\theta})\, d\theta,$$
then h_D is a harmonic function on D.
5. *If $|z| < e^{-1}$,*
$$|h_D(z) - h_D(0)| \leq c\, h_D(0)\, |z|. \tag{12}$$
6. *If $z \in D$,*
$$h_D(z) = \mathbb{E}^z\left[\log|B_\tau|\right] - \lim_{r\to\infty} r\, \mathbb{E}^z[h_D(B_{\sigma_{-r}}); \sigma_{-r} < \tau],$$
where $\tau = \tau_D = \inf\{t > 0 : B_t \notin D\}$. In particular, if D is bounded, then
$$h_D(z) = \mathbb{E}^z\left[\log|B_\tau|\right].$$
7. *If $D' \subset D$, then $G_{D'}(z) \leq G_D(z)$.*
8. *If $D_1 \subset D_2 \subset \cdots$ and $D = \bigcup_{n=1}^\infty D_n$, then for $z \in D$,*
$$G_D(z) = \lim_{n\to\infty} G_{D_n}(z). \tag{13}$$
9. *If $f : D \to f(D)$ is a conformal transformation with $f(0) = 0$, then $G_{f(D)}(f(z)) = G_D(z)$.*

Proof.

1. We have shown that $G_D(z) > 0$ if $z \in D$ and it is defined to be zero on ∂D. Suppose $U \subset D$ is a disk with $\overline{U} \subset D \setminus \{0\}$. Then by the definition of G_D we can see that
$$G_D(z) = \mathbb{E}^z\left[G_D(B_{\tau_U})\right] = \mathrm{MV}(z, G_D, U)$$
from which we see that G_D is harmonic on $D \setminus \{0\}$.
2. Note that (10) implies that $G_D(z) \leq 1/\rho$ for $|z| = 1$. If $|z| = e^{-r}$ with $r < s$, then
$$\mathbb{P}^z\{\sigma_s < \tau_D\} \leq \mathbb{P}^z\{\sigma_s < \sigma_0\} + \mathbb{P}^z\{\sigma_0 < \sigma_s < \tau_D\} \leq \frac{r}{s} + \frac{s-r}{s^2\rho}.$$
Letting $s \to \infty$, we see that $G_D(z) \leq r + (1/\rho)$.

3. Let z be a regular boundary point and let $\xi = \inf\{t : |B_t - z| = |z|/2\}$. Then for $|w-z| < |z|/2$,
$$G_D(w) \leq \alpha\, \mathbb{P}^w\{\xi < \tau_D\}$$
where $\alpha = \sup\{G_D(\zeta) : |z-\zeta| = |z|/2\} < \infty$. For every $\epsilon > 0$, we can find $\delta > 0$ such that $|w-z| < \delta$ implies that $\mathbb{P}^w\{\xi < \tau_D\} < \epsilon$ and hence $G_D(w) \leq \epsilon\, \alpha$.

4. For $z \in D \setminus \{0\}$, let
$$h_D(z) = G_D(z) + \log|z|.$$
Suppose $|z| < 1$. Then,
$$\begin{aligned} \mathbb{P}^z\{\sigma_s < \tau\} &= \mathbb{P}^z\{\sigma_s < \sigma_0\} + \mathbb{P}^z\{\sigma_0 < \sigma_s < \tau\} \\ &= \frac{-\log|z|}{s} + \int_{C_0} \mathbb{P}^w\{\sigma_s < \tau\}\, \mathrm{hm}_{\mathbb{A}_s}(z, dw). \end{aligned}$$
If we multiply both sides by s and take the limit as $s \to \infty$, we get
$$G_D(z) = -\log|z| + \int_{C_0} G_D(w)\, \mathrm{hm}_{\mathbb{D}}(z, dw) = -\log|z| + \frac{1}{\pi}\int_0^{2\pi} G_D(e^{i\theta})\, H_{\mathbb{D}}(z, e^{i\theta})\, d\theta.$$
In other words,
$$h_D(z) = \frac{1}{\pi}\int_0^{2\pi} G_D(e^{i\theta})\, H_{\mathbb{D}}(z, e^{i\theta})\, d\theta, \quad 0 < |z| < 1,$$
which can be extended to 0 by setting $z = 0$ on the right-hand side.

5. Using the exact form of the Poisson kernel, we can see that
$$H_{\mathbb{D}}(z, e^{i\theta}) = \frac{1}{2} + O(|z|),$$
and hence
$$|G_D(z) + \log|z| - h_D(0)| \leq c\,|z|\, h_D(0).$$

6. Let
$$\theta = \theta_z = \liminf_{r\to\infty} r\, \mathbb{P}^z\left\{\tau_D > \sigma_{-r}\right\}.$$
We claim that
$$\theta = \lim_{r\to\infty} r\, \mathbb{P}^z\left\{\tau_D > \sigma_{-r}\right\}.$$
To see this we first note that since ∂D is regular, if
$$q(r) = \sup_{|\zeta|=1} \mathbb{P}^\zeta\{\sigma_{-r} < \tau_D\},$$
then $q(r) \to 0$ as $r \to \infty$. Hence, if
$$p(r,s) = \sup_{|\zeta|=e^r} \{\sigma_{-s} < \tau_D\},$$

then for $|\zeta| = e^r$, and $s > r$,

$$\begin{aligned}\mathbb{P}^\zeta\{\sigma_{-s} < \tau_D\} &\leq \mathbb{P}^\zeta\{\sigma_{-s} < \sigma_0\} + \mathbb{P}^\zeta\{\sigma_0 < \sigma_{-s}\}\, \mathbb{P}^\zeta\{\sigma_s < \tau_D \mid \sigma_0 < \sigma_{-s}\} \\ &\leq \frac{r}{s} + q(r)\, p(r,s),\end{aligned}$$

which implies that

$$p(r,s) \leq \frac{r}{s\,(1-q(r))}.$$

Hence,

$$\begin{aligned}\mathbb{P}^z\{\sigma_{-s} < \tau_D\} &= \mathbb{P}^z\{\sigma_{-r} < \tau_D\}\, \mathbb{P}^z\{\sigma_{-s} < \tau_D \mid \sigma_{-r} < \tau_D\} \\ &\leq \mathbb{P}^z\{\sigma_{-r} < \tau_D\}\, \frac{r}{s\,(1-q(r))}.\end{aligned}$$

Therefore,

$$\limsup_{s\to\infty} s\,\mathbb{P}^z\{\sigma_{-s} < \tau_D\} \leq \liminf_{r\to\infty} \frac{r}{1-q(r)}\, \mathbb{P}^z\{\sigma_{-r} < \tau_D\} = \theta.$$

(If this argument seems familiar, we are really just proving the existence of the "Green's function at infinity" which can be obtained from the Green's function at zero by the map $z \mapsto 1/z$.)

Since h_D is bounded on $\{|z| \leq e^r\}$, we have

$$h_D(z) = \mathbb{E}^z[h_D(B_\tau); \tau < \sigma_{-r}] + \mathbb{E}^z[h_D(B_{\sigma_{-r}}); \sigma_{-r} < \tau].$$

Taking limits as $r \to \infty$ and using the monotone convergence theorem, we see that

$$h_D(z) = \mathbb{E}^z[h_D(B_\tau)] + \theta + G_D(\infty).$$

7. Monotonicity in D follows immediately from the definition.

8. Assume $D \in \mathcal{U}$. If $D_1 \subset D_2 \subset \cdots$ and $D = \cup D_n$, then monotonicity implies that the limit

$$G^*(z) := \lim_{n\to\infty} G_{D_n}(z)$$

exists and $G^*(z) \leq G_D(z)$. Assume that $\mathbb{D}_s \subset D_n$. Then for $|w| < e^{-s}$,

$$G_{D_n}(w) \geq G_{\mathbb{D}_s}(w) = G_{\mathbb{D}}(e^s w) = -\log|w| - s.$$

Therefore, for $m \geq n$, $u \geq s$, and all $z \in D_m$,

$$G_{D_m}(z) \geq (u-s)\, \mathbb{P}^z\{\sigma_u < \tau_{D_m}\}.$$

Letting $m \to \infty$, we see that

$$G^*(z) \geq (u-s)\, \lim_{m\to\infty} \mathbb{P}^z\{\sigma_u < \tau_{D_m}\} = (u-s)\, \mathbb{P}^z\{\sigma_u < \tau_D\}.$$

Letting $u \to \infty$, we get $G^*(z) \geq G_D(z)$.

9. Note that it follows from the definition, that for any $u > 0$,

$$\lim_{s\to\infty} s\,\mathbb{P}^z\{\sigma_{s+u} < \tau\} = G(z). \tag{14}$$

Let $C_s = \{\zeta : |\zeta| = e^{-s}\}$ and $\hat{\sigma}_s = \inf\{t : B_t \in f(C_s)\}$. Conformal invariance of Brownian motion implies that for $z \in D, |z| > e^{-s}$,

$$\mathbb{P}^z\{\sigma_s < \tau_D\} = \mathbb{P}^{f(z)}\{\hat{\sigma}_s < \tau_{f(D)}\}.$$

Let $\theta = -\log|f'(0)|$. If $\epsilon > 0$, then for all s sufficiently large,

$$\sigma_{s+\theta+\epsilon} \leq \hat{\sigma}_s \leq \sigma_{s+\theta-\epsilon}.$$

Hence, using (14),

$$G_{f(D)}(f(z)) = \lim_{s\to\infty} s\,\mathbb{P}^{f(z)}\{\sigma_s < \tau_{f(D)}\} = G_D(z).$$

$\square$

Definition If $D \in \mathcal{D}^*$, then the *Green's function* $G_D(z,w)$ is defined for distinct $z, w \in D$ by

$$G_D(z,w) = G_{D-z}(w-z).$$

In other words, if

$$\sigma_s(w) = \inf\{t : |B_t - w| \leq e^{-s}\},$$

then

$$G_D(z,w) = \lim_{s\to\infty} s\,\mathbb{P}^z\{\sigma_s(w) < \tau_D\}.$$

Lemma 3.3. *If $\tau = \tau_{\mathbb{D}}$, and $z \in \mathbb{D}$,*

$$\mathbb{E}^z[\tau_D] = \frac{1}{2}\,[1 - |z|^2].$$

Proof. Itô's formula shows that

$$M_t = |B_t|^2 - 2t$$

is a martingale. Since $\mathbb{E}^z[M_\tau] = \mathbb{E}[M_0] = |z|^2$,

$$2\,\mathbb{E}^z[\tau] = \mathbb{E}^z[B_{\tau_D}^2] - |z|^2 = 1 - |z|^2.$$

$\square$

Proposition 3.4. *If $D \in \mathcal{D}^*$, and $z, w \in D$ are distinct,*

$$\begin{aligned} G_D(z,w) &= \lim_{s\to\infty} e^{2s}\,\mathbb{E}^z\left[\int_0^{\tau_D} 1\{|B_t - w| \leq e^{-s}\}\,dt\right] \\ &= \lim_{s\to\infty} e^{2s} \int_0^\infty \mathbb{P}^z\{|B_t - w| \leq e^{-s}, B[0,t] \subset D\}\,dt. \end{aligned}$$

Proof. Let us first consider $D = \mathbb{D}_{-s}$, $w = 0$, and $|z| = 1$. Let $\tau_s = \tau_{\mathbb{D}_{-s}} = \inf\{t : |B_t| = e^s\}$ and

$$Y_s = \int_0^{\tau_s} 1\{|B_t - w| \le e^s\}\,dt, \qquad F(s) = \mathbb{E}^z[Y_s].$$

Suppose $0 < r < s$ and note that if $|\zeta| = e^r$,

$$\mathbb{E}^\zeta[Y_s] = \mathbb{P}^\zeta\{\tau_0 < \tau_s\}\,\mathbb{E}^\zeta[Y_s \mid \tau_0 < \tau_s] = \frac{s-r}{s}\,F(s),$$

$$F(s) = \mathbb{E}^z\left[Y_s\right] = \mathbb{E}^z\left[Y_r\right] + \mathbb{E}^z\left[Y_s - Y_r\right] = F(r) + \frac{s-r}{s}\,F(s).$$

Therefore, there exists c_0 such that $F(s) = c_0 s$. Using scaling, we see that if $z \in \mathbb{D} \setminus \mathbb{D}_s$, then

$$e^{2s} \int_0^\infty \mathbb{P}^z\{|B_t| \le e^{-s}; B[0,t] \subset \mathbb{D}\}\,dt = c_0\,[-\log|z|] = c_0\,G_{\mathbb{D}}(z,0). \tag{15}$$

To find c_0, note that

$$\begin{aligned} \mathbb{P}^0\{|B_t - z| \le e^{-s}; B[0,t] \subset (1-e^{-s})\mathbb{D}\} &= \mathbb{P}^z\{|B_t| \le e^{-s}; B[0,t] \subset \mathbb{D}\} \\ &\le \mathbb{P}^0\{|B_t - z| \le e^{-s}; B[0,t] \subset (1+e^{-s})\mathbb{D}\} \end{aligned}$$

From this we see (we omit the details) that

$$\lim_{s\to\infty} \int_{e^{-s} \le |z| \le 1} e^{2s} \int_0^\infty \mathbb{P}^z\{|B_t| \le e^{-s}; B[0,t] \subset \mathbb{D}\}\,dt\,dA(z) = \pi\,\mathbb{E}^0[\tau_{\mathbb{D}}] = \pi/2.$$

Also,

$$\int_{\mathbb{D}} G(0,w)\,dA(w) = \int_{\mathbb{D}} \log|w|\,dA(w) = \int_0^{2\pi}\int_0^1 (r\log r)dr\,d\theta = \pi\int_0^1 r\,dr = \frac{\pi}{2}.$$

Therefore $c_0 = 1$.

We now choose $D \in \mathcal{U}$ and allow constants to depend on D. Let

$$q = \sup_{|\zeta|=1} \mathbb{P}^\zeta\{\sigma_s < \tau_D\},$$

and recall that $q \le c/s$. Then if $|\zeta| = 1$, the strong Markov property and (15) imply that

$$\mathbb{E}^z\left[\int_0^{\tau_D} 1\{|B_t - w| \le e^{-s}\}\,dt\right] \le \sum_{j=1}^\infty q^j\,s\,e^{-2s} \le c\,e^{-2s}.$$

Hence if $|z| > e^{-s}$,

$$\begin{aligned} \mathbb{E}^z\left[\int_0^{\tau_D} 1\{|B_t - w| \le e^{-s}\}\,dt\right] &= \mathbb{P}^z\{\sigma_{-s} < \tau_D\}\,s\,e^{-2s}\,[1 + O(s^{-1})] \\ &= e^{-2s}\,G_D(z,0)\,[1 + O(s^{-1})]. \end{aligned}$$

This gives the result for $w = 0, U \in \mathcal{U}$, and the more general case follows from scaling and translation.

$\square$

Proposition 3.5. *If $D \in \mathcal{D}^*$ and $z, w \in D$,*

$$G_D(z,w) = G_D(w,z)$$

Proof. Let $D^\epsilon = \{\zeta \in D : \text{dist}(\zeta, \partial D) > \epsilon\}$. For a fixed t, let $W_r = w - B_t + B_{t-r}$, and note that $W_r, 0 \leq r \leq t$ is a Brownian motion starting at w. If $e^{-s} < \epsilon$, then

$$\mathbb{P}^z\{|B_t - w| \leq e^{-s}; B[0,t] \subset \tau_D\} \geq \mathbb{P}^w\{|W_t - z| \leq e^{-s}; W[0,t] \subset \tau_{D^\epsilon}\}.$$

By Proposition 3.4, we have $G_D(z,w) \geq G_{D^\epsilon}(w,z)$ and using (13) we get $G_D(z,w) \geq G_D(w,z)$. Similarly, $G_D(w,z) \geq G_D(z,w)$.

□

4 Riemann mapping theorem

We will prove one of the most important theorems in conformal mapping, the Riemann mapping theorem. We start with a lemma after which the proof will be rather short.

Lemma 4.1. *Suppose D is a domain containing the origin and $g : D \to \mathbb{D}$ is a holomorphic function satisfying $g(0) = 0$ with the following properties.*

1. *There exists $s > 0$ such for all $z \in \mathbb{D}_s$ there exists unique $w \in D$ with $g(w) = z$.*
2. *For each $s > 0$, there is a compact set $K_s \subset D$ such that $g^{-1}(\mathbb{D}_s) \subset K_s$.*

Then g is one-to-one and onto.

Proof. We recall a basic fact about holomorphic functions. If g is holomorphic at the origin, then

- If $g'(0) \neq 0$, then there is a neighborhood $\mathcal{N}$ of 0 for which g is one-to-one and onto the open set $g(\mathcal{N})$.
- If $g'(0) = 0$ and g is not a constant, then g is open and locally many-to-one, that is, there is neighborhood $\mathcal{N}$ of 0 such that $g(\mathcal{N})$ is open and each point in $g(\mathcal{N}) \setminus \{g(0)\}$ has at least two pre-images in $\mathcal{N}$.

Clearly g is not a constant function. We start by showing g is onto. Let r be the infimum of s such that there exists $z \in C_s \setminus g(D)$. The first condition shows that $r < \infty$. Suppose $r > 0$. Let $z \in C_r$. Then we can find a sequence $z_n \in \mathbb{D}_r$ with $z_n \to z$. There exist w_n with $g(w_n) = z_n$. Since $\{w_n\}$ lies in a compact set K_r, we can find a subsequence, which we also denote by w_n, that converges to $w \in K_r$. Continuity of g shows that $g(w) = z$. Hence $C_r \subset g(D)$. Around each $z \in C_r$ we can find open U_z contained in $g(D)$ and hence using compactness arguments, there exists $s < r$ with $D_s \subset g(D)$ which is a contradiction. Hence g is onto.

To show one-to-one, let r be the infimum of s such that there exists $z \in C_s$ with at least two preimages in D. The first condition shows that $r < \infty$. Suppose $r > 0$. Suppose first that there exists $z \in C_r$ and distinct $w_1, w_2 \in D$ with $g(w_1) = g(w_2) = z$. Let $\mathcal{N}_1, \mathcal{N}_2$ be nonintersecting neighborhoods of w_1, w_2 and let $U_j = g(\mathcal{N}_j)$. Then $U_1 \cap U_2$ is an open neighborhood of z such that all points have at least two preimages. This contradicts the value of r. Now suppose z_n is a sequence of points in $\mathbb{D}$ with $|z_n| \to e^{-r}$ and such that each z_n has two distinct preimages w_n and ζ_n.

By taking subsequences if necessary we can assume that $w_n \to w, \zeta_n \to \zeta$, $z_n \to z$ with $w, \zeta \in K_{r-1}$ and $z \in C_r$. Since z has only one preimage, we must have $w = \zeta$. In a small neighborhood about w, f must be locally one-to-one or locally many-to-one. Since points in $\mathbb{D}_r$ have only one preimage, it must be locally one-to-one. But this contradicts the definition of the sequences w_n, ζ_n.

□

Theorem 3 (Riemann mapping theorem)**.** *Suppose D is a simply connected strict subdomain of $\mathbb{C}$ containing the origin. Then there exists a unique conformal transformation $f : D \to \mathbb{D}$ with $f(0) = 0, f'(0) > 0$.*

Proof. Uniqueness in the case $D = \mathbb{D}$ follows as a consequence of the Schwarz lemma. More generally, if $f : D \to \mathbb{D}, g : D \to \mathbb{D}$ are two such transformation then $h := f \circ g^{-1}$ is a conformal transformation of $\mathbb{D}$ onto itself with $h(0) = 0, h'(0) > 0$, and hence h is the identity. The work is to show existence.

We will construct f using the Green's function $G_D(z) = G_D(z, 0)$. Recall that we can write

$$G_D(z) = -\log|z| + u(z),$$

where $u(z) = u_D(z)$ is a harmonic function in D, Since D is simply connected, there is a unique holomorphic function $h : D \to \mathbb{D}$ such that $\mathrm{Re}h = -u$ and $\mathrm{Im}h(0) = 0$. Let

$$f(z) = z\, e^{h(z)}.$$

Then f is holomorphic on D with $f(0) = 0, f'(0) = e^{h(0)} = e^{-u(0)} > 0$. Also $|f(z)| = e^{-G_D(z)}$. This will be the map f.

Since the Green's function goes to zero at the boundary we see that for all $r > 0$,

$$K_s := \{z \in D : G_D(z, 0) \geq s\}$$

is a compact set and $f^{-1}(\mathbb{D}_s) \subset K_s$. Also, since $f'(0) > 0$, there exist a neighborhood $\mathcal{N}$ of 0 such that f restricted to $\mathcal{N}$ is one-to-one and onto. If we choose s sufficiently large so that $G_D(z, 0) \leq s$ on $D \setminus \mathcal{N}$, we see that each $z \in \mathbb{D}_s$ has a unique preimage in D. Hence f satisfies the conditions of Lemma 4.1 and is one-to-one and onto.

□

The basic idea of this proof will be used for proving conformal equivalence of other domains. The idea is to assume that a transformation exists and try to construct the function. After a candidate is found, we then try to see if the candidate works.

The lemma can be considered a special case of what is known as the "argument principle" which is related to Rouché's theorem. Note that the lemma did not assume that D was simply connected; this comes as a consequence.

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5 Analytic boundary points and arcs

While boundaries of domains can be very rough, there are times that we would like to restrict to nice "smooth" boundaries. It will suffice for our purposes to consider very smooth boundaries given by analytic arcs. For these we can do calculations in the upper half plane near zero with (an interval of) the real line as the boundary and make use of Schwarz reflection (see Proposition 2.7). If a conformal transformation is analytic near zero, then these calculations apply to the image as well.

Definition Let $\mathcal{K}_0$ denote the set of domains $D \subset \mathbb{H}$ such that $\mathrm{dist}(0, \mathbb{H} \setminus D) > 0$. Let $\mathcal{K}$ denote the set of domains $D \in \mathcal{K}_0$ with $\mathrm{dist}(0, \mathbb{H} \setminus D) > 1$.

Definition Suppose D is a domain.

- A point $z \in \partial D$ is called an *analytic (boundary) point* of D (or of ∂D) if there exists $D' \in \mathcal{K}$ and a conformal transformation $f : D' \to D$ with $f(0) = z$ that has an extension as an analytic function on $D' \cup \mathbb{D}$.
- A simple curve $\eta : (a, b) \to \partial D$ is called an *analytic arc* of D (or of ∂D) if $\eta(t)$ is an analytic point for each $a < t < b$.

In the definition of $\mathcal{K}_0$ and $\mathcal{K}$ it is not assumed that D is simply connected. The extension of f in the definition of analytic point must be an analytic function but it is not required to be one-to-one on $D' \cup \mathbb{D}$.

In the upper half plane we have (see Section 11.3)

$$H_{\mathbb{H}}(z, 0) = -\mathrm{Im}\left[\frac{1}{z}\right] = \frac{\mathrm{Im}(z)}{|z|^2}, \quad G_{\mathbb{H}}(z, w) = \log|z - \overline{w}| - \log|z - w|.$$

We can use the right-hand side to extend $w \mapsto G(z, w)$ to the disk of radius $|z|$ about the origin (or we could use Proposition 2.7), and direct calculation shows that

$$H_{\mathbb{H}}(z, 0) = \frac{1}{2}\,\partial_y\, G_{\mathbb{H}}(z, 0).$$

If $x \neq 0$ is real, we also have the boundary Poisson kernel

$$H_{\mathbb{H}}(0, x) = \partial_y H_{\mathbb{H}}(0, x),$$

where the derivative on the right can be taken with respect to either component. More generally, if $D \in \mathcal{K}$,

$$H_D(z, 0) = H_{\mathbb{H}}(z, 0) - \mathbb{E}^z[H_{\mathbb{H}}(B_\tau, 0)], \quad G_D(z, w) = G_{\mathbb{H}}(z, w) - \mathbb{E}^z[G_{\mathbb{H}}(B_\tau, w)],$$

where $\tau = \tau_D$. By integrating under the expectation we see that

$$H_D(z, 0) = \frac{1}{2}\,\partial_y G_D(z, 0). \tag{16}$$

Also if $\mathrm{dist}(x, \mathbb{H} \setminus D) > 0$, $H_D(x,0) = \partial_y H_D(x,0)$.

We really want to generalize this to consider "analytic prime ends". As an example, suppose that $D = \mathbb{C} \setminus [-1,1]$. Then D is a simply connected domain of the Riemann sphere $\hat{\mathbb{C}}$, and we can find a conformal transformation $f : \mathbb{H} \to D$ that sends 0 to what we will call 0^+, the "positive-y" side of 0 in D. (This is including ∞ in D; if we do not want to include ∞ we can consider f restricted to $\mathbb{H} \setminus f^{-1}(\infty)$. By scaling if necessary, we can assume that $|f^{-1}(\infty)| > 1$ and hence $\mathbb{H} \setminus f^{-1}(\infty) \in \mathcal{K}$.) This map can be extended to an analytic map in a neighborhood of 0 (this extension is not one-to-one on D). Hence 0^+ is an analytic boundary point (prime end). Note that 0^- is also an analytic boundary point, but it is considered as a different point. One can check that a point $z \in \partial D$ can correspond to at most two analytic prime ends and for convenience we will just use the term analytic point.

If z is an analytic boundary point, then there is a well-defined inward unit normal derivative $\mathbf{n} = \mathbf{n}(z, D)$ pointing into D. (If z is a "two-sided" point, then each prime end has a normal derivative. Hence we consider $\mathbf{n}(z, D)$ as a function of the prime end z.) If $f : D' \to D$ is a map as above, then we write

$$f(iy) = z + y\,|f'(0)|\,\mathbf{n} + O(|y|^2), \quad y \downarrow 0.$$

If ϕ is a harmonic function on D with boundary value 0 in a neighborhood of z, then we define $\tilde{\phi}$ on $D' = f^{-1}(D)$ by $\tilde{\phi}(w) = \phi(f^{-1}(w))$. Note that $\tilde{\phi}$ is harmonic with boundary value 0 in an interval $[-\delta, \delta]$ and hence

$$\partial_y \tilde{\phi}(0) = \lim_{y \downarrow 0} y^{-1}\, \tilde{\phi}(iy),$$

is well defined. We define

$$\partial_n \phi(z) = \lim_{y \downarrow 0} y^{-1}\, \phi(z + y\mathbf{n}) = \lim_{y \downarrow 0} y^{-1}\, \tilde{\phi}(|f'(0)|^{-1}\, yi + O(y^2)) = |f'(0)|^{-1}\, \partial_y \tilde{\phi}(z).$$

An immediate consequence of this and (16) is the following.

Proposition 5.1. *If w is an analytic boundary point of D and $z \in D$, then the Poisson kernel $H_D(z,w)$ exists and*

$$H_D(z,w) = \frac{1}{2}\, \partial_{\mathbf{n}_w} G_D(z,w),$$

where $\mathbf{n}_w$ denotes the inward unit normal. If w' is another analytic boundary point, then the boundary Poisson kernel $H_D(w,w')$ exists and

$$H_D(w,w') = \partial_{\mathbf{n}_w} H_D(w,w') = \partial_{\mathbf{n}_{w'}} H_D(w,w').$$

Suppose $f : D \to D'$ is a conformal transformation, $z \in D$ and w, w' are distinct analytic boundary points of D such that $f(w)$ and $f(w')$ are analytic boundary points of D'. Then

$$H_D(z,w) = |f'(w)|\, H_{f(D)}(f(z), f(w)), \quad H_D(w,w') = |f'(w)|\, |f'(w')|\, H_{f(D)}(f(w), f(w')).$$

As another check to see that the constant is correct, recall that we have normalized our quantities so that

$$H_{\mathbb{D}}(0,1) = \frac{1}{2}, \quad G_{\mathbb{D}}(0,x) = -\log x,$$

and hence $\partial_{\mathbf{n}} G_{\mathbb{D}}(0,1) = 1$.

Proposition 5.2. *Suppose $D \in \mathcal{K}$. If $z, w \in D$ with $|z|, |w| \leq 1/2$,*

$$H_D(z,0) = H_{\mathbb{H}}(z,0)\ [1 + O(|z|)]\,.$$

$$G_D(z,w) = G_{\mathbb{H}}(z,w)\ [1 + O(|z|)]\,.$$

Proof. Note that

$$H_{\mathbb{D}_+}(z,0) \leq H_D(z,0) \leq H_{\mathbb{H}}(z,0), \quad G_{\mathbb{D}_+}(z,w) \leq G_D(z,w) \leq G_{\mathbb{H}}(z,w).$$

Using conformal transformation (see Section 11.4), we can show the estimates for $D = Disk_+$.

$$H_{\mathbb{D}_+}(z,0) = H_{\mathbb{H}}(z,0)\ [1 + O(|z|)]\,, \quad G_{\mathbb{D}_+}(z,w) = G_{\mathbb{H}}(z,w)\ [1 + O(|z|)]\,.$$

□

Proposition 5.3. *Suppose $D \in \mathcal{K}$ and $h : D \to \mathbb{R}$ is harmonic with $h \equiv 0$ on $[-x, x]$ for some $x > 1$. Then*

$$\partial_y h(0) = \frac{2}{\pi} \int_0^\pi h(e^{i\theta})\, \sin\theta\, d\theta.$$

Proof. By Schwarz reflection, we can extend h to a harmonic function on $D \cup \{z : |z| < x\}$ and from this we see that h is bounded and continuous on $\{z : |z| \leq 1\}$. The optional sampling theorem implies that if $z \in \mathbb{D}_+$, then

$$h(z) = \mathbb{E}^z \left[h(B_{\tau_{\mathbb{D}_+}})\right] = \int_{\partial \mathbb{D}_+} h(w)\, \mathrm{hm}_{\mathbb{D}_+}(z, dw) = \frac{1}{\pi} \int_0^\pi h(e^{i\theta})\, H_{\partial \mathbb{D}_+}(z, e^{i\theta})\, d\theta.$$

Using conformal invariance (see Section 11.4) we can see that

$$H_{\partial \mathbb{D}_+}(iy, e^{i\theta}) = 2y\, \sin\theta\ [1 + O(y)]\,, \quad y \downarrow 0.$$

□

Proposition 5.4. *Suppose $D \in \mathcal{K}$. If $0 < \epsilon \leq 1/2$, let $D_\epsilon = D \cap \{|z| > \epsilon\}$ and $\tau_\epsilon = \tau_{D_\epsilon}$. Then for $z \in D$ with $|z| \geq 1$,*

$$H_{D_\epsilon}(z, \epsilon e^{i\theta}) = \frac{\pi}{2}\, \mathbb{P}^z\{|B_{\tau_\epsilon}| = \epsilon\}\, \epsilon^{-1}\, \sin\theta\ [1 + O(\epsilon)]\,.$$

In other words, if $\psi(\theta; z, \epsilon, D)$ is the density of $\arg(B_{\tau_\epsilon})$ given $|B_{\tau_\epsilon}| = \epsilon$. Then,

$$\psi(\theta; z, \epsilon, D) = \frac{\sin\theta}{2}\, [1 + O(\epsilon)].$$

In particular, if ϕ is a nonnegative function defined on ∂D_ϵ that vanishes on ∂D, and $|z| \geq 1$,

$$\mathbb{E}^z\, [\phi(B_{\tau_\epsilon})] = [1 + O(\epsilon)]\ \mathbb{P}^z\{|B_{\tau_\epsilon}| = \epsilon\} \int_0^\pi \phi(\epsilon e^{i\theta})\, \frac{\sin\theta}{2}\, d\theta.$$

Proof. We fix $0 < \theta_1 < \theta_2 < \pi$, let $V_\epsilon = \{\epsilon e^{i\theta} : \theta_1 \leq \theta \leq \theta_2\}$, and let $p = (\cos\theta_1 - \cos\theta_2)/2$. Let $U_\epsilon = \{w \in \mathbb{H} : \epsilon < |w| < 1\}$ and let $\eta_\epsilon = \tau_{U_\epsilon}$. Using conformal invariance (see Section 11.2), we can see that if $\zeta \in U_\epsilon$ with $|\zeta| = 3/4$, then

$$\mathbb{P}^\zeta \{B_{\eta_\epsilon} \in V_\epsilon \mid |B_{\eta_\epsilon}| = \epsilon\} = p\,[1 + O(\epsilon)].$$

But $\mathbb{P}\{B_{\tau_\epsilon} \in V_\epsilon \mid |B_{\tau_\epsilon}|$ obviously lies between the infimum and the supremum of this quantity over $|\zeta| = 3/4$.

□

Proposition 5.5. *If $D \in \mathcal{K}$ and $|z| > 1$, then*

$$H_D(z,0) = \frac{1}{\pi}\int_0^\pi G_D(e^{i\theta}, z)\, \sin\theta\, d\theta.$$

Proof. By the strong Markov property, we can see that

$$G_D(z, iy) = \frac{1}{\pi}\int_0^\pi G_D(e^{i\theta}, z)\, H_{\mathbb{D}_+}(iy, e^{i\theta})\, d\theta.$$

Letting $y \downarrow 0$, we get (see Section 11.4)

$$H_D(z,0) = \frac{1}{2\pi}\int_0^\pi G_D(e^{i\theta}, z)\, H_{\mathbb{D}_+}(0, e^{i\theta})\, d\theta = \frac{1}{\pi}\int_0^\pi G_D(e^{i\theta}, z)\, \sin\theta\, d\theta.$$

□

5.1 Excursion measure

If D is a domain and z is an analytic boundary point, we define the *(point-to-set) excursion measure* $\mathcal{E}_D(z, \cdot)$ to be the derivative of the harmonic measure,

$$\mathcal{E}_D(z, V) = \partial_n \mathrm{hm}_D(z, V).$$

If D is an open set, not necessarily connected, we define $\mathcal{E}_D(z, V)$ to be $\mathcal{E}_{D'}(z, V)$ where D' is the connected component containing z on the boundary. The measure $\mathcal{E}_D(z, \cdot)$ is an infinite measure on ∂D, but if $\mathrm{dist}(z, V) > 0$, then $\mathcal{E}_D(z, V) < \infty$. If V is an analytic arc, we can write

$$\partial_n \mathrm{hm}_D(z, V) = \frac{1}{\pi}\int_V \partial_n H_D(z, w)\, |dw| = \frac{1}{\pi}\int_V H_{\partial D}(z, w)\, |dw|.$$

Note that if $f : D \to f(D)$ is a conformal transformation that is analytic in a neighborhood of z,

$$\mathcal{E}_D(z, V) = |f'(z)|\, \mathcal{E}_{f(D)}(f(z), f(V)).$$

If V_1, V_2 are two analytic arcs, we define the *(set-to-set) excursion measure*

$$\mathcal{E}_D(V_1, V_2) = \int_{V_1} \mathcal{E}_D(z, V_1)\, |dz| = \int_{V_1}\int_{V_2} H_{\partial D}(z, w)\, |dw|\, |dz|.$$

An important fact is that the set-to-set excursion measure is a conformal invariant.

Proposition 5.6. *If $f : D \to f(D)$ is a conformal transformation that is analytic on the arcs $V_1, V_2 \subset \partial D$, then*

$$\mathcal{E}_D(V_1, V_2) = \mathcal{E}_{f(D)}(f(V_1), f(V_2)).$$

The term (point-to-set) excursion measure is used both for a measure on the boundary $\mathcal{E}_D(z, \cdot)$ and also for the measure on paths starting at z, ending at ∂D, otherwise in D, corresponding to "Brownian motion conditioned to begin and end at the boundary". In this case $\mathcal{E}_D(z, V)$ is the total mass of curves that end at V. We can define a (set-to-set) excursion measure similarly.

We can also define point-to-point excursion measure which has total mass $H_{\partial D}(z, w)/\pi$.

The excursion measure viewed as a measure on paths is also conformally invariant provided that we change time as in the conformal invariance of Brownian motion.

Since the set-to-set excursion measure is a conformal invariant it is well defined even if the boundary is not analytic. For example, if V_1, V_2 are on the same connected component of the boundary, we can first map D to $\mathbb{H}$ mapping this component to the real line. If they are in different components K_1, K_2, then we can first map $\hat{\mathbb{C}} \setminus (K_1 \cup K_2)$ to an annulus.

Proposition 5.7. *Suppose D is a domain and z is a locally analytic point. Suppose $D' \subset D$ and D, D' agree in a neighborhood of z. Let $w \in D \setminus \overline{D}'$. Then*

$$H_D(w, z) = \frac{1}{2} \int_{\partial D'} G_D(\zeta, w)\, \mathcal{E}_{D'}(z, d\zeta).$$

If $\partial D' \cap D$ is analytic, we can write

$$H_D(w, z) = \frac{1}{2\pi} \int_{\partial D'} G_D(\zeta, w)\, H_{\partial D'}(z, \zeta)\, |d\zeta|.$$

As an example (and a check on the constants), suppose that $D = \mathbb{D}, D' = A_r = \{e^{-r} < |z| < 1\}$, $z = 1, w = 0$. Then $H_{\mathbb{D}}(0, 1) = 1/2$, and

$$\mathcal{E}_{A_r}(1, C_r) = \frac{1}{r}, \qquad G_{\mathbb{D}}(e^{-r+i\theta}, 0) = r, \qquad H_{A_r}(1, e^{-r+i\theta}) = \frac{e^r}{2r}\, [1 + o(1)].$$

Proof. We first consider the case with $z = 0$, $D, D' \in \mathcal{K}$. The function $h(\zeta) := G(\zeta, w)$ is a bounded harmonic function on D'. Therefore, for $y < 1$,

$$h(iy) = \mathbb{E}^{iy}\left[h(B_{\tau_{D'}})\right] = \int_{\partial D'} G_D(\zeta, w)\, \mathrm{hm}_{D'}(iy, d\zeta).$$

Letting $y \to 0$, we get

$$2H_D(0, w) = \partial_n G(0, w) = \partial_n \mathbb{E}^{iy}\left[h(B_{\tau_{D'}})\right] = \int_{\partial D'} G_D(\zeta, w)\, \mathcal{E}_{D'}(0, d\zeta).$$

For the more general case, since z is an analytic point we can find $\tilde{D} \in \mathcal{K}$ and conformal transformation $f : \tilde{D} \to D$ with $f(0) = z$. By scaling earlier, we can find such $\tilde{D}, F$ such that $\tilde{D}' := f^{-1}(D') \in \mathcal{K}$. We then use

$$H_D(w,z) = |f'(0)|\, H_{\tilde{D}}(f^{-1}(w), 0), \quad \mathcal{E}_D(z, V) = |f'(0)|\, \mathcal{E}_{\tilde{D}}(0, f^{-1}(V)),$$
$$G_D(\zeta, w) = G_{\tilde{D}}(f^{-1}(\zeta), f^{-1}(w)).$$

□

Examples

- If $\mathbb{A}_r = \{e^{-r} < |z| < 1\}$ is the annulus with boundaries C, C_r, then $\mathcal{E}_{\mathbb{A}_r}(e^{i\theta}, C_r) = \frac{1}{r}$, and hence
$$\mathcal{E}_{\mathbb{A}_r}(C, C_r) = \int_0^{2\pi} \mathcal{E}_{\mathbb{A}_r}(e^{i\theta}, C_r)\, d\theta = \frac{2\pi}{r}.$$
In particular, we see that if $r \neq s$, then $\mathcal{E}_{\mathbb{A}_r}(C, C_r) \neq \mathcal{E}_{\mathbb{A}_s}(C, C_s)$. From this we can see that $\mathbb{A}_r$ and $\mathbb{A}_s$ are not conformally equivalent.

- Let $\mathcal{R}_L = \{x + iy : 0 < x < L, 0 < y < \pi\}$ and let $\partial_1 = [0, i\pi], \partial_2 = \partial_{2,L} = [L, L + i\pi]$ be the vertical boundaries. Let h be the harmonic function on $\mathcal{R}_L$ with boundary value 1 on ∂_2 and 0 on $\partial \mathcal{R}_L \setminus \partial_2$. This can be found by separation of variables,
$$h(x+iy) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sinh(nx)\, \sin(ny)}{n \,\sinh(nL)},$$
$$\partial_x h(iy) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(ny)}{\sinh(nL)} = \frac{8 \,\sin y}{\pi}\, e^{-L}\, \left[1 + O(e^{-L})\right], \;\; L \to \infty.$$
$$\mathcal{E}_{\mathcal{R}_L}(\partial_1, \partial_2) = \int_0^{\pi} \partial_x h(iy)\, dy = \frac{16}{\pi}\, e^{-L}\, \left[1 + O(e^{-L})\right], \;\; L \to \infty.$$
Although it is not immediately obvious from the last expression, one can use the definition to see that the function $L \mapsto \mathcal{E}_{\mathcal{R}_L}(\partial_1, \partial_2)$ is strictly decreasing.

- Let $D = D(a,b) = \{a < \operatorname{Im}(z) < b\}$ with boundaries $I_a = \{\operatorname{Im}(z) = a\}, I_b = \{\operatorname{Im}(z) = b\}$. Then, the gambler's ruin estimate implies that if $x + ia \in I_a$, then $\mathcal{E}_D(x+ia, I_b) = 1/(b-a)$, and hence if $V \subset I_a$,
$$\mathcal{E}_D(V, I_b) = \mathcal{E}_D(I_b, V) = \frac{1}{b-a}\, \ell(V),$$
where ℓ denotes Lebesgue measure. More generally, suppose that $D = \mathbb{H} \setminus K$ is a domain with $K \subset \{\operatorname{Im}(z) < a\}$. The strong Markov property, implies that if $b > a$, then
$$\mathcal{E}_D(I_b, V) = \frac{1}{b-a} \int_{-\infty}^{\infty} \operatorname{hm}_{D \cap \{\operatorname{Im}(z) < b\}}(x + ia, V)\, dx,$$
and hence,
$$\lim_{b \to \infty} b\, \mathcal{E}_D(I_b, V) = \int_{-\infty}^{\infty} \operatorname{hm}_D(x + ia, V)\, dx \tag{17}$$

Definition

- We say two domains D_1, D_2 are *conformally equivalent* if there exists a conformal transformation $f : D_1 \to D_2$.
- A domain D is a *conformal annulus* if $\hat{\mathbb{C}} \setminus D$ consists of two connected components each larger than a single point.

The Riemann mapping theorem states that any two simply connected domains other than the entire complex plane are conformally equivalent. We have used excursion measure to see that if $r \neq s$, then the annuli $\mathbb{A}_r$ and $\mathbb{A}_s$ are not conformally equivalent. The next proposition will show that every conformal annulus is conformally equivalent to $\mathbb{A}_s$ for some (necessarily unique) s.

Proposition 5.8. *If D is a conformal annulus with boundary components ∂_1, ∂_2, then D is conformally equivalent to $\mathbb{A}_r$ where $r = 2\pi/\mathcal{E}_D(\partial_1, \partial_2)$.*

Proof. Let V_1, V_2 denote the connected components of $\hat{\mathbb{C}} \setminus D$. Let $D' = \hat{\mathbb{C}} \setminus V_1$. By the Riemann mapping theorem, we can conformally transform D' onto the unit disk. For this reason, without loss of generality, we will assume that $D \subset \mathbb{D}$ and $\partial_1 = C$. Let

$$q(z) = q_D(z) = \mathbb{P}^z\{B_{\tau_D} \in \partial_2\}.$$

We let $\mathbf{n}$ denote inward normals in this proof.

Let $r > 0$, and suppose that $f : D \to A_r$ is a conformal transformation with $f(C) = C$. By conformal invariance,

$$\mathcal{E}_D(C, \partial_2) = \mathcal{E}_{A_r}(C, C_r) = \frac{2\pi}{r}.$$

Hence, $r = 2\pi/\mathcal{E}_D(C, \partial_2)$.

Note that $u(z) := r q_D(z)$ is a harmonic function on D with

$$\int_C \partial_{\mathbf{n}} u(z)\, |dz| = r\, \mathcal{E}_D(C, \partial_2) = 2\pi. \tag{18}$$

Suppose γ is a simple curve in D separating ∂_2 from C. Using the fact that u is harmonic, we see that (18) implies that

$$\int_\gamma \partial_{\mathbf{n}} u(z)\, |dz| = 2\pi. \tag{19}$$

We can find a harmonic function $h(z) = u(z) + iv(z)$ locally around each z, and let $f(z) = \exp\{-h(z)\}$. Using (19), we can see that f is well defined globally. This gives a map $f : D \to \mathbb{A}_r$. We need to show that f is one-to-one and onto.

As in the simply connected case, we can see that for each $0 < q < 1$, the sets $V_q = \{z : u(z) = q\}, \{z : u(z) > q\}, \{z : u(z) < q\}$ are connected, and using this we get that $f'(z) \neq 0$ for every z. To show global injectivity, consider the point smallest q for which f is not one-to-one on V_q. □

We say two domains D_1, D_2 are *conformally equivalent* if there exists a conformal transformation $f : D_1 \to D_2$. Let us call D a *conformal annulus* if D is connected and ∂D consists of two connected components each larger than a single point. Suppose D is a domain and V is a connected component of $\hat{\mathbb{C}} \setminus \mathbb{C}$ containing more than one point. Then, $\hat{\mathbb{C}} \setminus V$ is a simply connected subset of the Riemann sphere $\hat{\mathbb{C}}$ and hence can be mapped conformally onto the disk or conformally onto $\mathbb{H}$. For this reason, when we consider multiply connected domains it will suffice to consider subdomains of $\mathbb{D}$ (or $\mathbb{H}$) for which C (or $\mathbb{R}$) are contained in the boundary. Similarly, if V_1, V_2 are two connected components of $\hat{\mathbb{C}} \setminus \mathbb{C}$ containing more than one point, we can start by mapping $\hat{\mathbb{C}} \setminus (V_1 \cup V_2)$ to an annulus.

5.2 Poisson kernel

If D is a regular domain and z is an analytic boundary point, then the Poisson kernel is defined, up to a multiplicative constant, as a positive harmonic function f whose boundary value is zero everywhere except for z (here we are interpreting z in terms of a prime end. Suppose $D' \in \mathcal{K}$ and $f : D' \to D$ is a conformal transformation. If we define

$$h(w) = H_{\partial D'}(f^{-1}(w))$$

then h satisfies these conditions on D. Hence, we do not need a nice boundary point to have such a function.

If z, w are both analytic boundary points, then we define the boundary Poisson kernel $H_{\partial D}(z, w)$ by

$$H_{\partial D}(z, w) = \partial_{n_z} H_D(z, w) = 2\, \partial_{n_z}\, \partial_{z_w}\, G_D(z, w),$$

where we write n_z, z_w for the derivative at the inward normal at z, w, respectively. The second expression shows that $H_{\partial D}(z, w) = H_{\partial D}(w, z)$

6 Extremal length and reflecting Brownian motion

Suppose D is a domain and ∂_1, ∂_2 are disjoints subsets of ∂D. One conformally invariant way to measure the "distance" between ∂_1 and ∂_2 is in terms of Brownian excursion measure $\mathcal{E}_D(\partial_1, \partial_2)$. Roughly speaking, this gives the measure of the set of Brownian motions starting at ∂_1 that exit D at ∂_2. (Strictly speaking, this measure is zero, so the actual definition is in terms of the boundary Poisson kernel.)

There is a different conformally quantity, which we will denote by $\mathcal{E}^*_D(\partial_1, \partial_2)$, that is more commonly used in the complex variable literature. Its reciprocal is called *extremal length* or *extremal distance*. It is defined in the same way as $\mathcal{E}_D(\partial_1, \partial_2)$ except that the Brownian motions are not killed when they hit $\partial D \setminus (\partial_1 \cup \partial_2)$ rather, they are reflected orthogonally.

Defining reflected Brownian motion can be tricky for rough domains, but because it is a conformally invariant quantity we can restrict ourselves to a rather simple set of domains that will suffice for our purposes. Let $\mathcal{D}_{\rm ref}$ denote the set of domains $(D, \partial_1, \partial_2)$ such that: $D \subset \mathbb{H}$; ∂_1, ∂_2 are disjoint closed subsets of ∂D with a finite number of connected components each larger than a single point; and $I_D := \partial D \setminus (\partial_1 \cup \partial_2)$ consists of a finite or countable number of disjoint subsets of $\mathbb{R}$. If $(D, \partial_1, \partial_2) \in \mathcal{D}_{\rm ref}$, we let D^* be the reflected domain

$$D^* = D \cup I_D \cup \{\overline{z} : z \in D\}.$$

and $\partial_1^*, \partial_2^*$ be the corresponding closed subsets of ∂D^*. We let $\tilde{\mathcal{D}}_{\rm ref}$ be the set of conformal images of $(D, \partial_1, \partial_2) \in \mathcal{D}_{\rm ref}$.

Under these assumptions, we construct W_t, Brownian motion started at $z \in D \cup I_D$, reflected orthogonally on I_D, stopped when it reaches $\partial_1 \cup \partial_2$ by:

- Let $B_t = X_t + iY_t$ be a standard Brownian motion stopped at time τ, the first time it leaves D^*.
- Let $W_t = X_t + i|Y_t|$, $0 \le t \le \tau$.
- The reflected excursion measure is given by

$$\mathcal{E}^*_D(\partial_1, \partial_2) = \mathcal{E}_{D^*}(\partial_1, \partial_2^*) = \frac{1}{2}\,\mathcal{E}_{D^*}(\partial_1^*, \partial_2^*).$$

We extend $\mathcal{E}^*_D(\partial_1, \partial_2)$ to $D \in \tilde{\mathcal{D}}_{\mathrm{ref}}$ by conformal invariance. Let $f(z)$ denote the probability that the reflected Brownian motion starting at z in D reaches ∂_2 before reaching ∂_1. This is the same as the probability that the usual Brownian motion starting at z in D^* reaches ∂_2^* before reaching ∂_1^*. Then f is harmonic in D, $f \equiv 1_{\partial_2}$ on $\partial_1 \cup \partial_2$, and it satisfies the reflecting boundary condition

$$\partial_n f(x) = 0, \quad x \in I_D.$$

Here ∂_n denotes partial with respect to the interior normal derivative which for $x \in I_D$ is just the partial with respect to y. If ∂_1 is sufficiently smooth, we have

$$\mathcal{E}^*_D(\partial_1, \partial_2) = \int_{\partial_1} \partial_n f(z)\,|dz|.$$

Examples.

- Let $D = \mathcal{R}_L = (0, L) \times i(0, \pi)$ be the $L \times \pi$ rectangle and let $\partial_1 = i[0, \pi], \partial_2 = L + i[0, \pi]$ be the vertical edges. Then $(D, \partial_1, \partial_2) \in \tilde{\mathcal{D}}_{\mathrm{ref}}$. In this case

$$f(x + iy) = x/L, \quad \partial_n f(iy) = 1/L, \quad \mathcal{E}^*_D(\partial_1, \partial_2) = \frac{\pi}{L}.$$

  This is easy because the reflection only affects the imaginary part, and so the calculation boils down to the gambler's ruin estimate. Note that $\mathcal{E}_D(\partial_1, \partial_2) \asymp e^{-L}$ and hence decays much faster.

- Let D be the annulus $\mathbb{A}_s = \{e^{-s} < |z| < 1\}$ and let $\partial_1 = C_0, \partial_2 = C_s$ be the two boundary circles. Then $(D, \partial_1, \partial_2) \in \tilde{\mathcal{D}}_{\mathrm{ref}}$ (use a map that sends $\{|z| > e^{-s}\}$ on the Riemann sphere to $\mathbb{H}$). In this case there is no reflection, and hence

$$\mathcal{E}^*_D(\partial_1, \partial_2) = \mathcal{E}_D(\partial_1, \partial_2) = \frac{2\pi}{s}.$$

  Note that $f(z) = -s^{-1} \log |z|$.

- Let D be the half-annulus $\mathbb{A}_s^+ = \{z \in \mathbb{H} : e^{-s} < |z| < 1\}$ and let ∂_1, ∂_2 be as in the last example. Then $(D, \partial_1, \partial_2) \in \mathcal{D}_{\mathrm{ref}}$ with $I_D = (-1, -e^{-s}) \cup (e^{-s}, 1)$. Since $D \subset \mathbb{H}$ with reflection on the real axis, this is a domain of the type in the definition. Note that $D^* = \mathbb{A}_s, \partial_1^* = C_0, \partial_2^* = C_s$. This domain is also conformally equivalent to $\mathcal{R}_s$ with the circles being sent to the vertical edges of $\mathcal{R}_s$. Therefore,

$$\mathcal{E}^*_D(\partial_1, \partial_2) = \frac{\pi}{s}, \quad \mathcal{E}_D(\partial_1, \partial_2) \asymp e^{-s}.$$

Another common name for various analogs and generalizations of excursion measure and its reciprocal are *conductance* and *resistance*.

We now describe the more classical way of defining $\mathcal{E}^*_D(\partial_1, \partial_2)$. Let $\mathcal{K} = \mathcal{K}_D(\partial_1, \partial_2)$ denote the set of piecewise C^1 curves γ from ∂_1 to ∂_2 otherwise lying in D. We say that a positive function $\rho : D \to [0, \infty)$ is *admissible* (with respect to $\mathcal{K}$), if for every $\gamma \in \mathcal{K}$,

$$\int_\gamma \rho(z)\,|dz| \,\geq 1. \tag{20}$$

If $\mathcal{K}$ is a set of piecewise C^1 curves in a domain D (with endpoints perhaps on ∂D), we define the *modulus* of $\mathcal{K}$ by

$$\Lambda(\mathcal{K}) = \inf \int_D \rho(z)^2\, dA(z),$$

where the infimum is over all admissible functions ρ. The reciprocal of the modulus is called the *extremal length* or *extremal distance*. As an example, suppose that $\mathcal{K} = \mathcal{K}(D, \partial_1, \partial_2)$ where $D = \mathcal{R}_L$ and ∂_1, ∂_2 are the vertical edges. Then for any admissible ρ and $0 < y < \pi$,

$$\frac{1}{L}\int_0^L \rho(x+iy)^2\, dx \geq \left[\frac{1}{L}\int_0^L \rho(x+iy)\, dx\right]^2 = \frac{1}{L^2},$$

and hence,

$$\int \rho(z)^2\, dA(z) = \int_0^\pi \int_0^L \rho(x+iy)^2\, dx\, dy \geq \frac{\pi}{L}.$$

Since the function $\rho(x + iy) = 1/L$ is acceptable we can see that $\Lambda(\mathcal{K}) = \pi/L$ and the constant function $\rho(z) = 1/L$ is the minimizer.

Proposition 6.1. *The modulus is a conformal invariant. That is, if $\mathcal{K}$ is a collection of curves in D and $g : D \to f(D)$ is a conformal transformation, then $\Lambda(\mathcal{K}) = \Lambda(g \circ \mathcal{K})$ where $g \circ \mathcal{K} = \{g \circ \gamma : \gamma \in \mathcal{K}\}$.*

Proof. Suppose that ρ is admissible for D. Define ρ_g on $g(D)$ by $\rho_g(g(z)) = |g'(z)|^{-1}\,\rho(z)$. If $\gamma \in \mathcal{K}_D(\partial_1, \partial_2)$, let $g \circ \gamma \in \mathcal{K}_{f(D)}(f(\partial_1), f(\partial_2)))$ be the corresponding curve. (The parametrization of the curve is not important.) Then

$$\int_{g\circ\gamma} \rho_g(z)\,|dz| = \int_\gamma \rho_g(g(w))\,|g'(w)|\,|dw| = \int_\gamma \rho(w)\,|dw| \geq 1.$$

Also,

$$\int_{g(D)} \rho_g(z)^2\, dA(z) = \int_D [\rho(w)\,|g'(w)|^{-1}]^2\,|g'(w)|^2\, dA(w) = \int_D \rho(w)^2\, dA(w).$$

Taking infimums, we see that $\Lambda(g \circ \mathcal{K}) \leq \Lambda(\mathcal{K})$. Applying the same argument to g^{-1} gives the other direction.

□

Proposition 6.2. *If D is simply connected with $(D, \partial_1, \partial_2) \in \tilde{\mathcal{D}}_{\rm ref}$ and ∂_1 is connected, then the minimizing ρ for $\mathcal{K}_D(\partial_1, \partial_2)$ is given by $|\nabla f(z)|$ where f is the unique harmonic function on D with boundary conditions $f \equiv 1_{\partial_2}$ on $\partial_1 \cup \partial_2$ and $\partial_n f \equiv 0$ on I_D. In particular,*

$$\Lambda[\mathcal{K}_D(\partial_1, \partial_2)] = \int_D |\nabla f(z)|^2 \, dA(z).$$

Proof. We will assume that $D = \mathcal{R}_L, \partial_1 = i[0, \pi]$ and

$$\partial_2 = A_1 \cup A_2 \cup \cdots \cup A_{k+1},$$

where A_j are disjoint nontrivial closed subintervals of the right vertical boundary $L + i[0, \pi]$, with the intervals ordered counterclockwise (imaginary parts increasing), and $L \in A_1, L + i\pi \in A_{k+1}$. We let $l_1, \ldots, l_k$ be the open intervals in between, so that

$$I_D = l_1 \cup \cdots \cup l_k \cup (0, L) \cup [(0, L) + i\pi].$$

Every $(D, \partial_1, \partial_2) \in \tilde{\mathcal{D}}_{\rm ref}$ is conformally equivalent to such a domain, and the representation is unique. We let $f(z)$ be the probability that Brownian motion reflected off of I_D hits ∂_2 before ∂_1.

We will call $\hat{D}$ a "comb" domain if it is of the form

$$\hat{D} = \mathcal{R}_{L'} \setminus (l_1' \cup \cdots \cup l_k')$$

where $l_1', \ldots, l_k'$ are disjoint intervals of the form

$$l_j' = \{x + iy_j : x_j \le x \le 1\},$$

where $0 < x_j < 1$, $0 < y_j < L'$. We set $\hat{\partial}_1, \hat{\partial}_2$ to be the vertical boundaries. If $\hat{f}$ is the corresponding function, then $\hat{f}(x + iy) = x/L$, the same as for $\mathcal{R}_{L'}$, since the reflection is always in the y-direction and is independent of the real part. Using the same argument as for $\mathcal{R}_{L'}$, we see that $\Lambda[\mathcal{K}(\hat{D}, \partial_1, \partial_2)] = L'$. Similarly, $\mathcal{E}^*_{\hat{D}}(\partial_1, \partial_2) = L'$.

We claim that we can find a comb domain $\hat{D}$ and a conformal transformation $g : D \to \hat{D}$ so that $g \circ l_j = l_j'$. To see this, we will first determine what the parameters $L, x_1, \ldots, x_k, y_1, \ldots, y_k$ would need to be. Since we need $\mathcal{E}^*_D(\partial_1, \partial_2) = \mathcal{E}^*_{\hat{D}}(\hat{\partial}_1, \hat{\partial}_2)$, we choose L' satisfying

$$\pi / L' = \mathcal{E}^*_{\mathcal{R}_{L'}}(\partial_1, \partial_2) = \int_0^\pi \partial_x f(iy) \, dy.$$

We then choose $0 = y_0 < y_1 < \cdots < y_k < y_{k+1} = \pi$ uniquely so that

$$\mathcal{E}^*_{\mathcal{R}_{L'}}(\partial_1, L' + i[y_{k-1}, y_k]) = \mathcal{E}^*_D(\partial_1, A_k).$$

Finally, for $z = L + iy \in I_D \cap \partial_2$, we must have $f(z) = \hat{f}(g(z))$. This leads to the choice

$$x_j = \min\{f(z) : z \in l_j\}.$$

This defines $\hat{D}$ in terms of D, and given this we define g to be the unique holomorphic function on $\mathcal{R}_L$ with $g(0) = 0$ and $\mathrm{Re}[g(z)] = L' \, f(z)$.

Once we have this transformation, we know that $\rho = |\nabla \hat{f}|$ is the minimizer in $\hat{D}$ and be conformal transformation, $|\nabla f|$ is the minimizer in D.

□

The next proposition is almost immediate using the definition of modulus but would be much harder use the definition coming from reflected Brownian motion.

Proposition 6.3 (Monotonicity)**.** *If $D' \subset D, \partial_1' \subset \partial_1, \partial_2' \subset \partial_2$, then $\mathcal{K}(D', \partial_1', \partial_2') \subset \mathcal{K}(D, \partial_1, \partial_2)$, and hence*

$$\Lambda[\mathcal{K}(D', \partial_1', \partial_2')] \leq \Lambda[\mathcal{K}(D, \partial_1, \partial_2)].$$

Proof. This follows immediately from the definition of modulus because if $\mathcal{K}' \subset \mathcal{K}$, then any ρ that is admissible for $\mathcal{K}$ is also admissible for $\mathcal{K}'$. □

7 Toolbox for conformal maps

Here we develop some of the classical tools for dealing with conformal transformations. One can get very far having three results in one's pocket: the Koebe-1/4 theorem, the distortion theorem, and the Beurling estimate. We will do these here. We call a function f on a domain D *univalent* if it is holomorphic and one-to-one.

7.1 Beurling estimate

The Beurling estimate is a uniform upper bound on the probability that a Brownian motion avoids a connected set. As an example suppose $K = [0,1]$ and B_t is a Brownian motion starting at $-\epsilon$ and $D = \mathbb{D} \cap \mathbb{H}$. Then by considering the square root map that takes $\mathbb{D} \setminus K$ to D,

$$\begin{aligned} \mathbb{P}^{-\epsilon}\{B[0, \tau_{\mathbb{D}}] \cap K = \emptyset\} &= \mathbb{P}^{-i\sqrt{\epsilon}}\{B_{\tau_D} \notin \mathbb{R}\} \\ &= \frac{1}{\pi}\int_0^\pi H_D(i\sqrt{\epsilon}, e^{i\theta})\, d\theta \sim \frac{4\sqrt{\epsilon}}{\pi}, \quad \epsilon \downarrow 0 \end{aligned}$$

If we replace $[0,1]$ with a different curve from 0 to the unit disk, we would expect that the probability for a Brownian motion to avoid the set would decrease. This statement is made precise in the Beurling projection theorem. From a practical perspective, what is used is the fact that the probability is bounded by $c\sqrt{\epsilon}$. This latter statement is often referred to as the Beurling estimate.

We will state and prove the Beurling projection theorem in this section. If $K \subset \mathbb{H}$ is a closed subset, we write

$$K_+ = K \cap \{\mathrm{Im}(w) \geq 0\}, \quad K_- = K \cap \{\mathrm{Im}(w) \leq 0\},$$

$$K^* = \{\overline{w} : w \in K\}, \quad K' = K_+ \cup K_-^*.$$

In other words, K' is obtained from K by reflecting the elements of K below the real line to the upper half plane. Note that $K \cap \mathbb{R} = K' \cap \mathbb{R} = (K \cap K^*) \cap \mathbb{R}$, and, more generally, $\mathrm{dist}(x, K) = \mathrm{dist}(x, K') = \mathrm{dist}(x, K \cup K^*)$ for all $x \in \mathbb{R}$.

Lemma 7.1. *Suppose $K \subset \mathbb{D}$ is closed and let K' be as above. Let $\tau = \tau_{\partial\mathbb{D}}$ and ρ, ρ' the first times to visit K, K' respectively. If $-1 < x < 1$, then*

$$\mathbb{P}^x\{\rho < \tau\} \geq \mathbb{P}^x\{\rho' < \tau\}.$$

Proof. We assume $x \notin K$ and write $\mathbb{P}$ for $\mathbb{P}^x$ throughout this proof. We will give an increasing sequence of stopping times. Let $\delta_0 = \operatorname{dist}(x, K \cup \partial\mathbb{D}) = \operatorname{dist}(x, K' \cup \partial\mathbb{D}) = \operatorname{dist}(x, K^* \cup \partial\mathbb{D})$, and

$$S_0 = \inf\{t : |B_t - x| = \delta_0\},$$

$$T_0 = \inf\{t \geq S_0 : B_t \in \mathbb{R}\}.$$

More generally, if $j \geq 1$, we set

$$\delta_j = \operatorname{dist}\left(B_{T_{j-1}}, K \cup \partial\mathbb{D}\right),$$
$$S_j = \inf\{t \geq T_{j-1} : |B_t - B_{T_{j-1}}| = \delta_j\},$$
$$T_j = \inf\{t \geq S_j : B_t \in \mathbb{R}\}.$$

It is possible that $B(T_j) \in K$ for some j in which case $S_k = T_k = T_j$ for $k \geq j$. However, if $B(T_j) \notin K$, then with probability one $T_j < S_{j+1} < T_{j+1}$. Note that on the event $\{B[0,\tau] \cap (K \cap \mathbb{R}) = \emptyset, B(\tau) \notin \mathbb{R}\}$, there exists j with $\{T_j < \tau < T_{j+1}\}$. Hence it suffices to show that for every $j \geq 0$,

$$\mathbb{P}\{T_j < \tau < T_{j+1}; B[0,\tau] \cap K = \emptyset\} \leq \mathbb{P}\{T_j < \tau < T_{j+1}; B[0,\tau] \cap K' = \emptyset\}. \tag{21}$$

It will be useful to add some randomness to the process. Let $J_0, J_1, \ldots$ be independent random variables, independent of the Brownian motion $B_t = B_t^1 + i\, B_t^2$ with $\mathbb{P}\{J_j = 1\} = \mathbb{P}\{J_j = -1\} = 1/2$. Define W_t by

$$W_t = B_t^1 + i\, J_j\, B_t^2 \quad T_{j-1} \leq t < T_j.$$

(Here $T_{-1} = 0$.) If $B_{T_j} \in \mathbb{R} \cap K$ so that $T_{j+1} = T_j$, we stop the process W_t at time T_j. Note that

$$\mathbb{P}\{T_j < \tau < T_{j+1}; B[0,\tau] \cap K = \emptyset\} = \mathbb{P}\{T_j < \tau < T_{j+1}; W[0,\tau] \cap K = \emptyset\},$$

and similarly with K' replacing K. Let $\mathcal{F}$ denote the σ-algebra generated by the Brownian motion B only, so that $\mathcal{F}$ is independent of the J_j. We claim that for each j,

$$\mathbb{P}\{T_j < \tau < T_{j+1}; W[0,\tau] \cap K = \emptyset \mid \mathcal{F}\} \leq \mathbb{P}\{T_j < \tau < T_{j+1}; W[0,\tau] \cap K' = \emptyset \mid \mathcal{F}\}.$$

Let us fix a j. The event $\{T_j < \tau < T_{j+1}\}$ is $\mathcal{F}$-measurable. On this event, we can write

$$\mathbb{P}\{T_j < \tau < T_{j+1}; W[0,\tau] \cap K = \emptyset \mid \mathcal{F}\} = I_0\, \mathbb{E}\left[I_1 \cdots I_j\, \hat{I}_{j+1} \mid \mathcal{F}\right],$$

where

$$I_k = 1\{J_k = 1\}\, 1\{B[S_k, T_k] \cap K = \emptyset\} + 1\{J_k = -1\}\, 1\{B[S_j, T_k] \cap K^* = \emptyset\},$$
$$\hat{I}_k = 1\{B[S_k, \tau] \cap K = \emptyset\} + 1\{J_k = -1\}\, 1\{B[S_k, \tau] \cap K^* = \emptyset\},$$

We get a similar expression for K' in terms of $I_k', \hat{I}_k'$, obtained by replacing K, K^* with $K', (K')^*$. The random variables $I_0, I_1, \ldots$ are conditionally independent given $\mathcal{F}$, and hence it suffices to show for every k,

$$\mathbb{P}\{I_k = 1 \mid \mathcal{F}\} \leq \mathbb{P}\{I_k' = 1 \mid \mathcal{F}\},$$
$$\mathbb{P}\{\hat{I}_k = 1 \mid \mathcal{F}\} \leq \mathbb{P}\{\hat{I}_k' = 1 \mid \mathcal{F}\},$$

We will show the first; the second is done in the same way. If $B(T_k) \in K$, then $I_k = 0$, so let us assume that $B(T_k) \notin K$. Consider the events

$$E_1 = \{B(S_k, T_k) \cap K_+ = \emptyset\}, \quad E_2 = \{B(S_k, T_k) \cap K_- = \emptyset\},$$

$$E_3 = \{B(S_k,T_k) \cap (K_+)^* = \emptyset\}, \quad E_4 = \{B(S_k,T_k) \cap (K_-)^* = \emptyset\}.$$

We can write

$$\mathbb{P}\{I_k = 1 \mid \mathcal{F}\} = \frac{1}{2}\, 1_{E_1 \cap E_2} + \frac{1}{2}\, 1_{E_3 \cap E_4},$$

$$\mathbb{P}\{I_k' = 1 \mid \mathcal{F}\} = \frac{1}{2}\, 1_{E_1 \cap E_4} + \frac{1}{2}\, 1_{E_3 \cap E_2}.$$

We therefore get

$$2\,\left[\mathbb{P}\{I_k' = 1 \mid \mathcal{F}\} - \mathbb{P}\{I_k = 1 \mid \mathcal{F}\}\right] = 1_{E_1 \cap (E_4 \setminus E_2)} + 1_{E_3 \cap (E_2 \setminus E_4)} - 1_{E_1 \cap (E_2 \setminus E_4)} - 1_{E_3 \cap (E_4 \setminus E_2)}.$$

Note that on the event $E_4 \setminus E_2$, $B(S_j,T_j)$ lies in the lower half-plane and hence $1_{E_1} = 1$. Similarly, on the event $E_2 \setminus E_4$, we have $1_{E_3} = 1$. Hence,

$$2\,\left[\mathbb{P}\{I_k' = 1 \mid \mathcal{F}\} - \mathbb{P}\{I_k = 1 \mid \mathcal{F}\}\right] = 1_{E_4 \setminus E_2} + 1_{E_2 \setminus E_4} - 1_{E_1 \cap (E_2 \setminus E_4)} - 1_{E_3 \cap (E_4 \setminus E_2)} \geq 0.$$

□

Proposition 7.2. *Under the assumptions above, if D, D' are the connected components of $\mathbb{C} \setminus K, \mathbb{C} \setminus K'$ containing the origin, and $x \in \mathbb{R} \setminus \{0\}$,*

$$G_D(x) \leq G_{D'}(x).$$

Proof. Since $\mathbb{R} \cap D = \mathbb{R} \cap D'$, the result is trivial for $x \in \mathbb{R} \setminus D$, so we assume $x \in D$. Let s be sufficiently small so that $\mathbb{D}_s \subset D$ and $|x| > e^{-s}$. Then we can follow the proof as above, to show that

$$\mathbb{P}^x\{\sigma_s < \tau_D\} \leq \mathbb{P}^x\{\sigma_s < \tau_{D'}\}.$$

Letting $s \to \infty$ gives the result. □

Theorem 4 (Beurling projection theorem)**.** *Suppose K is a connected, closed subset of $\overline{\mathbb{D}}$ such that for each $\epsilon \leq r \leq 1$*

$$K \cap \{|B_t| = r\} \neq \emptyset.$$

Then,

$$\mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap K = \emptyset\} \leq \mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap [\epsilon, 1] = \emptyset\}.$$

In particular,

$$\mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap K = \emptyset\} \leq 2\,\epsilon^{1/2}.$$

By conformal invariance, we can see that as $\epsilon \downarrow 0$,

$$\mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap [\epsilon, 1] = \emptyset\} = \frac{4}{\pi}\,\epsilon^{1/2} + O(\epsilon).$$

Although it is not optimal, we can write

$$\mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap [\epsilon, 1] = \emptyset\} \leq 2\,\epsilon^{1/2}, \quad 0 < \epsilon \leq 1.$$

Proof. Fix $\epsilon > 0$. For any K, let K' denote the set $K_+ \cup (K_-)^*$ as above. Then,

$$\mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap K = \emptyset\} \leq \mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap K' = \emptyset\}.$$

Similarly, we can reflect the negative real part of K' onto the positive real axis and increase the probabiilty. By doing this trick repeatedly and rotating, we see that for any $\delta > 0$ and any K we can find $K_\delta \subset \{0 \leq \arg(z) \leq \delta\}$ with

$$\mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap K = \emptyset\} \leq \mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap K_\delta = \emptyset\}.$$

For fixed ϵ, we can use connectivity of K_δ to see that

$$\lim_{\delta \downarrow 0} \mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap K_\delta = \emptyset \mid B[0,\tau_{\mathbb{D}}] \cap [\epsilon, 1] \neq \emptyset\} = 0.$$

Therefore,

$$\limsup_{\delta \downarrow 0} \mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap K_\delta = \emptyset\} \leq \mathbb{P}\{B[0,\tau_{\mathbb{D}}] \cap [\epsilon, 1] = \emptyset\}.$$

□

In applications one generally uses a corollary of the Beurling projection theorem often called the Beurling estimate.

Corollary 7.3 (Beurling estimate)**.** *There exists $c < \infty$ such that if D is domain with $0 \notin D$ and such that the connected component of $\mathbb{C} \setminus D$ containing the origin intersects the unit circle, then for all $|z| \geq 2$,*

$$\mathrm{hm}_D(z, \partial D \cap \{|\zeta| \leq \epsilon\}) \leq c\sqrt{\epsilon}.$$

Proof. By making D larger if necesssary, we can assume that $D = \mathbb{C} \setminus K$ where K is a compact connected subset of $\overline{\mathbb{D}}$ intersecting the unit circle. Let B_t be a Brownian motion starting at z and let $\rho = \inf\{t : |B_t| \leq \epsilon\}$. Then, $\mathrm{hm}_D(z, \partial D \cap \{|\zeta| \leq \epsilon\}) \leq \mathbb{P}^z\{\rho \leq \tau\}$. Now consider $f(w) = \epsilon/w$ and use

$$\mathbb{P}^z\{\rho \leq \tau\} = \mathbb{P}^{f(z)}\{\tau_{\mathbb{D}} \leq \tau_{f(D)}\}.$$

□

7.2 Koebe distortion theorems

The Riemann mapping theorem implies that there is a one-to-one correspondence between simply connected domains $D \subsetneq \mathbb{C}$ containing the origin and univalent functions f on the unit disk with $f(0) = 0, f'(0) > 0$. We let $\mathcal{S}^*$ denote the set of all such function and $\mathcal{S}$ the set of $f \in \mathcal{S}^*$ with $f'(0) = 1$. Any function $f \in \mathcal{S}$ can be written as a power series

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots .$$

One example of such a function is the *Koebe function* f_{Koebe},

$$f_{\mathrm{Koebe}}(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4} = z + 2z^2 + 3z^3 + \cdots .$$

Using the second expressions, we can see that f_{Koebe} is a composition of conformal transformations, and hence is a conformal transformation, with $f_{\mathrm{Koebe}}(\mathbb{D}) = \mathbb{C} \setminus (-\infty, -1/4]$. The Koebe function is an extremal function in $\mathcal{S}$, and a big problem in the twentieth century was to prove the Bieberbach conjecture:

- If $f \in \mathcal{S}$, then $|a_n| \leq n$ for all n.

This was proved by de Branges. Fortunately, for most applications one does not need this result (and for this reason we do not need to go through the proof).

Lemma 7.4. *Suppose* $f : \mathbb{D} \to D$ *is a conformal transformation with* $f(0) = 0, f'(0) > 0$ *and* $\mathrm{dist}(0, \partial D) \geq 1$. *Then*

$$\log f'(0) = \frac{1}{2\pi} \int_0^{2\pi} G_D(e^{i\theta})\, d\theta. \tag{22}$$

Proof. Fix f, D and we allow constants and $O(\cdot)$ error terms to depend on f, D. Let k equal the right-hand side of (22). We know from (12) that

$$G_D(z) = -\log |z| + k + O(|z|), \quad z \to 0.$$

However, as $z \to 0$,

$$\begin{aligned} -\log|z| = G_{\mathbb{D}}(z) = G_D(f(z)) &= G_D(f'(0)\, z + O(|z|^2)) \\ &= -\log\left[f'(0)\, z + O(|z|^2)\right] + k + O(|z|) \\ &= -\log|z| - \log f'(0) + k + O(|z|). \end{aligned}$$

□

Lemma 7.5. *Suppose* D *is a regular domain containing the origin. Let* $T = \inf\{t \geq 0 : B_t \in \mathbb{R}\}$. *Then for every* $z \in D$,

$$G_D(z, 0) = \mathbb{E}^z \left[G_D(B_T); T < \tau_D\right].$$

Proof. We write $G(z) = G_D(z, 0)$. If $z \in \mathbb{R}$ the result is immediate. Assume that $\mathrm{Im}(z) > 0$ (the case $\mathrm{Im}(z) < 0$ is done similarly). We allow constants to depend on z, D. Let s be sufficiently large so that $\mathrm{dist}(0, \partial D) > e^{-s}$ and let $\xi_s = \tau_D \wedge T \wedge \sigma_s$. Since $M_t = G(B_{t\wedge\xi_s})$ is a continuous, bounded martingale,

$$\begin{aligned} G(z) &= \mathbb{E}^z\left[G(B_{\xi_s})\right] \\ &= \mathbb{E}^z\left[G(B_T); T < \tau_D \wedge \xi_s\right] + \mathbb{E}^z\left[G(B_{\sigma_s}); \sigma_s < \tau_D \wedge T\right]. \end{aligned}$$

We know that for $|\zeta| = e^{-s}$ that $G_D(\zeta) \leq c\, s$. Also, using the Poisson kernel in $\mathbb{H}$, we see as $s \to \infty$,

$$\mathbb{P}^z\{\sigma_s < \tau_D \wedge T\} \leq \mathbb{P}^z\{\sigma_s < T\} \leq c\, e^{-s}.$$

Therefore,

$$\lim_{s\to\infty} \mathbb{E}^z\left[G(B_{\sigma_s}); \sigma_s < \tau_D \wedge T\right] = 0,$$

and, hence, by the monotone convergence theorem,

$$G(z) = \lim_{s\to\infty} \mathbb{E}^z\left[G(B_T); T < \tau_D \wedge \xi_s\right] = \mathbb{E}^z\left[G(B_T); T < \tau_D\right].$$

□

Proposition 7.6. *Suppose $f : \mathbb{D} \to D$ is a conformal transformation with $f(0) = 0, f'(0) > 0$ and* $\mathrm{dist}(0, \partial D) = 1$. *Then*

$$1 \le f'(0) \le 4. \tag{23}$$

Proof. The first inequality follows from (22) (or from the Schwarz lemma applied to f^{-1}, considered as a map from $\mathbb{D}$ into $\mathbb{D}$). To get the second inequality, we show that the right hand side of (22) is maximized if $D = \mathbb{C} \setminus [1, \infty)$. This is done similarly as in the Beurling inequality. Suppose that $D = \mathbb{C} \setminus K$, and as before we write

$$K_+ = \{z \in K : \mathrm{Im}(z) \ge 0\}, \quad K_- = \{z \in K : \mathrm{Im}(z) \le 0\},$$

$$(K_-)^* = \{z : \overline{z} \in K_-\}, \quad K' = K_+ \cup \bar{(}K_-)^*.$$

Let $D' = C \setminus K'$ and note that D is simply connected. We claim that

$$\frac{1}{2\pi} \int_0^{2\pi} G_D(e^{i\theta})\, d\theta \le \frac{1}{2\pi} \int_0^{2\pi} G_{D'}(e^{i\theta})\, d\theta. \tag{24}$$

To see this let $\sigma = \inf\{t : |B_t| = 1\}$ and $T = \inf\{t \ge \sigma : B_t \in \mathbb{R}\}$. Then, using Lemma 7.5, we see that

$$\frac{1}{2\pi} \int_0^{2\pi} G_D(e^{i\theta})\, d\theta = \mathbb{E}\left[G_D(B_T)\; 1\{B[0,T] \cap K = \emptyset\}\right],$$

and similarly for D', K'.

As in the proof of Lemma 7.1, we let $W_t = B^1_t + i\, J\, B^2_t$ where J is a random variable independent of B with $\mathbb{P}\{J = \pm 1\} = 1/2$. Clearly W is a Brownian motion with $W_T = B_T$, and hence

$$\frac{1}{2\pi} \int_0^{2\pi} G_D(e^{i\theta})\, d\theta = \mathbb{E}\left[G_D(W_T)\; 1\{B[0,T] \cap K = \emptyset\}\right],$$

Let

$$E_1 = \{B[0,T] \cap K_+ = \emptyset\}, \quad E_2 = \{B[0,T] \cap K_- = \emptyset\},$$

$$E_3 = \{B[0,T] \cap (K_+)^* = \emptyset\}, \quad E_4 = \{B[0,T] \cap (K_-)^* = \emptyset\}.$$

Arguing as in that proof, conditioned on $B[0,T]$,

$$\mathbb{P}\{W[0,T] \cap K = \emptyset \mid B[0,T]\} \le \mathbb{P}\{W[0,T] \cap K' = \emptyset \mid B[0,T]\}.$$

Proposition 7.2 tells us that $G_D(x) \le G_{D'}(x)$ for $x \in \mathbb{R}$. This gives (24).

Given (24), we can do the argument in Theorem 4 to see that we can choose a maximizing D so that K lies in a wedge $\{0 \le \arg(w) \le \delta\}$ of arbitrarily small width, and then we argue as in the Beurling estiamate to see that the supremum is taken on by a slit domain $D = \mathbb{C} \setminus (-\infty, -1]$ for which $f(z) = 4\, f_{\mathrm{Koebe}}(z)$.

□

Corollary 7.7 (Koebe (1/4)-theorem)**.** *Let $f \in \mathcal{S}^*$ and let $d = \mathrm{dist}(0, \partial f(\mathbb{D}))$. Then*

$$d \le f'(0) \le 4d.$$

In particular, if $f \in \mathcal{S}$, then $f(\mathbb{D})$ contains the open disk of radius $1/4$ about the origin.

The next proposition is a slightly weaker version of the "growth theorem".

Proposition 7.8. *There exists $c < \infty$ such that if $f \in \mathcal{S}$, then*

$$|f(z)| \leq c\,[1-|z|]^{-2}.$$

Proof. All constants c in this proof are independent of f, that is, the estimates hold for all $f \in \mathcal{S}$. Let $D = f(\mathbb{D})$ and recall that $G_{\mathbb{D}}(z) = -\log|z| \geq 1-|z|$. By the Schwarz lemma applied to f^{-1}, we can see that there exists $z \in \partial\mathbb{D} \setminus f(\mathbb{D})$. We claim that there exists $c < \infty$ such that for all $f \in \mathcal{S}$ and all $z \in \partial\mathbb{D}$, $G_D(z) \leq c_3$. Indeed, if one goes to the proof of Proposition 3.1, especially (9), (10) we see that $G_D(z) \leq 1/\rho$ where $\rho = \rho_D$ is the infimum over $z \in \partial\mathbb{D}$ of the probability that a Brownian motion leaves D before reaching $C_1 := \{|w| = e^{-1}\}$. If $D = f(\mathbb{D})$ for $f \in \mathcal{S}$, this probability is greater than the probability that the Brownian motion makes a closed loop in $\{1 < |z| < e\}$ before reaching C_1.

Using Theorem 4, we can see that there exists c' such that if $|\zeta| > 1$, $\mathbb{P}^{\zeta}\{\sigma_0 < \tau_D\} \leq c'\,|\zeta|^{-1/2}$. Hence,

$$G_D(\zeta) \leq \mathbb{P}^{\zeta}\{\sigma_0 < \tau_D\} \sup_{|w|=1} G_D(w) \leq c''\,|\zeta|^{-1/2},$$

and $|\zeta| \leq c\,G_D(\zeta)^{-2}$ for some c. Hence, for all $z \in \mathbb{D}$ with $|f(z)| \geq 1$,

$$|f(z)| \leq c\,G_D(f(z))^{-2} = c\,[G_{\mathbb{D}}(z)]^{-2} \leq c\,[1-|z|]^{-2}.$$

□

We will now prove a form of the "distortion theorem". This is not as strong as the standard version, but this is easy to prove now and is that is needed for most arguments. The key fact is that the constants $c = C(D,V)$ can be chosen uniformly over $\mathcal{S}$.

Proposition 7.9 (Distortion Principle)**.** *Suppose D is a domain and $V \subset D$ is compact. Then there exists $c = c(D,V) < \infty$ such that if $f : D \to f(D)$ is a conformal transformation, then*

$$|f'(z)| \leq c\,|f'(w)|, \quad z,w \in V.$$

Proof. We first assume that $D = \mathbb{D}$. Since f is uniformly bounded on $\{|z| \leq 1/2\}$, the Cauchy integral formula gives a uniform bound on $|f''|$ for $|z| \leq 1/4$, and this implies that there exists $c < \infty$ such that

$$|f'(z) - 1| \leq c\,|z|, \quad |z| \leq 1/4.$$

In particular, we can find δ such that $1/2 \leq |f'(z)| \leq 2$ for $|z| \leq \delta$ and hence

$$|f'(w)| \leq 4\,|f'(z)|, \quad |z|,|w| \leq \delta. \tag{25}$$

Let us define a metric $\rho_D(z,w)$ on D to be the minimum integer k such that we can write down a sequence

$$z = \zeta_0, \zeta_1, \ldots, \zeta_k = w,$$

such that for $j = 1, \ldots, k$,

$$|\zeta_j - \zeta_{j-1}| < \delta \, \max\{\mathrm{dist}(\zeta_{j-1}, \partial D), \mathrm{dist}(\zeta_j, \partial D)\},$$

where δ is as in the last proof. Then, we have $|f'(z)| \leq 4^{\rho_D(z,w)}\,|f'(w)|$. Arguing as in the proof of Proposition 2.6, we can see that for all compact V, $\max\{\rho_D(z,w) : z,w \in V\} < \infty$.

□

These arguments measure the "closeness" of z and w in D tp be the number of balls $\{\zeta : |\zeta - z_j| \leq \operatorname{dist}(z_j, \partial D)\}$ are needed to "connect" z to w. This measure of distance is closely related to hyperbolic distance. This definition in the last proof is valid for all domains.

We end by stating the more precise distortion estimate. Usually we do not need the precision in this statement, but since the optimal constants are known, it is generally nicer to use them.

Theorem 5 (Distortion Theorem)**.** *If $f \in \mathcal{S}$ and $|z| < 1$, then*

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}.$$

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

Corollary 7.10. *There exists $c_0 > 0$ such that if $f : \mathbb{H} \to f(\mathbb{H})$ is a conformal transformation and $x \in \mathbb{R}$, $r > 0$,*

$$\frac{|f'(ri)|}{c_0\,(x^4+1)} \leq |f'(rx+ri)| \leq c_0\,(x^4+1)\,|f'(ri)|, \tag{26}$$

$$|f(rx+ri) - f(ri)| \leq \; c_0\, r\,(|x|^4+1)\,|f'(ri)|. \tag{27}$$

Proof. For (26), without loss of generality assume that $r = 1$, $f(i) = 0, f'(i) = -2i$. Let us also assume $|x| \geq 1$; otherwise we use the distortion principle immediately. Let $F(z) = (z-i)/(z+i)$ which maps $\mathbb{H}$ onto $\mathbb{D}$ with $F(i) = 0, F'(i) = -2i$, and let $g = f \circ F^{-1} \in \mathcal{S}$. Note that $|F(x+i)| \leq 1 - c\,x^{-2}$, and hence the distortion theorem implies that

$$\frac{c}{x^2} \leq |g'(F^{-1}(F(x+i)))| = \frac{|f'(x+i)|}{|F'(x+i))|}.$$

We check directly that $|F'(x+i)| \asymp x^{-2}$. Therefore $|f'(x+i)| \geq c\,x^{-4}$. This gives the first inequality in (26) and the second follows from the first applied to $\tilde{f}(z) = f(z-x)$. The estimate (27) follows from $|g'(F(z))| \leq c\,(1-|F(z)|)^{-2}$.

$$|f(rx+ri) - f(ri)| \leq \int_0^{rx} |f'(s+ri)|\,ds.$$

It is clear that by doing this proof slightly more carefully we could find an explicit c_0, but we will not need it. $\square$

8 Loewner differential equation

We will give a number of versions of what are called Loewner differential equations. These equations describe the dynamics of conformal maps as a domain is perturbed. As a start we will describe one version of the half-plane equation. Suppose $\gamma : (0, \infty) \to \mathbb{H}$ is a simple curve with $\gamma(0+) = 0$. (For us curve means a continuous image of the real line and simple means that the function is one-to-one. For each t, let $H_t = \mathbb{H} \setminus \gamma[0,t]$. The Riemann mapping theorem tells us that there exist conformal maps $g_t : H_t \to \mathbb{H}$. There are many such maps, but as we will see we can specify a unique one by requiring that

$$g_t(z) = z + o(1), \quad z \to \infty.$$

For fixed $z \in \mathbb{H}$, we can consider the flow $t \mapsto g_t(z)$. If $z \notin \gamma(0,\infty)$, then this flow exists for all times. If $z = \gamma(t)$ then the flow stops at time t at which $g_t(z) \in \mathbb{R}$.

The main result is that *if we reparametrize γ appropriately* then $g_t(z)$ is a a C^1 function of t that satisfies

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z,$$

where $U_t = g_t(\gamma(t))$.

We will first derive the equation in the case γ is a simple curve and show that $t \mapsto U_t$ is a continuous real-valued function. We then will consider the equation as an initial value problem for a given continuous U_t and discuss the solutions of the differential equation.

8.1 A class of transformations

We will consider simply connected subdomains of $\mathbb{H}$ of the form $\mathbb{H} \setminus K$ where K is a bounded set. We will be making estimates that are valid over all such domains so it useful to set up some notation. Recall that we write $\mathrm{rad}(K) = \mathrm{rad}(0,K) = \sup\{|z| : z \in K\}$.

Definition Let $\mathcal{J}_q$ denote the set of subdomains $D \subset \mathbb{H}$ with $\mathrm{rad}(\mathbb{H} \setminus D) \leq q$, and let $\mathcal{J}'_q$ be the set of real translations $D = D + x, D \in \mathcal{J}_q, x \in \mathbb{R}$. Let $\mathcal{J} = \mathcal{J}_1, \mathcal{J}' = \mathcal{J}'_1$.

We allow multiply connected domains in $\mathcal{J}$. Note that $D \in \mathcal{J}$ if and only if $f(D) \in \mathcal{K}$ where $f(z) = -1/z$. Suppose $D \in \mathcal{J}_q, D' \in \mathcal{J}_{q'}$ and $g : D \to D'$ is a conformal transformation such that $g(\infty) = \infty$ (that is, if $z \to \infty$, then $g(z) \to \infty$) and such that for $x \in \mathbb{R} \setminus [x_1, x_2]$,

$$\lim_{y \downarrow 0} g(x + iy) \in \mathbb{R}.$$

Then we can use Schwartz reflection to extend g to a conformal transformation of

$$D^* := D \cup \{\overline{z} : z \in D\} \cup (-\infty, x_1) \cup (x_2, \infty).$$

The map

$$f(z) = \frac{1}{g(1/z)}$$

is a univalent function in a neighborhood $\mathcal{N}$ of the origin with $f(0) = 0$, and hence has a power series expansion

$$f(z) = a_1 \, z + a_2 \, z^2 + \cdots .$$

Since $f(\mathcal{N} \cap \mathbb{R}) \subset \mathbb{R}$, we can see that $a_j \in \mathbb{R}$, and since $f(\mathcal{N} \cap \mathbb{H}) \subset \mathbb{H}$, we can see that $a_1 > 0$. Using this we see that g has an expansion at infinity

$$g(z) = b_{-1} \, z + b_0 + b_1 \, z^{-1} + b_2 \, z^{-2} + \cdots , \quad b_{-1} > 0, \quad b_j \in \mathbb{R}.$$

We will write $g'(\infty) = 1$ if $b_{-1} = 1$. If g has an expansion as above, and $\tilde{g}(z) = (g(z) - b_0)/b_{-1}$, then $\tilde{g}(z) = z + o(1)$ as $z \to \infty$.

One might expect that we should define $g'(\infty) = b_{-1}$. However, for reasons that we will discuss later, there are good reasons to define $g'(\infty) = 1/b_{-1}$. We will avoid this issue for the moment by only defining "$g'(\infty) = 1$" which is the same under both definitions.

Definition Let $\mathcal{Q}_q$ denote the set of conformal transformations $g : D \longrightarrow g(D)$ where $D \in \mathcal{J}_q, g(D) \in \mathcal{J}'$ and such that

$$g(z) - z \to 0, \quad z \to \infty.$$

Let $\mathcal{Q} = \mathcal{Q}_1$. Transformations in $\mathcal{Q}$ are sometimes said to satisfy the *hydrodynamic normalization.*

Lemma 8.1. *Suppose $D \in \mathcal{J}_q$. Then there is a unique positive harmonic function v_D on D such that $v_D \equiv 0$ on ∂D and $v_D(z) = \operatorname{Im}(z) + O(1)$ as $z \to \infty$. It is given by*

$$v_D(z) = \operatorname{Im}(z) - \mathbb{E}^z\left[\operatorname{Im}(B_{\tau_D})\right] = \lim_{n\to\infty} n\mathbb{P}^z\{T_n < \tau_D\},$$

where $T_n = \inf\{t : \operatorname{Im}(B_t) = n\}$. Moreover, there exist $c = c_D < \infty$ such that for $|z| \geq 2q$,

$$|\operatorname{Im}(z) - v_D(z)| \leq \frac{c\operatorname{Im}(z)}{|z|^2}.$$

To be more precise, we mean that if we extend v_D to ∂D by setting $v_D(z) = 0$ for $z \in \partial D$, then v_D is continuous at the regular points of ∂D.

Proof. If v_D is such a function, then $\operatorname{Im}(z) - v_D(z)$ is a bounded harmonic function on D, and hence,

$$\operatorname{Im}(z) - v_D(z) = \mathbb{E}^z\left[h(B_{\tau_D})\right] = \mathbb{E}^z\left[\operatorname{Im}(B_{\tau_D})\right].$$

This gives existence and uniqueness of v_D. Since $\operatorname{Im}(w)$ is a bounded harmonic function on $\{0 < \operatorname{Im}(w) < n\}$, if $0 < \operatorname{Im}(z) < n$,

$$\operatorname{Im}(z) = \mathbb{E}^z\left[\operatorname{Im}(B_{\tau_D \wedge T_n})\right] = \mathbb{E}^z\left[\operatorname{Im}(B_{\tau_D}); \tau_D < T_n\right] + n\,\mathbb{P}^z\{T_n < \tau_D\}.$$

Using the monotone convergence theorem, we therefore see that

$$\begin{aligned}\lim_{n\to\infty} n\,\mathbb{P}^z\{T_n < \tau_D\} &= \operatorname{Im}(z) - \lim_{n\to\infty} \mathbb{E}^z\left[\operatorname{Im}(B_{\tau_D}); \tau_D < T_n\right] \\ &= \operatorname{Im}(z) - \mathbb{E}^z\left[\operatorname{Im}(B_{\tau_D})\right].\end{aligned}$$

To get the final assertion, note that

$$|\operatorname{Im}(z) - v_D(z)| \leq \mathbb{P}^z\{B_{\tau_D} \notin \mathbb{R}\}\,\sup\{\operatorname{Im}(w) : w \in \mathbb{H} \setminus D\} \leq c_D\,\mathbb{P}^z\{B[0, \tau_{\mathbb{H}}] \cap (q\mathbb{D}) \neq \emptyset\}.$$

The probability on the right can be computed by conformal invariance. We omit the details. $\square$

- One can consider $\mathcal{Q}$ as a half-plane analogue of the schlicht functions $\mathcal{S}$.
- If $g \in \mathcal{Q}_q$ with domain D, let $\tilde{D} = q^{-1}D$ and $\tilde{g}(z) = q^{-1}g(qz)$. Then $\tilde{g} \in \mathcal{Q}$ with $\tilde{g}'(z/q) = g'(z)$. We will focus on estimates for $g \in \mathcal{Q}$, but these immediately imply estimates for $g \in \mathcal{Q}_q$.

- Every $g \in \mathcal{Q}$ has an expansion at infinity
$$g(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots, \quad b_j \in \mathbb{R}.$$

- If we write $g(z) = u(z) + iv(z)$, then $h(z) := \mathrm{Im}(z) - v(z)$ is a bounded harmonic function on D such that $h(z) \to 0$ as $z \to \infty$. Note that h is a harmonic function on D^*.

- If $g : D \to g(D)$ is in $\mathcal{Q}$ then so is $g^{-1} : g(D) \to D$ with expansion
$$g^{-1}(z) = z - b_1 z^{-1} + O(|z|^{-2}).$$

- If $D \in \mathcal{J}_q$ is simply connected, there exists unique $g \in \mathcal{Q}_q$ such that $g(D) = \mathbb{H}$. The existence of conformal transformations $f : D \to \mathbb{H}$ follows from the Riemann mapping theorem, and if $\tilde{f}$ is another such a transformation, then $\tilde{f} = T \circ f$ where T is a Möbius transfromation of $\mathbb{H}$. There is exactly one such T such that $T \circ f \in \mathcal{Q}$. In this case $\mathrm{Im} g = v_D$ where v_D is the function from Lemma 8.1.

- If $g = u + iv \in \mathcal{Q}$ and $z = x + iy$ with $|z| > 1$, we can use the Cauchy-Riemann equations to write
$$\begin{aligned} u(x, iy) &= \lim_{y_1 \to \infty} [u(x, iy_1) + u(x, iy) - u(x, iy_1)] \\ &= \lim_{y_1 \to \infty} \left[u(x, iy_1) - \int_y^{y_1} \partial_x v(x + it)\, dt \right] \\ &= x - \int_y^\infty \partial_x v(x + it)\, dt \\ &= x + \int_y^\infty \partial_x h(x + it)\, dt. \end{aligned}$$

  To see that the integral is well defined, note that for $|z| \geq 2$, h (extended to D^*) is a bounded harmonic function in the disk of radius $|z|/2$ about z and is bounded by $c\,|z|^{-1}$. Therefore by Proposition 2.5, $|\nabla h(z)| \leq c\,|z|^{-2}$.

Proposition 8.2. *Suppose $D \in \mathcal{J}$ and h is a positive harmonic function on D that is bounded on $\{|z| \geq 1\}$ and equals zero on $\{x \in \mathbb{R} : |x| \geq 1\}$. Then for $|z| \geq 2$,*
$$h(z) = H_{\mathbb{H}}(z, 0)\, h_\infty\, [1 + O(|z|^{-1})] = -\mathrm{Im}(1/z)\, h_\infty\, [1 + O(|z|^{-1})],$$
where
$$h_\infty = \lim_{y \to \infty} y\, h(iy) = \frac{2}{\pi} \int_0^\pi h(e^{i\theta}) \sin\theta\, d\theta. \tag{28}$$
Moreover, if $y > 1$,
$$h_\infty = \frac{1}{\pi} \int_{-\infty}^\infty h(x + iy)\, dx. \tag{29}$$

The condition $|z| \geq 2$ is put into the proposition to make the estimates uniform over z. We could replace it with $|z| \geq r$ for any $r > 1$, but then the implicit constant could depend on r.

As $|x| \to \infty$, $h(x+iy) \leq O(x^{-2})$ which shows that the integral in (29) is finite.

Proof. Let $O_+ = O \cap \mathbb{H} = \{z \in \mathbb{H} : |z| > 1\}$. Since h is a bounded harmonic function on O_+, the optional sampling theorem implies that

$$h(z) = \mathbb{E}^z\left[h(B_+)\right] = \frac{1}{\pi}\int_{\partial O_+} H(z,w)\,h(w)\,|dw|.$$

Using conformal invariance (see (58)), we can see that

$$H_{O_+}(z,e^{i\theta}) = 2\,\mathrm{Im}[-1/z]\,\sin\theta\,[1+O(|z|^{-1})] = 2\,H_{\mathbb{H}}(z,0)\,\sin\theta\,[1+O(|z|^{-1})].$$

Since $h(x) = 0$ for $x \in \mathbb{R}\cap\partial O_+$,

$$h(z) = \frac{2\,H_{\mathbb{H}}(z,0)}{\pi}\,\left[1+O(|z|^{-1})\right]\int_0^\pi h(e^{i\theta})\,\sin\theta\,d\theta$$

This gives (28). Let $U_y = \{x+is : s > y\}$. Then, if $y' > y$,

$$y'\,h(iy') = \frac{y'}{\pi}\int_{\partial U_y} H_{U_y}(iy',x+iy)\,h(x+iy)\,dx = \frac{y'}{\pi(y'-y)}\int_{-\infty}^{\infty}\frac{(y'-y)^2}{x^2+(y'-y)^2}\,h(x+iy)\,dx$$

Letting $y' \to \infty$, we get (29). $\square$

We note that the Harnack principle implies that there exists c_1, c_2 such that for all such h,

$$c_1\,h_\infty \leq h(2i) \leq c_2\,h_\infty.$$

We can write

$$h_\infty = \frac{4}{\pi}\lim_{z\to\infty}\mathbb{E}^z\left[h(B_\tau)\mid |B_\tau| = 1\right].$$

We can then view the results as two separate estimates:

$$\mathbb{E}^z\left[h(B_\tau)\mid |B_\tau| = 1\right] = \frac{\pi}{4}\,h_\infty\,[1+O(|z|^{-1})],$$

$$\mathbb{P}^z\{|B_\tau| = 1\} = \frac{4\mathrm{Im}(z)}{\pi|z|^2}\,\left[1+O(|z|^{-1})\right].$$

Definition Suppose $K \subset \mathbb{H}$ is bounded such that $D = \mathbb{H}\setminus K$ is a domain. Then the *half-plane capacity* $\mathrm{hcap}(K)$ is defined by

$$\mathrm{hcap}(K) = \lim_{y\to\infty} y\,\mathbb{E}^{iy}\left[\mathrm{Im}[B(\tau_D)]\right].$$

Suppose $D = \mathbb{H} \setminus K, \tau = \tau_D$, and let $h(z) = \mathbb{E}^z[\mathrm{Im}(B_\tau)]$ be the bounded harmonic function on D with boundary value $\mathrm{Im}(z)$. If $D \in \mathcal{J}$, that is, if $K \subset \{|z| \leq 1\}$, then in the notation of Proposition 8.2,

$$\mathrm{hcap}(K) = h_\infty = \frac{2}{\pi}\int_0^\pi \mathbb{E}^{e^{i\theta}}[\mathrm{Im}(B_\tau)] \sin\theta \, d\theta.$$

Also, for $|z| \geq 2$,

$$h(z) = -\mathrm{hcap}(K)\,\mathrm{Im}(1/z)\,[1 + O(|z|^{-1})].$$

In particular, $\mathrm{hcap}(K) \asymp \mathbb{E}^{2i}[\mathrm{Im}(B_\tau)]$.

Proposition 8.3. *Suppose $K \subset \mathbb{H}$ is bounded such that $D = \mathbb{H} \setminus K$ is a domain.*

1. *If $r > 0$, then* $\mathrm{hcap}(rK) = r^2\,\mathrm{hcap}(K)$.
2. *If $x \in \mathbb{R}$, then* $\mathrm{hcap}(x + K) = \mathrm{hcap}(K)$.

Proof.

1. Let $D_r = \mathbb{H} \setminus rK$. Then, by conformal invariance

   $$\mathbb{E}^{iry}\left[\mathrm{Im}(B_{\tau_{D_r}})\right] = r\,\mathbb{E}^{iy}\left[\mathrm{Im}(B_{\tau_D})\right].$$

   Therefore,

   $$\mathrm{hcap}(rK) = \lim_{y\to\infty} ry\,\mathbb{E}^{iry}\left[\mathrm{Im}(B_{\tau_{D_r}})\right] = r^2 \lim_{y\to\infty} y\,\mathbb{E}^{iy}\left[\mathrm{Im}(B_{\tau_D})\right] = r^2\,\mathrm{hcap}(K).$$

2. Let $D_x = \mathbb{H} \setminus (K + x) = D + x$. Then,

   $$\mathbb{E}^{iy}\left[\mathrm{Im}(B_{\tau_{D_x}})\right] = \mathbb{E}^{-x+iy}\left[\mathrm{Im}(B_{\tau_{D_x}})\right].$$

   Using, for example, Proposition 8.2 (or just derivative estimates for harmonic functions), we can see that for fixed x, as $y \to \infty$,

   $$\mathbb{E}^{-x+iy}\left[\mathrm{Im}(B_{\tau_{D_x}})\right] \sim \mathbb{E}^{iy}\left[\mathrm{Im}(B_{\tau_{D_x}})\right].$$

□

There is another notion of capacity that we will consider that scales differently from hcap.

Definition Suppose V is a compact subset of $\overline{\mathbb{H}}$. For a Brownian motion B_t and let $D = \mathbb{H} \setminus V$. Then

$$\mathrm{cap}_{\mathbb{H}}(V) = \lim_{y\to\infty} \mathbb{P}^{iy}\{B_\tau \in V\} = \lim_{y\to\infty} y\,\mathrm{hm}_{\mathbb{H}\setminus V}(iy, V).$$

Note that we allow V to be a subset of reals. The quantity $\mathrm{cap}_{\mathbb{H}}(V)$ is a normalized form of the point-to-set excursion measure as we now show. Let

$$f(z) = \frac{z - i}{z + i}$$

which is a conformal transformation of $\mathbb{H}$ onto $\mathbb{D}$. Then conformal invariance implies that

$$\mathrm{hm}_{\mathbb{H}\setminus V}(iy, V) = \mathrm{hm}_{\mathbb{D}\setminus f(V)}(f(iy), f(V)) = \mathrm{hm}_{\mathbb{D}\setminus f(V)}\left(\frac{y-1}{y+1}, f(V)\right).$$

Therefore,

$$\mathrm{cap}_{\mathbb{H}}(V) = 2\lim_{y\to\infty}\frac{y}{2}\,\mathrm{hm}_{\mathbb{D}\setminus f(V)}\left(1-\frac{2}{y+1}, f(V)\right) = 2\,\mathcal{E}_{\mathbb{D}\setminus f(V)}(1, f(V)).$$

- If $T = \inf\{t : B_t \in \mathbb{R}\}$, we can use the Poisson kernel in $\mathbb{H}$ to see that

$$\mathrm{cap}_{\mathbb{H}}([0,x]) = \lim_{y\to\infty} y\mathbb{P}^{iy}\{0 \le B_T \le x\} = \lim_{y\to\infty} y\int_0^x \frac{y\,dy}{\pi(t^2+y^2)}\,dt = \frac{x}{\pi}.$$

- If $V \subset \{|z| \le 1\}$ and $y \ge 2$,

$$\mathrm{cap}_{\mathbb{H}}(V) \asymp y\,\mathbb{P}^{iy}\{B[0,T]\cap V \ne \emptyset\} = y\,\mathrm{hm}_{\mathbb{H}\setminus V}(2i, V).$$

- Using conformal invariance, we get the following relations:

$$\mathrm{cap}_{\mathbb{H}}(V+x) = \mathrm{cap}_{\mathbb{H}}(V), \quad \mathrm{cap}_{\mathbb{H}}(rV) = r\,\mathrm{cap}_{\mathbb{H}}(V).$$

- Suppose V is the disk of radius ry about $z = x+iy$ where $0<r<1$. We claim that

$$\mathrm{cap}_{\mathbb{H}}(V) = 2y\,\left[\log(1/r) + O(r)\right]^{-1} \quad r \to 0. \tag{30}$$

  It suffices to prove this for $y=1, x=0$ for which it follows from

$$G_{\mathbb{H}}(iy, i) = \log\frac{y+1}{y-1} = 2y^{-1} + O(y^{-2}), \quad y\to\infty.$$

Since $\mathrm{cap}_{\mathbb{H}}$ scales linearly, one might expect that $\mathrm{cap}_{\mathbb{H}}$ of a connected set to be comparable to the diameter of a set. Indeed this is true if the set touches the boundary, but is not correct for "interior" sets.

Lemma 8.4. *There exists c_1, c_2 such that if $V \subset \mathbb{H}$ is compact and connected and $d = \mathrm{diam}(V)$, $y = \sup\{\mathrm{Im}(z); z\in V\}$, then*

$$c_1\,(d+y)\,[1+\log_+(y/d)]^{-1} \le \mathrm{cap}_{\mathbb{H}}(V) \le c_2\,(d+y)\,[1+\log_+(y/d)]^{-1}.$$

In particular, if both V and $V\cup\mathbb{R}$ are connected, then $\mathrm{cap}_{\mathbb{H}}(V) \asymp \mathrm{diam}(V)$.

Proof. By scaling and translation we may assume that $y=1$ and that $\min\{\mathrm{Re}(z) : z \in V\} = 0$. Let D denote the unbounded connected component of $\mathbb{H}\setminus V$. As noted above,

$$\mathrm{cap}_{\mathbb{H}}(V) \asymp (d+1)\,\mathrm{hm}_D(2(d+1)i, V). \tag{31}$$

If $d \ge 4$, let $s = d-1 \ge 3$. The upper bound follows immediately from (31). For the lower bound, note that there exists $z \in V$ with $\mathrm{Re}(z) = s$. Consider the square $\{x+iy : 0\le x\le s, 0\le y\le 1\}$.

We know that V is connected and contains points on both vertical sides, $[0, i]$ and $[s, s+i]$, If a Brownian motion B_t starting at $2(d+1)i$ exits $\mathbb{H}$ at $(0,1)$ without hitting either of the vertical sides $[0, i]$ or $[s, s+i]$, then either the curve hits V or the curve disconnects V. Since we know that V is connected, the former must then hold. Let $\sigma = \inf\{t : \mathrm{Im}(B_t) = 1\}, x = \mathrm{Re}(B_\sigma)$. If $x \in [1, s-1]$, then there is a probability of $1/4$ that the continuation of the path will exit the square $[x-1, x+1] \times [x-1+2i, x+1-2i]$ at $[x-1, x+1]$. Therefore,

$$\mathrm{hm}_D(2(d+1)i, V) \geq \frac{1}{4}\, \mathbb{P}^{2(d+1)i}\{\mathrm{Re}(B_\sigma) \in [1, d-2]\}.$$

Using the exact form of the Poisson kernel in the upper half plane, we can see that

$$\inf_{d \geq 4} \mathbb{P}^{2(d+1)i}\{\mathrm{Re}(B_\sigma) \in [1, d-2]\} > 0.$$

If $d \leq 1/2$, let $z = x+i$ be a point in V with maximal imaginary part and note that $0 \leq x \leq 1/2$. Let $\mathcal{B}_r$ denote the closed disk of radius r centered at z with boundary ∂_r. Since $\mathrm{cap}_{\mathbb{H}}(V) \leq \mathrm{cap}_{\mathbb{H}}(\mathcal{B}_d)$, the upper bound follows from (30). The connected set V intersects $\partial_{d/2}$. Let $q > 0$ be the probability that a Brownian motion starting at $|z| = 1/2$ makes a closed loop about the origin before reaching the unit circle. Suppose that a Brownian motion starting at $2(d+1)i$ reaches $\partial_{d/4}$ before leaving $\mathbb{H}$. Then there is a probability q, that it will make a loop about z before reaching the circle of radius $d/2$. If that happens, the curve must hit V. From this we get the inequality

$$\mathrm{cap}_{\mathbb{H}}(V) \geq q\, \mathrm{hm}_{\mathbb{D} \setminus \mathcal{B}_{d/4}}(2(d+1)i, \mathcal{B}_{d/4}) \asymp q\, \mathrm{cap}_{\mathbb{H}}(\mathcal{B}_{d/4}).$$

This and (30) give the lower bound.

If $1/2 \leq d \leq 4$ we can use the $d = 1/2$ estimate for a lower bound and the $d = 4$ estimate for an upper bound.

□

The estimate above is useful in studying the boundary behavior of conformal maps. For future reference we state a disk version of the proposition that can be proved in the same way. We will only give the boundary version.

Proposition 8.5. *There exist $0 < c_1 < c_2 < \infty$ such that if $V \subset \overline{\mathbb{D}}$ is a connected compact set with $V \cap \partial\mathbb{D} \neq 0$, then*

$$c_1\, \mathrm{diam}(V) \leq \mathbb{P}\{B[0, \tau_{\mathbb{D}}] \cap V \neq \emptyset\} \leq c_2\, \mathrm{diam}(V).$$

Roughly speaking, the quantity $\mathrm{cap}_{\mathbb{H}}(V)$ is the normalized probability that a Brownian motion "starting at infinity" exits $\mathbb{H}\setminus V$ at V. It is a version of excursion measure. The quantity $\mathrm{hcap}(K)$ is a normalized probability that a Brownian motion "starting at infinity and conditioned to leave $\mathbb{H}$ at infinity" hits K. This is only nonzero if $K \subset \mathbb{H}$, and if K is very close to the real line it is near zero. It is analogous to what we will call boundary bubbles.

8.2 Compact hulls

Definition We call a compact $K \subset \overline{\mathbb{H}}$ a *compact* $\mathbb{H}$*-hull* if

- $K \cap \mathbb{R} \neq \emptyset$.
- $K \cup \mathbb{R}$ is connected.

We have chosen to make connectedness of $K \cup \mathbb{R}$ one of the conditions for being a hull. This is not always done. We choose this definition for convenience.

For any such K, let D_K denote the unbounded component of $\mathbb{H} \setminus K$ and note that D_K is simply connected. Let $x_-(K) = \min\{x : x \in K \cap \mathbb{R}\}, x_+(K) = \max\{x : x \in K \cap \mathbb{R}\}$. We define the *fill* of K by $\mathrm{fill}(K) = (\mathbb{H} \setminus D_K) \cup [x_-(K), x_+(K)]$. Note that $\mathrm{fill}(K)$ is a compact $\mathbb{H}$-hull. Let $R_K = \sup\{|z| : z \in K\} = \sup\{|z| : z \in \mathrm{fill}(K)\}$ and

$$D^* = \mathbb{C} \setminus [\mathrm{fill}(K) \cup \{\overline{z} : z \in \mathrm{fill}(K)\}] .$$

Note that

$$\mathrm{hcap}(K) = \mathrm{hcap}(\mathrm{fill}(K)).$$

Sometimes, in an abuse of notation, we will refer to a bounded, but not closed, $K \subset \mathbb{H}$ as a compact $\mathbb{H}$-hull. In this case the implicit hull is the union of $\overline{K}$ and the smallest closed line segment in $\mathbb{R}$ needed to make the union connected.

Proposition 8.6. *There exists $c_0 < \infty$ such that the following holds. Suppose that K is a compact $\mathbb{H}$-hull, $D = D_K$, $R = R_K$, $a = \mathrm{hcap}(K)$.*

1. *There exists a unique conformal transformation $g = g_K : D \longrightarrow \mathbb{H}$ such that*
$$\lim_{z \to \infty} [g(z) - z] = 0.$$
It extends by Schwarz reflection to a conformal transformation $g : D^* \longrightarrow \mathbb{C} \setminus [x_1, x_2]$ for some $-\infty < x_1 < x_- \leq x_+ < x_2 < \infty$. For $z \in D$, $\mathrm{Im} g(z)$ is the same as $v_D(z)$ from Lemma 53.

2. *The expansion of g at infinity is*
$$g(z) = z + \frac{a}{z} + \sum_{j=2}^{\infty} b_j\, z^{-j}, \quad b_j \in \mathbb{R}.$$

3. $$x_1 = \lim_{y \to \infty} \pi\, y \left[\frac{1}{2} - \mathbb{P}^{iy}\{B_{\tau_D} \in (-\infty, x_-)\}\right],$$
$$x_2 = \lim_{y \to \infty} \pi\, y \left[\frac{1}{2} - \mathbb{P}^{iy}\{B_{\tau_D} \in (x_+, \infty)\}\right].$$
In particular, $x_1 \leq x_- \leq x_+ \leq x_2$.

4. *If $r > 0, x \in \mathbb{R}$, then*
$$g_{rK}(z) = r\, g(z/r), \qquad g'_{rK}(z) = g'(z/r),$$
$$g_{K+x}(z) = x + g_K(z - x).$$

5. *If $K \subset K'$, $g_{K'} = g_{g(K')} \circ g$. In particular,*
$$\mathrm{hcap}(K') = \mathrm{hcap}(K) + \mathrm{hcap}[g(K')]. \tag{32}$$
Here by $g(K')$ we mean the hull $\overline{g(K' \setminus K)}$.

6. *If $|z| \geq 2R$, then*
$$|g'(z) - 1| \leq c_0 \, \frac{a}{|z|^2}.$$

7. *If $|z| \geq 2R$, then*
$$\left| g_K(z) - z - \frac{a}{z} \right| \leq c_0 \, \frac{a\,R}{|z|}. \tag{33}$$

Proof.

1. The existence of the map was shown in the previous section.

2. Note that as $y \to \infty$,
$$g(iy) = i\left[y - \frac{b_1}{y}\right] + O(y^{-2}) = iv(iy) + O(y^{-2}).$$
Therefore, using Lemma 8.1 and the definition of hcap, we see that
$$b_1 = \lim_{y\to\infty} y\,[y - v(iy)] = \lim_{y\to\infty} y\,\mathbb{E}^{iy}\left[\mathrm{Im}(B_\tau)\right] = \mathrm{hcap}(K).$$

3. Using the Poisson kernel in $\mathbb{H}$, we see that
$$\lim_{y\to\infty} \pi\, y \left[\frac{1}{2} - \mathbb{P}^{iy}\{B_{\tau_{\mathbb{H}}} \in [x_+, \infty)\}\right] = x_+.$$
Conformal invariance implies that
$$\mathbb{P}^{iy}\{B_{\tau_D} \in [x_+, \infty)\} = \mathbb{P}^{g(iy)}\{B_{\tau_{\mathbb{H}}} \in [x_2, \infty)\}.$$
We know that $g(iy) = iy - iay^{-1} + O(y^{-2})$ and derivative estimates for harmonic function show that
$$\mathbb{P}^{g(iy)}\{B_{\tau_{\mathbb{H}}} \in [x_2, \infty)\} = \mathbb{P}^{i(y-ay^{-1})}\{B_{\tau_{\mathbb{H}}} \in [x_2, \infty)\} + O(y^{-2}).$$
Therefore,
$$\begin{aligned}
\lim_{y\to\infty} \pi\, y \left[\frac{1}{2} - \mathbb{P}^{iy}\{B_{\tau_D} \in [x_+, \infty)\}\right] &= \lim_{y\to\infty} \pi\, y \left[\frac{1}{2} - \mathbb{P}^{i(y-ay^{-1})}\{B_{\tau_{\mathbb{H}}} \in [x_2, \infty)\} + O(y^{-2})\right] \\
&= \lim_{y\to\infty} \pi\,(y - ay^{-1}) \left[\frac{1}{2} - \mathbb{P}^{i(y-ay^{-1})}\{B_{\tau_{\mathbb{H}}} \in [x_2, \infty)\}\right] \\
&= x_2.
\end{aligned}$$
Since $\mathbb{P}^{iy}\{B_{\tau_D} \in [x_+, \infty)\} \leq \mathbb{P}^{iy}\{B_{\tau_{\mathbb{H}}} \in [x_+, \infty)\}$, we see that $x_2 \geq x_+$. The argument for x_1 is the same.

4. Note that $\tilde{g}(z) := r g_D(z/r)$ is a conformal transformation of $\mathbb{H} \setminus (rK)$ onto $\mathbb{H}$ satisfying $\tilde{g}(z) = z + o(1), \quad z \to \infty$. By uniqueness $\tilde{g} = g_{rK}$. We argue similarly for $\hat{g}(z) = x + g_K(z-x)$.

5. It is easy to see that $\tilde{g} := g_{g(K_2)} \circ g_{K_1}$ is a conformal transformation of D_{K_2} onto $\mathbb{H}$ satisfying $\tilde{g}(z) = z + o(1)$.

6. We first assume that $R = 1$. Let $h(z) = \mathrm{Im}[z - g_D(z)] = \mathrm{Im}(z) - v_D(z)$ which we consider as a harmonic function on $D^* \supset \{|z| > 1\}$. Using Lemma 8.2, we see there exists universal c such that
$$|h(z)| \leq c\, a\, |z|^{-1}.$$
If $|z| \geq 2$, then $h(z)$ is a harmonic function defined on the disk of radius $|z|/2$ bounded by $caO(|z|^{-1})$. Hence using Proposition 2.5, we see that
$$|\nabla h(z)| \leq c\, a\, |z|^{-2},$$
and hence
$$|1 - g_D'(z)| = \sqrt{[\partial_x v(z)]^2 + [1 - \partial_y v(z)]^2} \leq c\, a\, |z|^{-2}.$$
For more general R, $g_{RD}'(z) = g_D'(z/R)$, and hence for $|z| \geq 2R$,
$$|1 - g_{RD}'(z)| = |1 - g_D'(z/R)| \leq \; c\, R^2\, a\; |z|^{-2} = c\, \mathrm{hcap}(RK)\; |z|^{-2}.$$

7. Assume $R = 1$, let
$$f(z) = g(z) - z - \frac{a}{z}.$$
and let
$$v_f(z) = \mathrm{Im} f(z) = v(z) - \mathrm{Im}(z) - a\, \mathrm{Im}(1/z).$$
Using Proposition 8.2 with $h(z) = z - v(z)$, we see that
$$|v_f(z)| \leq c\, \frac{a\, \mathrm{Im}(z)}{|z|^3}.$$
Using the fact that v_f (extended to D^*) is a harmonic function on $\{|w| \leq |z|/2\}$ bounded by $c\, a\, |z|^{-2}$, we see that
$$|f'(z)| = |\nabla v_f(z)| \leq c\, a\, |z|^{-3}.$$
Using $f(\infty) = 0$, we see that for $|z| \geq 2$, $|f(z)| \leq c\, a\, |z|^{-2}$.

For more general R, recall that $g_{RK}(z) = R\, g_K(z/R)$ and hence
$$\begin{aligned} |g_{RK}(z) - z - \mathrm{hcap}(RK) z^{-1}| &= |R g_K(z/R) - z - R^2 a z^{-1}| \\ &= R|g_K(z/R) - (z/R) - a(z/R)^{-1}| \\ &\leq c\, R\, a\, |z/R|^{-2} \\ &= c\, R\, \mathrm{hcap}(RK)\, |z|^{-2}. \end{aligned}$$

□

Examples.

- Let $K = \overline{\mathbb{D}_+}$. Then
$$g_K(z) = z + \frac{1}{z}.$$
In particular, $\mathrm{hcap}(\overline{\mathbb{D}_+}) = 1$.

- Let K be the vertical line segment $[0, i]$. Then
$$g_K(z) = \sqrt{z^2+1} = z + \frac{1}{2z} + \cdots .$$
To be more precise, note that if $z \in \mathbb{H} \setminus [0, i]$, then $z^2 + 1$ is not on the positive real line. Hence. we can take the branch of the square root with values in the positive half plane. This shows that $\mathrm{hcap}([0, i]) = 1/2$.

If K is a compact $\mathbb{H}$-hull, then $\mathrm{hcap}(K)$ is the coefficient of z^{-1} in the expansion of g_K from infinity. Indeed, that is how some people *define* the quantity. However, this definition does not work for compact K for which $K \cup \mathbb{R}$ is not connected.

As a slight abuse of notation, we write
$$g_D(x_-) = s \quad g_D(x_+) = t.$$
If K is disconnected it is possible that g_D can be extended to a slightly larger domain, but we will not need to consider this extension.

Lemma 8.7.
$$-2R \leq g_D(x-) \leq g_D(x_+) \leq 2R.$$

Proof. We do the case $R = 1$; the other cases can be handled by scaling. Recall from Proposition 8.6 that
$$g_D(x_+) = \lim_{y\to\infty} \pi\, y \left[\frac{1}{2} - \mathbb{P}^{iy}\{B_{\tau_D} \in [x_+, \infty)\}\right].$$
The right-hand side is maximized (under the constraint $R = 1$) when $D = \mathbb{H} \setminus \overline{\mathbb{D}_+}$ in which case
$$g_D(z) = z + \frac{1}{z}, \quad g_D(1) = 2.$$
□

It follows that $g_D(x_+) - x_+ \leq 3R$. However, we can get arbitrarily close to $3R$. If we let D be the maximizing domain for $R = 1$, then we can take
$$D_\epsilon = D \setminus \{x + iy : -1 < x \leq 1 : 0 < y < \epsilon(x+1)\}$$
for which $x_+ = -1$ and $g(x_+) \to 3$ as $\epsilon \to 0$.

8.3 Boundary behavior

The behavior of conformal transformations near the boundary is a delicate topic. We will consider here the case where K is a compact $\mathbb{H}$-hull contained in the closed unit disk, $D = \mathbb{H} \setminus K \in \mathcal{J}$, and $g = g_K$ is the unique conformal transformation $g : D \to \mathbb{H}$ with $g(z) - z \to 0$ as $z \to \infty$. We will write f for the inverse map $f = g^{-1} : \mathbb{H} \to D$. The question is whether or not f extends to a map on the $\overline{\mathbb{H}}$. If we only assume that D is the form above, then the situation can be difficult. As a bad example to consider as we go along, let

$$\hat{K} = \left[-\frac{1}{2}, \frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{1}{2} + \frac{i}{2}\right] \cup \left[-\frac{1}{2}, \frac{1}{2} + \frac{i}{2}\right] \cup \bigcup_{n=1}^{\infty} \left(\left[-\frac{1}{2} + \frac{i}{2^{2n-1}}, \frac{1}{4} + \frac{i}{2^{2n-1}}\right] \cup \left[-\frac{1}{4} + \frac{i}{2^{2n}}, \frac{1}{2} + \frac{i}{2^{2n}}\right] \right),$$

and $\hat{D} = \mathbb{H} \setminus \hat{K}$. Fortunately, such bad behavior will not arise if we assume K is the image of a curve.

Definition

- If D is a domain, then a *(simple) crosscut* is a simple curve $\eta : (0, t_0) \to D$ with such that the limits $\eta(0) = \eta(0+), \eta(1) = \eta(1-)$ exist and are on ∂D. (We allow $\eta(0) = \eta(1)$.)
- We say that a simple curve $\eta : [0, t_0] \to \mathbb{C}$ is an *accessing curve* for D if $\eta(0, t_0) \subset D$ and $\eta(0) \in \partial D$. We say that η accesses z if $\eta(0) = z$. The point $z \in \partial D$ is *accessible* if there exists at least one curve accessing η.

Note that under our definition, crosscuts (or their reversal) are accessing curves for both endpoints. In our pathological example $\hat{D}$, the origin is not an accessible point for $\hat{D}$. The Beurling estimate implies that following.

Proposition 8.8. *There exists $c < \infty$ such that if $D = \mathbb{H} \setminus K \in \mathcal{J}$, and η is a curve accessing $z \in \partial D$, then if* $\operatorname{diam}(\eta_t) \leq 1$,

$$\operatorname{diam}[g \circ \eta_t] \leq c \sqrt{\operatorname{diam}(\eta_t)}. \tag{34}$$

Here $\eta_t = \eta[0, t]$. In particular, the limit

$$\lim_{t \downarrow 0} g(\eta(t))$$

exists.

Proof. The proof is the same as that of Lemma 8.15. □

What makes the last proposition true is that if a curve in the upper half plane has a large diameter then there is a good chance that it will be hit by a Brownian motion. "Hit by Brownian motion", that is, harmonic measure, is a conformal invariant. However, we do not get a lower bound on $\operatorname{diam}[g \circ \eta_t]$ in terms of $\operatorname{diam}(\eta_t)$. If ∂D is very rough, or even it just has some protected "fjords", it is possible for $\operatorname{diam}(\eta)$ to be large but the harmonic measure of $c\eta$ to be small.

Proposition 8.9. *Suppose that η is a crosscut of $D = \mathbb{H} \setminus K \in \mathcal{J}$ whose endpoints are distinct. Then $g \circ \eta$ is a crosscut of $\mathbb{H}$ with distinct endpoints.*

Proof. The fact that $g \circ \eta$ is a crosscut follows from the previous proposition. To see that the endpoints are distinct, note that if $w \in D \setminus \eta$, then there is a positive probability that a Brownian motion starting at w hits $\mathbb{R}$ before hitting η and hence leaves D before hitting η. By conformal invariance this must hold for the image $g \circ \eta$. But if the endpoints of $g \circ \eta$ were the same, this would not be true for w in the bounded component of $\mathbb{H} \setminus (g \circ \eta)$. □

For each $z \in \partial D$, let $\mathbb{D}_s(z)$ denote the open ball of radius e^{-s} about z with boundary $C_s(z)$. The set $C_s(z) \cap D$ is the disjoint union of a finite or countably infinite number of *crosscuts* of D. The image of each crosscut under g is a crosscut of $\mathbb{H}$ and Proposition 8.8 implies that $g \circ l$ is a crosscut of $\mathbb{H}$ with $\mathrm{diam}[g \circ l] \leq c\, r^{1/2}$ for some universal constant c. (One needs to be careful here; although the image of each crosscut is small, the images of different crosscuts may not be close to each other so the diameter of the union of the crosscuts can be large.) The last proposition implies that the endpoints of $g \circ l$ are distinct.

Let us fix z and assume that z is accessible. Let $\mathcal{B}_s = \mathbb{D}_s(z)$ and let $U_1^s, U_2^s, \ldots$ denote the connected components of $D \setminus C_s$ that contain z on its boundary. Accessibility implies that there is at least one such component. (In the example $\hat{D}$ above, there are no such components for $z = 0$; however, this point is not accessible.) Typically there will not be many such components, but it is possible for there to be a countable number. For each of these components U_j^s, there is a unique crosscut l_j^s of D such that $l_j^s \subset \partial U_j^s$ and the component of $D \setminus l_j^s$ containing U_j^s is a bounded component. (It is useful to draw pictures. The bounded component of $D \setminus l_j^s$ need not be contained in $\mathcal{B}_s$.) Let us call this bounded component V_j^s. It can be characterized as follows. Suppose η is a curve as in Proposition 8.8. Then for all t sufficiently small either $\eta(0,t) \subset V_j^s$ or $\eta(0,t) \cap V_j^s = \emptyset$. For each s we have an equivalence relations on η with $\eta^1 \equiv_s \eta^2$ if they end up in the same component V_j^s. Note that this is monotonic: if $\eta^1 \equiv_s \eta^2$ then $\eta^1 \equiv_r \eta^2$ for all $r < s$. Hence we can write $\eta^1 \equiv \eta^2$ if $\eta^1 \equiv_s \eta^s$ for some s.

Definition The equivalence classes of accessing curves for D are called the *prime ends*. The prime ends at $z \in \partial D$ are the equivalence classes of curves that access z.

We summarize our discussion in a proposition.

Proposition 8.10. *A boundary point $z \in D$ is accessible if and only if there is a prime end at z. If η^1, η^2 are two curves accessing z in D, then*

$$\lim_{t \downarrow 0} g(\eta^1(t)) = \lim_{t \downarrow 0} g(\eta^2(t)),$$

if and only if η^1, η^2 are equivalent as prime ends.

Proposition 8.11. *Suppose $\gamma : (0,1] \to \mathbb{H}$ is a simple curve with $\gamma(0+) = x \in \mathbb{H}$, and let $\eta(t) = f(\gamma(t))$. Suppose that*

$$\lim_{\epsilon \downarrow 0} \mathrm{diam}[\eta(0,\epsilon)] = 0.$$

Then

$$\lim_{t \downarrow 0} f(\gamma(t)) = z$$

exists and is in ∂D. The curve η accesses z in D. If $\tilde{\gamma} : (0,1] \to \mathbb{H}$ is another simple curve with $\gamma(0+) = x \in \mathbb{H}$, then

$$\lim_{t \downarrow 0} f(\tilde{\gamma}(t)) = z.$$

If l is a crosscut on $\mathcal{B}_{r,z}$, then $D \setminus l$ has two components, one bounded and one unbounded. If U is the bounded component, then we can see that

$$\operatorname{diam}[g(U)] \leq c\, r^{1/2}.$$

However, even if r is very small, it is possible for $\operatorname{diam}[U]$ to be of order 1. As an example, consider the example $\hat{D}$ above. Let η_n be the crosscut formed by the vertical line segment from $2^{-n}i$ to $2^{-(n_1)i}$. Then $\operatorname{diam}(\eta_n) = 2^{-(n+1)}$. However, the diameter of the bounded component of $\hat{D} \setminus \eta_n$ is greater than 1 for each n. In order to prevent this from happening, we can require that $\mathbb{C} \setminus D$ be locally connected.

Definition The set V is *(uniformly) locally connected* if there exists a function $\epsilon(\delta)$ with $\epsilon(0+) = 0$ such that if $z, w \in V$ with $|z - w| \leq \delta$, then there exists a closed connected set $V' \subset V$ containing z, w of diameter at most $\epsilon(\delta)$.

Indeed, suppose we knew that $\mathbb{H} \setminus D$ were locally connected with function $\epsilon(\cdot)$. Let η be a crosscut of D connecting boundary points z, w with $\delta = \operatorname{diam}(\eta)$, and let U be the bounded component of $D \setminus \eta$. Since $|z - w| \leq \delta$, there exists closed $V \subset \mathbb{H} \setminus D$ containing z, w with $\operatorname{diam}(V) \leq \epsilon(\delta)$. Note that U is contained in a bounded component of $\mathbb{C} \setminus (\eta \cup V)$, and hence

$$\operatorname{diam}(U) \leq \operatorname{diam}(V \cup \eta) \leq \delta + \epsilon(\delta).$$

The next topological lemma shows that the domains that we will be studying have locally connected complements.

Lemma 8.12. *If $\gamma = \gamma[0,1]$ is the image of a curve with $\gamma(0) = 0$, then γ and $\mathbb{R} \cup \operatorname{fill}[\gamma]$ are locally connected.*

Proof. Let $z \in \gamma$ and $\epsilon > 0$. let $T = \gamma^{-1}(z)$ which is a nonempty compact subset of $[0,1]$. For each $t \in T$, there exists an open interval I_t containing t such that $|\gamma(s) - z| < \epsilon/4$ for $s \in I_t$. By compactness, we can find $I_{t_1}, \ldots, I_{t_n}$ such that $I := I_{t_1} \cup \cdots \cup I_{t_n}$, covers T. Let $2\delta = \min\{|\gamma(s) - z| : s \in [0,1] \setminus I\} > 0$. If $w \in \gamma$ with $|w - z| < 2\delta$, then $w = \gamma(s)$ for some $s \in I_{t_j}$. Then $\gamma(I_{t_j})$ is a connected subset of γ containing w, z that has diameter at most $\epsilon/2$. Hence, for every $z \in \gamma$, there exists $\delta_z > 0$ such that if $|w - z| < \delta_z$, then for every w' with $|w' - w| < \delta_z$, we can find a connected subset of γ (in fact, the union of two subpaths each going through z) of diameter at most ϵ. Using compactness of γ, we can find $z_1, \ldots, z_m$ such that the open disks of radius δ_{z_j} cover γ. Let $\delta = \min \delta_{z_j}$. Then if $w, w' \in \gamma$ with $|w - w'| < \delta$, we find z_j with $|w - z_j| < \delta_j$. Since $|w - w'| < \delta \leq \delta_j$, we can find a connected subset of γ including w, w' of diameter at most ϵ. Note that we made no assumptions about double points for the curve. Suppose $\operatorname{diam}\gamma \leq R$. Then $[-2R, 2R] \cup \gamma$ is the image of a curve (start at $-2R$ go to $2R$ come back to 0 and then proceed along γ) and so $\gamma \cup [-2R, 2R]$ is locally connected. With this, showing that $am\gamma \cup \mathbb{R}$ is locally connected is easy.

Finally, suppose $w, w' \in \mathbb{R} \cup \operatorname{fill}[\gamma]$ with $|w - w'| < \delta$. If $\operatorname{dist}(w, \gamma \cap \mathbb{H}) \geq \delta$ or $\operatorname{dist}(w', \gamma \cap \mathbb{H}) \geq \delta$, then we can connect w, w' by the straight line segment of length $|w - w'|$. Otherwise, we connect w, w' to z, z' in $\mathbb{R} \cup \operatorname{fill}[\gamma]$ with line segments length less than δ. Therefore $|z - z'| < 3\delta$ and we can find a connected subset of $\mathbb{R} \cup \operatorname{fill}[\gamma]$ of diameter at most $\epsilon(3\delta)$ containing z, z'. The union of this subset and the two line segments is a connected subset of diameter at most $2\delta + \epsilon(3\delta)$ connecting w and w'.

∎

Theorem 6. *Suppose $D = \mathbb{H} \setminus K \in \mathcal{J}$ and $g : D \to \mathbb{H}$ is a conformal transformation with $g(\infty) = \infty$. Suppose that $\mathbb{C} \setminus D$ is locally connected. Then g^{-1} can be extended to a continuous function from $\mathbb{H}$ to $\overline{D}$.*

Proof. Let $\epsilon(\delta)$ be the function as in the definition for $V = \mathbb{C} \setminus D$. Note that if η is a crosscut of D, then the bounded component of $D \setminus \eta$ must have diameter at most $\epsilon(\mathrm{diam}(\eta))$. Let $f = g^{-1}$.

Let $l_{r,x}$ denote the crosscut in $\mathbb{H}$ given by the half-circle $l_{r,x}(t) = x + re^{it}, 0 \le t \le \pi$. We claim there exists $c < \infty$ such that for every $x \in \mathbb{R}$ and every $\rho < 1$ there exists $r = r(x, \rho)$ with $\rho \le r \le \sqrt{\rho}$ such that

$$\mathrm{diam}(f \circ l_{r,x}) \le \frac{c}{\sqrt{\log(1/\rho)}}. \tag{35}$$

To see this, we first note that there exists $c_0 < \infty$ such that for all x, $\mathrm{area}[f(\{z \in \mathbb{H} : |z - x| \le 1)] \le c_0$. Let $\Gamma = \Gamma_{\rho,x} = \{z \in \mathbb{H} : \rho \le |z - x| \le \sqrt{\rho}\}$. With aid of the Cauchy-Schwarz inequality, we see that

$$\begin{aligned}
c_0 \ge \mathrm{area}[f(\Gamma)] &= \int_\Gamma |f'(z)|^2 \, dA(z) \\
&= \int_\rho^{\sqrt{\rho}} \left[\int_0^\pi |f'(re^{i\theta})|^2 \, d\theta \right] r \, dr \\
&\ge \int_\rho^{\sqrt{\rho}} \left[\frac{1}{\pi} \left(\int_0^\pi |f'(re^{i\theta})| \, d\theta \right)^2 \right] r \, dr \\
&\ge \int_\rho^{\sqrt{\rho}} \left[\frac{1}{\pi} \left(\int_0^\pi r \, |f'(re^{i\theta})| \, d\theta \right)^2 \right] r^{-1} \, dr \\
&\ge \frac{\log(1/\rho)}{2\pi} \inf_{\rho \le r \le \sqrt{\rho}} \left[\int_0^\pi r \, |f'(re^{i\theta})| \, d\theta \right]^2 \\
&\ge \frac{\log(1/\rho)}{2\pi} \inf_{\rho \le r \le \sqrt{\rho}} [\mathrm{diam}(f \circ l_{r,x})]^2 .
\end{aligned}$$

This establishes the claim. This estimate was valid for all f (even if $\mathbb{C} \setminus D$ is not locally connected). If $|z - x| < r$, then $f(z)$ is in the bounded component of $f \circ l_{r,x}$. However, in our case we can conclude that diameter of this component is bounded above by

$$\epsilon\left(\frac{c}{\sqrt{\log(1/\rho)}} \right).$$

Therefore, for z, w in the bounded component of $\mathbb{H} \setminus l_{\rho,x}$,

$$|f(z) - f(w)| \le \frac{c}{\sqrt{\log(1/\rho)}} + \epsilon\left(\frac{c}{\sqrt{\log(1/\rho)}} \right),$$

which goes to zero as ρ goes to zero.

$\square$

An important technical result was used in the last proof. From (35), we see that we can find half-circles l_r about the origin of radius r so that $\mathrm{diam}(f \circ l_r) \leq c\sqrt{\log(1/r)}$. However, one must be careful with this. Although $\mathrm{diam}(f \circ l_r)$ is small, it is not always true that the image of the disk of radius r has small diameter.

We have restricted our consideration to domains in $\mathcal{J}$, but the argument for the last theorem is all local. Using the same argument we can get this more traditional statement of the theorem.

Theorem 7. *Suppose $f : \mathbb{D} \to D$ is a conformal transformation where D is a bounded domain with $\mathbb{C} \setminus D$ locally connected. Then f extends to a continuous function on $\overline{\mathbb{D}}$.*

Corollary 8.13. *Suppose that $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ and H_t is the unbounded component of $\mathbb{H} \setminus \gamma_t$. Then the inverse map $g_t^{-1} : \mathbb{H} \to D$ can be extended continuously to $\partial\mathbb{H}$. Moreover, all points of ∂H_t are accessible.*

Definition

- A curve $\gamma : [0, t_0] \to \mathbb{C}$ is called a Jordan curve, if $\gamma(0) = \gamma(t_0)$ and $\gamma(s) \neq \gamma(t)$ for $0 \leq s < t < t_0$.
- A *Jordan domain* is a bounded domain D whose boundary is a Jordan curve.

The Jordan curve theorem which we will not prove here states that if γ is a Jordan curve, then $\mathbb{C} \setminus \gamma$ consists of two connected components. The bounded component is a Jordan domain.

If f in Theorem 7 is one-to-one on $\overline{\mathbb{D}}$, then $t \mapsto f(e^{it})$ gives a parameterization of ∂D as a Jordan curve. In this case f is a homeomophism of $\overline{\mathbb{D}}$ onto $\overline{D}$. (Continuity of $f^{-1} = g$ follows from the Beurling estimate as in Proposition 8.8.) Conversely, if we know that D is a Jordan domain, we can use Proposition 8.8 to see that f must be one-to-one on $\overline{\mathbb{D}}$. We end with a topological fact about domains generated by curves.

Proposition 8.14. *Suppose $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is a curve with $\gamma(0) = 0$. Let H_t denote the unbounded component of $\mathbb{H} \setminus \gamma_t$, and*

$$H_{t-} = \bigcap_{s<t} H_s.$$

If $\gamma(t) \in H_{t-}$, then there is a single prime end of H_t associated to $\gamma(t)$.

Proof. Suppose $\gamma(t) \in H_{t-}$ and that there are at least two prime ends. We know that $\gamma(t)$ is an accessible point and hence there exists simple $\eta : (0,1) \to H_t$ with with $\eta(0+) = \eta(1-) = \gamma(t)$. Since $\gamma(t) \in H_{t-}$, we see that $\eta \subset H_s$ for all $s < t$. Let V be the bounded component of $\mathbb{C} \setminus \eta$. Since H_s is simply connected for $s < t$, we see that $\gamma_s \cap V = \emptyset$ for $s < t$ and hence $\gamma_t \cap V = \emptyset$. Since V is connected we see that either $V \subset H_t$ or $V \cap H_t = \emptyset$. If $V \cap H_t = \emptyset$, then since H_t is open, $\overline{V} \cap H_t = \emptyset$. In particular $\eta \cap H_t = \emptyset$ which is a contradiction. Therefore, we know that $V \subset H_t$.

Since $V \subset H_t$, if $\zeta \in V$, a Brownian motion starting at ζ cannot reach ∂H_t without hitting η. This must also be true for $g_t(\zeta)$ and $g_t \circ \eta$ which implies that $g_t(\eta(0+)) = g_t(\eta(1-))$. Hence both endpoints give the same prime end.

□

8.4 Curves

In this section, we let $\gamma : (0,\infty) \to \mathbb{C}$ be a simple curve with $\gamma(0+) = 0$. For each t, let $\gamma_t = \gamma[0,t]$ which is a compact $\mathbb{H}$-hull with $D_t = \mathbb{H} \setminus \gamma_t$ simply connected. Let $g_t = g_{\gamma_t}$ be the corresponding map which has an expansion at infinity

$$g_t(z) = z + \frac{a_t}{z} + O(|z|^{-2}).$$

This expression defines a_t; in fact, as we have seen $a_t = \mathrm{hcap}[\gamma_t]$. By (32), we see that a_t is strictly increasing in t. We will make the further assumption that

$$\lim_{t\to\infty} a_t = \infty.$$

This requires $\limsup_{t\to\infty} |\gamma(t)| = \infty$, although this last condition is not quite sufficient. Let $\tau_t = \tau_{D_t}$. The next proposition will show that a_t is a continuous function of t. It uses the Beurling estimate.

Lemma 8.15. *There exists $c < \infty$ such that for every γ, if $s < t$,*

$$\mathrm{diam}\,(g_s[\gamma_t \setminus \gamma_s]) \le c\,\sqrt{\mathrm{diam}(\gamma_t)}\,\sqrt{\mathrm{diam}(\gamma[s,t])}.$$

Proof. Let $V = V_{s,t} = g_s[\gamma_t \setminus \gamma_s], u = \mathrm{diam}[\gamma_t], r = \mathrm{diam}(\gamma[s,t]) \le u$. By Lemma 8.4, $\mathrm{cap}_{\mathbb{H}}(V) \asymp \mathrm{diam}(V)$. By definition,

$$\mathrm{cap}_{\mathbb{H}}(V) = \lim_{y\to\infty} y\,\mathbb{P}^{iy}\{B_{\tau_{\mathbb{H}\setminus V}} \in V\}.$$

Using the expansion of g_s at infinity and conformal invariance and the expansion $g_s(iy) = i[y - \mathrm{hcap}(\gamma_s)y^{-1}] + O(y^{-2})$, we see that

$$\lim_{y\to\infty} y\,\mathbb{P}^{iy}\{B_{\tau_t} \in \gamma[s,t]\} = \lim_{y\to\infty} y\,\mathbb{P}^{g_s(iy)}\{B_{\tau_{\mathbb{H}\setminus V}} \in V\} = \lim_{y\to\infty} y\,\mathbb{P}^{iy}\{B_{\tau_{\mathbb{H}\setminus V}} \in V\} = \mathrm{cap}_{\mathbb{H}}(V).$$

We will now estimate $\mathbb{P}^{iy}\{B_{\tau_t} \in \gamma[s,t]\}$ for large y. In order for $B_{\tau_t} \in \gamma[s,t]$, it is necessary for the Brownian motion starting at iy to reach the disk of radius $2u$ about the origin without leaving $\mathbb{H}$. The probability of this is $O(u/y)$. Given this, the Brownian motion must reach the disk of radius r about $\gamma(s)$ without leaving D_t. By the Beurling estimate, this probability is bounded by a constant times $\sqrt{r/u}$. Therefore

$$\lim_{y\to\infty} y\,\mathbb{P}^{iy}\{\tau_t < \tau_s\} \le c\,\sqrt{ru}.$$

$\square$

It follows that we have an estimate

$$a_t - a_s \le c\,\mathrm{diam}(\gamma_t)\,\mathrm{diam}(\gamma[s,t]).$$

In particular, a_t is a continuous function of t and we can reparametrize the curve so that $\mathrm{hcap}[\gamma_t] = 2t$.

Definition The curve γ has the *(standard) capacity parametrization* if $\mathrm{hcap}[\gamma_t] = 2t$ for all t.

The choice of the constant 2 is somewhat arbitrary although we will see reasons later why this is a natural choice. More generally, we will say that γ is parametrized by capacity with rate a if $\mathrm{hcap}[\gamma_t] = at$. For now assume that we have the standard capacity parametrization so that

$$g_t(z) = z + \frac{2t}{z} + O(|z|^{-2}), \quad z \to \infty,$$

Proposition 8.14 tells us that there is only one prime end associated to the tip $\gamma(t)$, that is, if $z_n \in D_t$ with $z_n \to \gamma(t)$, then the limit

$$g(\gamma_t(t)) = \lim_{n\to\infty} g_t(z_n)$$

exists and the limit is independent of the sequence. We will denote the limit by U_t.

Theorem 8 (Half plane Loewner differential equation)**.** *Suppose γ is a simple curve as above parameterized so that* $\mathrm{hcap}[\gamma_t] = 2t$. *Then every $z \in \mathbb{H}$ the flow $t \mapsto g_t(z)$ satisfies*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad 0 \le t < T_z,$$

where $U_t = g_t(\gamma(t)), T_z = \inf\{s : \gamma(s) = z\}$. Moreover, the function $t \mapsto U_t$ is continuous.

Proposition 8.14 shows that there is one prime end of $\mathbb{H} \setminus \gamma_t$ at $\gamma(t)$ and hence $g_t(\gamma(t))$ is well defined. Our estimate will focus on the right time derivative. In order to convert the result to a usual derivative we will use this easy lemma.

Lemma 8.16. *Suppose $u : [0,\infty) \to \mathbb{R}^n$ is a continuous function whose right derivative*

$$u'_+(t) = \lim_{s\downarrow 0} \frac{u(s+t) - u(t)}{s},$$

exists for all t. Suppose also that $t \mapsto u'_+(t)$ is continuous. Then f is differentiable with $u'(t) = u'_+(t)$.

Proof. It suffices to prove the result when $u(0) = 0, u'_+ \equiv 0$ for then (using continuity of u'_+) we can consider

$$f(t) = u(t) - u(0) - \int_0^t u_+(s)\,ds.$$

Let $\epsilon > 0$ and let $\sigma = \sigma_\epsilon = \inf\{t : |u(t)| > \epsilon\, t\}$. Since $u'_+(0) = 0$, we can see that $\sigma > 0$. Suppose $\sigma < \infty$. By continuity of u, we can see that $|u(\sigma)| = \epsilon\,\sigma$. However, since $u'_+(\sigma) = 0$, there exists $\delta > 0$ such that $|u(\sigma+s) - u(\sigma)| < \epsilon s$ for $0 \le s < \delta$. This implies that $|u(\sigma+s)| \le \epsilon(\sigma+s)$ for $0 < s < \delta$ which contradicts the definition of σ. Therefore $\sigma = \infty$. Since this is true for every ϵ, $u \equiv 0$. ☐

Proof of Theorem 8. Using Lemma 8.7, we can see that $\mathrm{diam}[g_t(\gamma_t)] \le 4\,\mathrm{diam}[\gamma_t]$ and hence $|U_t - U_0| \le 4\,\mathrm{diam}(\gamma_t)$. More generally, if $s < t$,

$$|U_t - U_s| \le 4\,\mathrm{diam}[g_s(\gamma_t \setminus \gamma_s)]).$$

Combining this with Lemma 8.15, we see that $t \mapsto U_t$ is continuous. Similarly, we see that for fixed z, $g_t(z)$ is continuous in t. Therefore, by Lemma 8.16, it suffices to establish the result for the right derivative. But this follows from (33).

☐

8.5 Loewner differential equation

In the last section we started with a curve γ in the upper half plane which corresponded to a parametrized family of conformal maps. We then showed that the conformal maps satisfy a particular differential equation, In the next proposition, we start with a continuous function $t \mapsto U_t$ and find the appropriate maps. It will be useful to adopt the notation that dots refer to t-derivatives and primes refer to z-derivatives.

Theorem 9. *Suppose that $t \mapsto U_t$ is a continuous real valued function. For each $z \in \mathbb{C} \setminus \{0\}$, let $g_t(z)$ denote the solution to the initial value problem*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z. \tag{36}$$

- *For each z, the solution exists up to time $T_z \in (0, \infty]$ defined to be the smallest t such that*
$$\inf\{s < t : |g_s(z) - U_s|\} = 0.$$
- *If $z \in \mathbb{H}$, then $\mathrm{Im}[g_t(z)]$ decreases with t and if $t < T_z$, $\mathrm{Im}[g_t(z)] > 0$.*
- *For all z, $T_z = T_{\overline{z}}$ and if $t < T_z$, $g_t(\overline{z}) = \overline{g_t(z)}$.*
- *Let $H_t = \{z \in \mathbb{H} : T_z > t\}$. Then g_t is the unique conformal transformation of H_t onto $\mathbb{H}$ satisfying $g_t(z) - z \to 0$ as $z \to \infty$. Moreover g_t has the expansion*
$$g_t(z) = z + \frac{2t}{z} + O(|z|^{-2}), \quad z \to \infty. \tag{37}$$

Proof. We write $g_t(z) = u_t(z) + iv_t(z)$ and note that (36) can be written as

$$\dot{u}_t(z) = \frac{2[u_t(z) - U_t]}{|g_t(z) - U_t|^2}, \quad \dot{v}_t(z) = -\frac{2\, v_t(z)}{|g_t(z) - U_t|^2}.$$

In particular, $|v_t(z)|$ decreases with t. Since $|g_t(z) - U_t|^2 \geq v_t(z)^2$, these equations have solutions up to time T_z and we can write

$$g_t(z) = \int_0^t \frac{2\, ds}{g_s(z) - U_s},$$

Differentiating with respect to z gives

$$\dot{g}'_t(z) = -\frac{2\, g'_t(z)}{(g_t(z) - U_t)^2}, \quad g'_t(z) = \exp\left\{-\int_0^t \frac{2\, ds}{(g_s(z) - U_s)^2}\right\}.$$

This shows that g_t is holomorphic on $\{T_z > t\}$ and since

$$\partial_t\, [g_t(z) - g_t(w)] = \frac{2[g_t(w) - g_t(z)]}{[g_t(z) - U_t]\, [g_t(w) - U_t]},$$

we get

$$g_t(z) - g_t(w) = (w - z)\, \exp\left\{\int_0^t \frac{2\, ds}{[g_s(z) - U_s]\, [g_s(w) - U_s]}\right\},$$

from which we can deduce that g_t is one-to-one on $\{T_z > t\}$. If $z \in H_t$ and we define $\tilde{g}_t(\overline{z}) = \overline{g_t(z)}$, it is immediate that $\tilde{g}_t$ satisfies Loewner and hence $\tilde{g}_t(\overline{z}) = g_t(\overline{z})$.

To show that $g_t(H_t) = \mathbb{H}$ we "reverse the flow". For fixed t_0 and $z \in \mathbb{H}$, consider the differential equation

$$\dot{h}_t(z) = -\frac{2}{h_t(z) - V_t}, \quad h_0(0) = z.$$

where $V_t = U_{t_0 - t}, 0 \leq t \leq t_0$. Note that $\mathrm{Im}[h_t(z)]$ increases with t so this solution exists for all times t with

$$h_t(z) = z - \int_0^t \frac{2}{h_s(z) - V_s}\, ds.$$

Also $\phi_t(z) := h_{t_0 - t}(z)$ satisfies

$$\dot{\phi}_t(z) = \frac{2}{h_{t_0-t}(z) - V_{t_0-t}} = \frac{2}{\phi_t(z) - U_t},$$

with $\phi_0(z) = h_{t_0}(z), \phi_{t_0}(z)$. In other words, $g_{t_0}(h_{t_0}(z)) = z$.

To get the expansion at infinity note that

$$\dot{g}_t(z) = \frac{2}{g_t(z)}\ \left[1 + O(|z|^{-1})\right],$$

where the $O(\cdot)$ term depends on $U_s, 0 \leq s \leq t$. If we fix t and let $z \to \infty$ we get (37).

□

In the proof there was another construction which turns out to be useful, the *reverse Loewner equation*

$$\dot{h}_t(z) = -\frac{2}{h_t(z) - V_t}, \quad h_0(0) = z. \tag{38}$$

Theorem 10 (Reverse Loewner flow)**.** *Suppose $t \mapsto V_t$ is a continuous real-valued function and h_t is the solution to* (38)*. Then if $z \in \mathbb{H}$, the solution exists for all times t. Moreover, for each t, h_t is a conformal transformation $h_t : \mathbb{H} \to h_t(\mathbb{H})$ where $h_t(\mathbb{H}) \subset \mathbb{H}$ with $\mathbb{H} \setminus h_t(\mathbb{H})$ bounded. It satisfies*

$$h_t(z) = z - \frac{2t}{z} + O(|z|^{-2}), \quad z \to \infty.$$

Moreover, if $t_0 < \infty$, $U_t = V_{t_0} - t, 0 \leq t \leq t_0$, and g_t is the solution to (36)*, then $h_{t_0} = g_{t_0}^{-1}$.*

The proof of this is essentially in the last proof. We remark that if $x \in \mathbb{R}$, the solution of (38) exists up to some time T_x but it is possible (and usually true) that $T_x < \infty$. Note that we have found a way to get $g_{t_0}^{-1}$ using the reverse flow. However, t_0 was used in the definition of V_t, and h_t for other values of t does not equal g_t^{-1}.

There is another way to get the inverse function of g_t which we now demonstrate. If we let $f_t = g_t^{-1}$ and use the chain rule to differentiate the equation

$$f_t(g_t(z)) = z$$

with respect to t, we get

$$\dot{f}_t(g_t(z)) + f_t'(g_t(z))\, \dot{g}_t(z) = 0.$$

Here we are writing dots for t-derivatives and primes for z-derivatives. If g_t satisfies (36) and we write $w = g_t(z)$, then we get

$$\dot{f}_t(w) = -f_t'(w)\,\dot{g}_t(z) = -f_t'(w)\,\frac{2}{w - U_t}, \qquad f_0(w) = w. \tag{39}$$

We call this the *inverse Loewner equation.* At each time t, f_t is a conformal transformation of $\mathbb{H}$ onto a domain $f(\mathbb{H})$.

For future use, we prove the following proposition.

Lemma 8.17. *There exists $c < \infty$ such that if f is the solution to the inverse Loewner equation, then for all t, and all $w = x + iy \in \mathbb{H}$,*

$$e^{-10s/y^2}\,|f_t'(w)| \leq |f_{t+s}'(w)| \leq e^{10s/y^2}\,|f_t'(w)|.$$

$$|f_{s+t}(w) - f_t(w)| \leq \frac{\mathrm{Im}(w)\,[e^{10s/y^2} - 1]}{5}\,\left[|f_{s+t}'(w)| \wedge |f_t'(w)|\right].$$

Proof. By differentiating both sides of (39), we see that

$$\dot{f}_t'(w) = f_t'(w)\,\frac{2}{(w - U_t)^2} - f_t''(w)\,\frac{2}{w - U_t}.$$

The Bieberbach estimate on the second coefficient of schlicht functions, shows that if h is a univalent function on $\mathbb{D}$, then $|h''(0)| \leq 4\,|h'(0)|$. Applied to the disk of radius y about w, we see that $|f''(w)| \leq 4\,y^{-1}|f'(w)|$, and hence

$$|\partial_t \log |f_t'(w)|| \leq 10\,y^{-2}, \quad e^{-10s/y^2}\,|f_t'(w)| \leq |f_{t+s}'(w)| \leq e^{10s/y^2}\,|f_t'(w)|.$$

$$|\dot{f}_{t+s}(w)| \leq \frac{2\,|f_{t+s}'(w)|}{|U_{t+s} - w|} \leq 2\,e^{10s/y^2}\,y^{-1}\,|f_t'(w)|$$

$$\begin{aligned}
|f_{t+s}(w) - f_t(w)| &\leq \int_0^s |\dot{f}_{t+r}(w)|\,dr \\
&\leq 2\,\left(|f_t'(w)| \wedge |f_{t+s}'(w)|\right)\,y^{-1}\int_0^s e^{10r/y^2}\,dr \\
&= \left(|f_t'(w)| \wedge |f_{t+s}'(w)|\right)\,\frac{y\,[e^{10s/y^2} - 1]}{5}.
\end{aligned}$$

□

8.6 Loewner chains generated by a curve

Definition Suppose $U_t, 0 \leq t \leq T$ is a continuous real valued function.

- The collection of conformal maps g_t obtained from (36) is called a *Loewner chain.*
- The function U_t is called the *driving function* for the chain.

- We will say that t is an *accessible time* for the driving function U if the limit

$$\gamma(t) = \lim_{y \downarrow 0} g_t^{-1}(U_t + iy). \tag{40}$$

exists. We say that a driving function is *(everywhere) accessible* if all times are accessible.

- We say that z is a *pioneer point* for the chain at time $t > 0$ if $z \in H_s$ for all $s < t$ and $z \in \partial H_t$. The *pioneer set* γ_t at time t is the set of pioneer points for all $0 \leq s \leq t$.

- We say that g_t is *generated by a curve* or U_t *generates a curve* if U_t is everywhere accessible and $\gamma(t), 0 \leq t \leq T$ is a continuous function of t. Equivalently, $\gamma_t = \gamma[0, t]$.

- We will call a curve γ a *pioneer curve* if all times are pioneer points. This is equivalent to saying that γ is simple and $\gamma(0, T] \subset \mathbb{H}$.

The term "pioneer curve" is not standard but we do not want to have to say the phase "γ is simple and $\gamma(0, T] \subset \mathbb{H}$."

Recall that (40) holds if and only if there exists some simple curve $\eta : (0, 1] \to \mathbb{H}$ with $\eta(0+) = U_t$ such that

$$\gamma(t) = \lim_{s \downarrow 0} g_t^{-1}(\eta(s)).$$

It is not true that (40) holds for all t for all continuous functions U_t. It is also possible that the limit is not a continuous function of t. However, we will show that under some regularity assumptions on U_t it does. What we state will be sufficient conditions but not necessary conditions. A necessary and sufficient condition for a curve to be a pioneer curve is that for each s, $g_s(\gamma(s, T]) \subset \mathbb{H}$. Indeed, if $r < s$ and $\gamma(r) = \gamma(s)$ and $r < t < s$, then $g_t(\gamma(t, T]) \cap \mathbb{R} \neq \emptyset$,

Theorem 11. *Suppose $c_0 < 4$ and for all s, t,*

$$|U_s - U_t| \leq c_0 \, |t - s|^{1/2}. \tag{41}$$

Then the limit in (40) *exists for all t, and γ is a pioneer curve.*

To understand why the condition $|U_s - U_t| \asymp \sqrt{|t - s|}$ should be critical for the Loewner equation, consider the case $U_t \equiv 0$ for which $g_t(z) = \sqrt{z^2 + 4t}$. This is generated by the vertical curve $\gamma(t) = \sqrt{4t}\, i$. In time t, the curve moves distance $O(\sqrt{t})$ from the origin. Now suppose that U_t is not constant. If U_t grows slower than $\sqrt{t}$, then the horizontal effect will not be enough to bring the curve down to the real line. If $U_t \gg \sqrt{t}$, then there may be problems. This is why $\Delta(r)$ as defined in Proposition 8.18 is a natural quantity for driving functions of the Loewner equation.

Note that it suffices to assume that (41) holds for all s, t with $|t-s|$ sufficiently small. This theorem is not true for all values of c_0. One can find c_0 and driving function U_t satisfying (41) for which the limit (40) does not exists for all t. We will prove the theorem in a series of propositions. The first two will show that the chain is generated by a curve and the last will show that the curve is a pioneer curve. The The first proposition is stronger than we need for this section; however, the stronger version will be used when we consider the Schramm-Loewner evolution so we prove it now.

Proposition 8.18. *There exists $c < \infty$ such that the following holds. Suppose $U_s, 0 \leq s \leq 1$, satisfies (41) for all $0 \leq s, t \leq 1$, and let*

$$\Delta(r) = 1 + \max\left\{\frac{|U_t - U_s|}{\sqrt{t-s}} : 0 \leq s < t \leq 1,\ t - s \geq r\right\},$$

$$I(y) = \sup_{0 \leq t \leq 1} \int_0^y |f_t'(U_t + ir)|\, dr.$$

If $I(y) < \infty$, then the limit (40) exists for all $0 \leq t \leq 1$ and

$$|\gamma(t) - \gamma(s)| \leq c_1\, I(\sqrt{t-s})\, \Delta(t-s)^4, \quad 0 \leq s < t \leq 1.$$

In particular, if

$$\lim_{r \downarrow 0} I(\sqrt{r})\, \Delta(r)^4 = 0,$$

then γ is a curve.

Note that if U_t satisfies (41), then $\Delta(r)$ is uniformly bounded. Another important case for use will be when U_t is a Brownian motion path for which $\Delta(r) \leq O(\sqrt{\log(1/r)})$ as $r \downarrow 0$.

Proof. Let $\hat{f}_t(z) = f_t(U_t + z)$. The existence of the limit (40) follows immediately from finiteness of $I(y)$ with

$$|\gamma(t) - \hat{f}_t(iy)| \leq I(y).$$

The distortion theorem implies that $|\hat{f}_t'(iy')| \asymp |\hat{f}_t'(iy)|$ for $y/2 \leq y' \leq 2y$, and hence

$$I(y) \geq \int_{y/2}^y |\hat{f}_t'(ir)|\, dr \geq c_2\, y\, |\hat{f}_t'(iy)|.$$

Suppose $0 \leq s \leq t \leq s+\delta^2 \leq 1+\delta^2$. The triangle inequality implies that $|\gamma(s)-\gamma(t)|$ is bounded above by

$$|\gamma(s) - \hat{f}_s(i\delta)| + |\gamma(t) - \hat{f}_t(i\delta)| + |\hat{f}_s(i\delta) - \hat{f}_t(i\delta)| \leq 2I(\delta) + |\hat{f}_s(i\delta) - \hat{f}_t(i\delta)|,$$

$$|\hat{f}_s(i\delta) - \hat{f}_t(i\delta)| \leq |f_s(U_s + i\delta) - f_s(U_t + i\delta)| + |f_s(U_t + i\delta) - f_t(U_t + i\delta)|.$$

Lemma 8.17 shows that $|f_s(U_t + i\delta) - f_t(U_t + i\delta)| \leq c\,\delta\, |f_t'(U_t + i\delta)| \leq c\, I(\delta)$. Using the distortion theorem as in (27) and $|s-t| \leq \delta^2$, we see that

$$|f_s(U_s + i\delta) - f_s(U_t + i\delta)| \leq c\,\delta\, \left[1 + \frac{|U_t - U_s|^4}{\delta^4}\right]\, |f_s'(U_s + i\delta)| \leq c\, \Delta(\delta^2)^4\, I(\delta).$$

$\square$

We will consider the reverse Loewner flow

$$\dot{h}_t(z) = -\frac{2}{h_t(z) - U_t}$$

where U_t satisfies (41). If $z \in \mathbb{H}$, let $Z_t = Z_t(z) = h_t(z) - U_t$, $X_t = \mathrm{Re}[Z_t]$, $Y_t = \mathrm{Im}[Z_t]$. Then we can write the equation as

$$\partial_t\,[X_t + U_t] = -\frac{2\,X_t}{X_t^2 + Y_t^2}, \quad \partial_t Y_t = \frac{2\,Y_t}{X_t^2 + Y_t^2}. \tag{42}$$

Note that

$$\partial_t[\log h_t'(z)] = \frac{2}{Z_t^2}, \quad \partial_t[\log|h_t'(z)||] = \frac{2\,(X_t^2 - Y_t^2)}{|Z_t|^4}, \quad \log|h_t'(z)| = \int_0^t \frac{2\,(X_s^2 - Y_s^2)}{(X_s^2 + Y_s^2)^2}\,ds. \tag{43}$$

Example. If $U_t = 2b\sqrt{t}$ with $0 \le b < 2$, then the solution of (42) satisfying $X_0 = 0, Y_0 = 0$ is

$$X_t = -b\sqrt{t}, \quad Y_t = \sqrt{4 - b^2}\,\sqrt{t}.$$

Hence $h_\epsilon(0) = z_\epsilon + U_\epsilon := \sqrt{\epsilon}\,[b + i\sqrt{4 - b^2}]$. Let us write $h_t = h_{t,\epsilon} \circ h_\epsilon$. and hence if $t > \epsilon$,

$$|h_{t,\epsilon}'(z_\epsilon)| = \exp\left\{\int_\epsilon^t \frac{2\,(X_s^2 - Y_s^2)}{(X_s^2 + Y_s^2)^2}\,ds\right\} = \exp\left\{\int_\epsilon^t \frac{2\,(2b^2 - 4)}{16\,(s + \epsilon)}\,ds\right\} = \left(\frac{t}{\epsilon}\right)^{\frac{b^2-2}{4}}.$$

Using the reverse Loewner equation and the distortion principle, we can see that $|h_\epsilon'(\sqrt{\epsilon}i)| \asymp 1$ and $|h_{t,\epsilon}'(h_\epsilon(\sqrt{\epsilon}i))| \asymp |h_{t,\epsilon}'(z_\epsilon)|$, and therefore, $|h_t'(\sqrt{\epsilon}i)| \asymp |h_{t,\epsilon}'(z_\epsilon)|$. Ig g_t is a solution of the Loewner equation (36) with $U_t = 2b\sqrt{1-t}, 0 \le t \le 1$, then the distribution of $f_1 := g_1^{-1}$ is the same as that of h_1 above. In particular, using the distortion principle, we can see that

$$|f_1'(iy)| \asymp y^{\frac{2-b^2}{2}}, \quad y \downarrow 0.$$

Proposition 8.19. *For each $c_0 < 4$, there exists $\theta < 1$ and $c < \infty$ such that if U_t satisfies (41), then for all $0 \le t \le 1$ and all $y \le 1$,*

$$|f_t'(iy)| \le c\,y^{-\theta}, \quad I(y) \le c\,y^{1-\theta}, \quad \theta = 1 - \frac{c_0^2}{16}.$$

Proof. We write $c_0 = 2b$. Consider the equation,

$$\partial_t X_t = -\frac{2(b^2/4)}{X_t} - \partial + tU_t,$$

under the constraint $U_t \le 2b\,\sqrt{t}$. To maximize $|X_t|$ under these constraints, we choose U_t with constant sign and maximal absolute value. If we choose $U_t = 2b\sqrt{t}$, and let $R_t = X_t + U_t$, then the solution is $X_t = -b\,\sqrt{t}$. Hence for any U_t satisfying the condition, we have $X_t^2 \le b^2\,t$. If we assume that $X_0 = 0, Y_0 > 0$, then by induction, we see that for all t, $Y_t^2 \ge \frac{4-b^2}{b^2}\,X^2$, and hence $Y_t^2 \ge (4 - b^2)\,t$. The derivative estimate then follows as in (43) $\square$

Proposition 8.20. *Suppose U_t satisfies* (41) *with $c_0 < 4$ and $U_0 = 0$, and u_t satisfies*

$$\partial_t u_t = \frac{2}{u_t - U_t}, \quad 0 \le t \le r^2,$$

with $u_0 > 0$. Then $u_t > U_t$ for $0 \le t \le r^2$.

Proof. If $U_t, 0 \le t \le r^2$ satisfies (41) and $\tilde{u}_t = r^{-1}\, u_{r^2 t}(z/r)$, then

$$\partial_t \tilde{u}_t = \frac{2}{\tilde{u}_t - \tilde{U}_t},$$

where $\tilde{U}_t = r^{-1}\, U_{r^2\, t}$. Since, $\tilde{U}_t$ satisfies (40), it suffices to prove the result for $r^2 = 1$.

If $U_t \le 0$ for $0 \le t \le \delta$, then we can see that $u_t - U_t$ is locally strictly increasing; hence we can wait until it returns to value u_0 again. More generally, we can see that we may assume that U_t is nondecreasing. In order to minimize $u_t - U_t$ subject to $U_1 = \beta c_0$, we need to choose U_t minimal under the constraints of (40) and monotonicity. Therefore, the minimizer is given by a function

$$U_t = \begin{cases} 0, & t \le 1 - \beta^2 \\ c_0\,[\beta - \sqrt{1-t}], & 1 - \beta^2 \le t \le t. \end{cases}.$$

for some $0 < \beta \le 1$. Then $u_{1-\beta^2} = \sqrt{u_0^2 + 1 - \beta^2}$. If $X_t = u_t - U_t$, then

$$\partial_t X_t = \frac{2}{X_t} - \frac{(c_0/2)}{\sqrt{1-t}}, \quad 1 - \beta^2 \le t \le 1.$$

So we need to see that solutions to this with $X_{1-\beta^2} > 0$ satisfy $X_1 > 0$. Let $\phi(t) = X_t/\sqrt{1-t}$ which satisfies

$$\partial_t \phi(t) = \frac{1}{1-t}\left[\frac{2}{\phi(t)} - \frac{c_0}{2} + \frac{\phi(t)}{2(1-t)}\right] \ge \frac{2 - \frac{c_0}{2}}{1-t}.$$

Since $c_0 < 4$, $\phi(1-) = \infty$ and we can find t with $X_t \ge 2c_0\sqrt{1-t}$. Hence

$$X_1 \ge X_t - [U_t - U_t] \ge c_0\,\sqrt{1-t} > 0.$$

□

8.6.1 Non-crossing curves

In this section, we assume that $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is a continuous curve. We allow self-intersections and intersections with the boundary. As before, we write $\gamma_t = \gamma[0,t]$ and H_t for the unbounded component of $\mathbb{H} \setminus H_t$. Let $a(t) = \mathrm{hcap}(H_t)$ which is a continuous function of t. Let $g_t : H_t \to \mathbb{H}$ be the unique conformal transformation with $g_t(z) - z \to 0$ as $z \to \infty$.

- **Assumption 1.** The function $t \mapsto a(t)$ is strictly increasing with $a(t) \to \infty$ as $t \to \infty$.

If follows from this assumption that for all t and all δ, $\gamma(t, t+\delta] \cap H_t \ne \emptyset$. This assumption prevents the path from going into the complement of H_t and reappearing somewhere else. An example of a curve in the upper half plane that does not satisfy Assumption 1 is a Brownian excursion from 0 to ∞. If γ satisfies Assumption 1, then we can reparametrize γ so that it satisfies $a(t) = 2t$ for all t.

We also do not want the curve to jump from one side of a domain to another. This condition is expressed most easily in terms of prime ends. Since $\mathbb{C} \setminus H_t$ is locally connected, the point $\gamma(t)$ is accessible from H_t. (This essentially also follows from the fact that $\gamma(t,\infty)$ accesses $\gamma(t)$; however, since $\gamma(t,\infty)$ is not simple and may hit ∂H_t, we need a little more argument to prove accessibility.) Each prime end of H_t with endpoint $\gamma(t)$ is associated to a point on the real line by the map g_t.

- **Assumption 2.** For each t there is a prime end of H_t at $\gamma(t)$ which we associate to $U_t \in \mathbb{R}$ such that the following holds. For every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < s < \delta$ and $\gamma(t+s) \in H_t$, then $|U_t - g_t(\gamma(t+s))| < \epsilon$. (We write this as $g_t(\gamma(t+)) = U_t$.) Moreover, U_t is a continuous function of t.

Definition A function $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is called a *non-crossing* curve if it satisfies Assumptions 1 and 2.

If in Assumption 2 we had also put the condition that $\gamma(t, t+\delta) \cap H_t \neq \emptyset$ for all δ, then we would have $a(t)$ is strictly increasing. However, we would need to separately include the condition $a(t) \to \infty$, so we have made Assumption 1 as an assumption.

The following is proved in exactly the same way as Theorem 8. We do emphasize one difference. For a simple curve, the continuity of U_t was not assumed but rather was proved. For the theorem below, continuity of U_t is one of the assumptions.

Theorem 12. *Suppose $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is a non-crossing curve as above parameterized so that $\mathrm{hcap}[\gamma_t] = 2t$. Then every $z \in \mathbb{H}$ the flow $t \mapsto g_t(z)$ satisfies*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad 0 \le t < T_z,$$

where $U_t = g_t(\gamma(t+)), T_z = \inf\{s : \gamma(s) = z\}$.

The converse is the following.

Theorem 13. *Suppose U_t is a continuous real-valued function of t and g_t is the solution to the Loewner equation* (36)*. Suppose there exists a continuous function $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ such that for all t, H_t is the unbounded component of $\mathbb{H} \setminus \gamma_t$. Then γ is a non-crossing curve.*

8.7 Perturbation of Maps

In this section we will assume that $U_t, 0 \le t \le t_0$ is a continuous real valued function with $U_0 = 0$ and g_t is the solution to the Loewner equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Let H_t denote the corresponding domains and let $K_t = \overline{\mathbb{H} \setminus H_t}$ be the hulls. Let $\mathcal{N}$ be domain symmetric about the real axis containing K_{t_0} and suppose that $\Phi : \mathcal{N} \to \Phi(\mathcal{N})$ conformal transformation with $\Phi(\mathbb{R} \cap \mathcal{N}) \subset \mathbb{R}, \Phi(\mathbb{H} \cap \mathcal{N}) \subset \mathbb{H}$.

Let $K_t^* = \Phi(K_t)$ and $H_t^* = \mathbb{H} \setminus K_t^*$. (It might be tempting to write $H_t^* = \Phi(H_t)$ but we are not assuming that Φ is defined on all of H_t.) Let $g_t^* : H_t^* \to \mathbb{H}$ be the unique conformal transformation with $g_t^*(z) = z + o(1)$ as $z \to \infty$. Then,

$$g_t^*(z) = z + \frac{a^*(t)}{z} + O(|z|^{-2}),$$

where $a^*(t) = \mathrm{hcap}[K_t^*]$. As we will see below, it is not the case that $a^*(t) = 2t$. We define

$$\Phi_t(z) = g_t^* \circ \Phi \circ g_t^{-1}(z).$$

Using the crosscuts as in (35), we can see that there exists $\epsilon > 0$ such that for each $t \leq t_0$, the map Φ_t is a conformal transformation of $\{z \in \mathbb{H} : |z - U_t| < \epsilon\}$ with $\lim_{y \downarrow 0} \mathrm{Im}[\Phi_t(x + iy)] = 0$. Hence Φ_t extends to a conformal transformation of $\{z \in \mathbb{C} : |z - u_t| < \epsilon\}$. Let $U_t^* = \Phi_t(U_t)$.

It may be worth stopping and thinking about this. There exists a sequence of half circles η_n of smaller and smaller radius around U_t such that $\mathrm{diam}[g_t^{-1}(\eta_n)]$ goes to zero. Using the Beurling estimate, say, we can then see that

$$\mathrm{diam}\left[g_t^* \circ \Phi \circ g_t^{-1}(\eta_n)\right] \to 0,$$

and hence $g_t^* \circ \Phi \circ g_t^{-1}$ is well defined near U_t. However, the diameter of $g_t^{-1}\{z \in \mathbb{H} : |z - U_t| < r\}$ does not necessary go to zero as r goes to zero.

The Cauchy integral formula implies that for s sufficiently small,

$$\Phi'_{t+s}(U_t) = \frac{1}{\pi \epsilon i} \int_C \frac{\Phi_{t+s}(z)}{z - U_t}\, dz,$$

where C denotes the circle of radius $\epsilon/2$ about U_t. Using this, we can see that $s \mapsto \Phi'_{t+s}(U_t)$ is continuous (it is, in fact, differentiable), and from this we can see that $t \mapsto \Phi'_t(U_t)$ is continuous. (This latter function is not necessarily differentiable since the map $t \mapsto U_t$ does not have to be differentiable.)

Proposition 8.21. *Under the assumptions above,*

$$\partial_t a^*(t) = 2\, \Phi'_t(U_t)^2,$$

Proof. Since the right-hand side is continuous, it suffices to show that the right-derivative of $a^*(t)$ equals $2\, \Phi'_t(U_t)^2$. The argument is the same for all t, so for ease let us compute the right-derivative at the origin. By scaling and translation, we may assume that $\Phi = \Phi_0$ is defined in the unit disk with $\Phi(0) = 0$, $\Phi'_0(0) = 1$. This then reduces to the next lemma. $\square$

Lemma 8.22. *There exist $c < \infty$ such that the following holds. Let $\Phi : \mathbb{D} \to \Phi(\mathbb{D})$ be a conformal transformation with $\Phi(\mathbb{D} \cap \mathbb{H}) = \mathbb{H} \cap \Phi(\mathbb{D})$, $\Phi(0) = 0, \Phi'(0) = 1$. Then for every compact hull K with* $\mathrm{rad}(K) \leq 1/2$,

$$|\mathrm{hcap}[\Phi(K)] - \mathrm{hcap}(K)| \leq c\sqrt{\mathrm{rad}(K)}\,\mathrm{hcap}(K).$$

Proof. Let $h = \mathrm{hcap}(K), r = \mathrm{rad}(K), \tilde{K} = \Phi(K)$, $q = \sqrt{r}$. Distortion estimates tell us that there exists $c < \infty$ (uniform over all such Φ) such that for $|z| \leq 1/2$,

$$|\Phi(z) - z| \leq c\,|z|^2, \qquad |\mathrm{Im}[\Phi(z)] - \mathrm{Im}(z)| \leq c\,|z|\,\mathrm{Im}(z). \tag{44}$$

In particular, $\mathrm{rad}(\tilde{K}) = r + O(r^2)$.

Let τ (resp., $\tilde{\tau}$) be the first time that a Brownian motion B_t hits $K \cup \mathbb{R}$ (resp., $\tilde{K} \cup \mathbb{R}$). We know that

$$\mathrm{hcap}(K) = \frac{2r}{\pi}\int_0^\pi \mathbb{E}^{qe^{i\theta}}\left[\mathrm{Im}(B_\tau)\right]\sin\theta\,d\theta,$$

$$\mathrm{hcap}(\tilde{K}) = \frac{2r}{\pi}\int_0^\pi \mathbb{E}^{qe^{i\theta}}\left[\mathrm{Im}(B_{\tilde{\tau}})\right]\sin\theta\,d\theta.$$

Let $\sigma = \tau_\wedge \inf\{s : |B_s| = 1/10\}$ and similarly for $\tilde{\sigma}$. Note that for $|z| = q$,

$$\mathbb{E}^z\left[\mathrm{Im}(B_\tau)\right] = \mathbb{E}^z\left[\mathrm{Im}(B_\tau); \tau = \sigma\right]\,[1 + O(q)].$$

$$\mathbb{E}^z\left[\mathrm{Im}(B_{\tilde{\tau}})\right] = \mathbb{E}^z\left[\mathrm{Im}(B_{\tilde{\tau}}); \tilde{\tau} = \tilde{\sigma}\right]\,[1 + O(q)].$$

Here $O(q)$ is an upper bound for the probability that a Brownian motion starting on the circle of radius $1/10$ reaches the circle of radius q without hitting $\mathbb{R}$.

Using (44), we have

$$\mathrm{Im}[\Phi(z)] = \mathrm{Im}[z]\,\left[1 + O(r)\right], \qquad z \in K.$$

Hence we can write

$$\mathbb{E}^z\left[\mathrm{Im}(B_{\tilde{\tau}}); \tilde{\tau} = \tilde{\sigma}\right] = \mathbb{E}^z\left[\mathrm{Im}(\Phi^{-1}(B_{\tilde{\sigma}})); \tilde{\tau} = \tilde{\sigma}\right]\,[1 + O(q)].$$

Using (44) again, we see that $\Phi(z) = z + O(q^2)$ for $|z| = q$. Hence derivative estimates for harmonic functions give us for $|z| = q$,

$$\mathbb{E}^z\left[\mathrm{Im}(\Phi^{-1}(B_{\tilde{\sigma}})); \tilde{\tau} = \tilde{\sigma}\right] = \mathbb{E}^{\Phi^{-1}(z)}\left[\mathrm{Im}(\Phi^{-1}(B_{\tilde{\sigma}})); \tilde{\tau} = \tilde{\sigma}\right]\,[1 + O(q)].$$

Finally, conformal invariance of Brownian motion shows us that

$$\mathbb{E}^{\Phi^{-1}(z)}\left[\mathrm{Im}(\Phi^{-1}(B_{\tilde{\sigma}})); \tilde{\tau} = \tilde{\sigma}\right] = \mathbb{E}^{z)}\left[\mathrm{Im}(B_\sigma); \tau = \sigma\right]\,[1 + O(q)].$$

$\square$

With this result, we see that g_t^* satisfies the Loewner equation

$$\partial_t g_t^*(z) = \frac{2\,\Phi_t'(U_t)^2}{g_t^*(z) - U_t^*}. \tag{45}$$

Proposition 8.23. *Under the assumptions above, if $0 \leq t < t_0$ and $|z - U_t| < \epsilon$, then*

$$\dot{\Phi}_t(z) = 2 \left[\frac{\Phi_t'(U_t)^2}{\Phi_t(z) - \Phi(U_t)} - \frac{\Phi_t'(z)}{z - U_t} \right],$$

$$\dot{\Phi}_t'(z) = 2 \left[\frac{\Phi_t'(U_t)^2 \, \Phi_t'(z)}{(\Phi_t(z) - \Phi_t(U_t))^2} + \frac{\Phi_t'(z)}{(z - U_t)^2} - \frac{\Phi_t''(z)}{z - U_t} \right].$$

In particular,

$$\dot{\Phi}_t(U_t) = -3 \, \Phi_t''(U_t), \tag{46}$$

$$\dot{\Phi}_t'(U_t) = \frac{\Phi_t''(U_t)^2}{2 \, \Phi_t'(U_t)} - \frac{4 \, \Phi_t'''(U_t)}{3}. \tag{47}$$

Here we are writing $\dot{\Phi}_t(U_t)$ for $\dot{\Phi}_t(z)$ evaluated at $z = U_t$ and similarly for $\dot{\Phi}_t'(U_t)$.

Proof. We have noted that $t \mapsto \Phi_t'(U_t)$ is continuous so the right hand side is continuous in z, t. The first equation is an exercise in the chain rule, writing $\Phi_t = g_t^* \circ \Phi \circ g_t^{-1}$ and using (39) and (45). The second is obtained by differentiating with both sides with respect to z.

The limits are straightforward.

□

8.8 Radial Loewner equation

The Loewner equation in the upper half-plane is used to describe a curve connecting two boundary points in a simply connected domain. There is a similar equation called the *radial Loewner equation* that describes curves connecting a boundary point to an interior point. For ease, we choose the domain to be the unit disk and the interior point to be the origin. The proofs are similar to the upper half plane case, so we will not give all the details.

Suppose that $D = \mathbb{D} \setminus K$ is a simply connected subdomain of $\mathbb{D}$ containing the origin. The Riemann mapping theorem implies that there is a unique conformal transformation $g_D : D \to \mathbb{D}$ with $g_D(0) = 0, g_D'(0) > 0$. The construction of the map (see the proof of Theorem 3) shows that we can write $g(z) = z \, e^{f(z)}$ where $f(z) = \phi(z) + i\theta(z)$ is the holomorphic extension of the harmonic function

$$\phi(z) = \mathbb{E}^z \left[-\log |B_\tau| \right],$$

with $\theta(0) = 0$. Here B is a complex Brownian motion and $\tau = \inf\{t : B_t \notin D\}$. For $z \in D \setminus \{0\}$, we can write

$$\log g(z) = \log z + f(z),$$

provided that we interpret this correctly. We must either interpret g as a multi-valued function, or if we restrict to a simply connected neighborhood of z in $D \setminus \{0\}$ we can take a particular branch. Recall that

$$\frac{H_{\mathbb{D}}(z,w)}{H_{\mathbb{D}}(0,w)} = \frac{1 - |z|^2}{|z - w|}, \; ; \; |z| < 1, |w| = 1.$$

Proposition 8.24. *There exists $c < \infty$ such that if $D = \mathbb{D} \setminus K$ is a domain with $K \subset \{z : |z - w| \leq r\}$ and $z \in \mathbb{D}$ with $|z - w| \geq 2r$,*

$$\left| \phi(z) - \frac{1 - |z|^2}{|z - w|} \, \phi(0) \right| \leq \frac{c\phi(0)(1 - |z|^2)}{|z - w|}.$$

We can write this as

$$\phi(z) = \frac{1-|z|^2}{|z-w|}\,\phi(0)\,\left[1 + O\left(\frac{r}{|z-w|}\right)\right].$$

Sketch of proof. Let $U = U_{w,r} = \mathbb{D} \setminus \{\zeta : |w-\zeta| \le r\}$. Let $\xi = \xi_{r,w} = \inf\{t : |w - B_t| \le r\}$. Then for $|z-w| > r$,

$$\phi(z) = \mathbb{P}^z\{\xi < \tau\}\,\mathbb{E}^z\left[-\log|B_\tau| \mid \xi < \tau\right] = \frac{1}{2}\int_{\partial U} H_U(z,\zeta)\,\phi(\zeta)\,|d\zeta|.$$

Hence, it suffices to show that

$$H_U(z,\zeta) = H_U(0,\zeta)\,\frac{1-|z|^2}{|z-w|}\,\left[1 + O\left(\frac{r}{|z-w|}\right)\right].$$

This can be done either explicitly by mapping U to the $\mathbb{D}$ or by an argument similar to the chordal case. We leave the details to the reader.

□

Proposition 8.25. *Suppose $|w| = 1$ and $D_t = \mathbb{D} \setminus K_t$ is a decreasing sequence of domains with corresponding maps $g_t = g_{D_t}$ satisfying $g_t'(0) = e^{2at}$ and $\mathrm{rad}(K_t - w) \to 0$ as $t \downarrow 0$. Then,*

$$\lim_{t\downarrow 0} \frac{\phi_t(z)}{t} = 2a\,\frac{1-|z|^2}{|z-w|}.$$

Theorem 14. *Suppose $a > 0$, $\gamma : (0,t] \to \mathbb{D} \setminus \{0\}$ is a simple curve with $\gamma(0+) = w \in \partial\mathbb{D}$. Let $D_t = \mathbb{D} \setminus \gamma_t$ and let g_t be the corresponding conformal transformation Suppose g_t is parametrized so that $g_t'(0) = e^{2at}$. Then g_t satisfies the radial Loewner equation*

$$\dot{g}_t(z) = 2a\,g_t(z)\,\frac{w_t + g_t(z)}{w_t - g_t(z)}$$

where $w_t = g_t(\gamma(t)) \in \partial\mathbb{D}$. Moreover, the function $t \mapsto w_t$ is continuous. If we define h_t by

$$g_t(e^{2iz}) = \exp\left\{2ih_t(z)\right\},$$

then

$$\dot{h}_t(z) = a\,\cot(h_t(z) - X_t),$$

where X_t is a continuous process with $e^{2iX_t} = w_t$.

Proof. The full proof is similar to that of Theorem 9 so we do not give the details. Without loss of generality, assume that $2a = 1$. Note that this is clearly true at $z = 0$ and for $z \neq 0$, we can write

$$g_t(z) = z\,\exp\{\phi_t(z) + i\theta_t(z)\}.$$

Proposition 8.24 implies that

$$\partial_t[\log g_t(z)] = \frac{w_t + g_t(z)}{w_t - g_t(z)}, \tag{48}$$

$$\dot{\phi}_t(z) = \mathrm{Re}\left[\frac{w_t + g_t(z)}{w_t - g_t(z)}\right] = \frac{1 - |g_t(z)^2|}{|g_t(z) - w_t|}.$$

Since $\theta_t(0) = 1$ by the definition of g_t, we have $\dot{\theta}_t(0) = 0$. Hence we get (48). The chain rule gives

$$\dot{h}_t(z) = \frac{1}{2i}\,\frac{e^{2iX_t} + e^{2ih_t(z)}}{e^{2iX_t} - e^{2ih_t(z)}} = \frac{i}{2}\,\frac{e^{2i(h_t(z)-X_t)} + 1}{e^{2i(h_t(z)-X_t)} - 1} = \frac{1}{2}\,\cot(h_t(z) - X_t).$$

□

Writing points on the unit circle as $e^{2i\theta}$ rather than $e^{i\theta}$ makes some formulas nicer. For example $H_{\partial\mathbb{D}}(1, e^{2i\theta}) = (1/4)\,\sin^{-2}\theta$ and the sine of the argument of 0 with respect to 1 and $e^{2i\theta}$ is $\sin\theta$.

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9 Properties of the Loewner equation

In this section we suppose that U_t is a continuous real-valued function with $U_0 = 0$ and that g_t is the solution to the Loewner equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{49}$$

which for $z \in \mathbb{C} \setminus \{0\}$ is valid up to time $T_z \in (0, \infty]$. It is immediate that the flow is symmetric about the real axis, that is, $g_t(\overline{z}) = \overline{g_t(z)}$; in particular, $T_{\overline{z}} = T_z$. We let

$$H_t = \{z \in \mathbb{H} : T_z < \infty\}, \quad H_t^* = \{z \in \mathbb{C} \setminus \{0\} : T_z < \infty\}, \quad K_t = \mathbb{H} \setminus H_t.$$

If $z = x \in \mathbb{R}$, then (49) is a real differential equation for which it is easy to check that if $x < x'$ and $T_x, T_{x'} > t$, then $g_t(x) < g_t(x')$. From this we can see $\{x \in \mathbb{R} : T_x \leq t\}$ is a closed interval which we write as $[x_t^-, x_t^+]$ with $x_t^- \leq 0 \leq x_t^+$. Here the trivial interval $x_t^- = x_t^+ = 0$ is a possibility. Then we can write

$$H_t^* = H_t \cup \{z : \overline{z} \in H_t\} \cup (-\infty, x_t^-) \cup (x_t^+, \infty).$$

Recall that $g_t(H_t) = \mathbb{H}$. We define

$$D_t^* = g_t(H_t^*), \quad D_t^+ = g_t(H_t \cup \{z : \overline{z} \in H_t\} \cup (x_t^+, \infty)),$$

and we define D_t^- similarly. We can write

$$D_t^* = \mathbb{C} \setminus [g_t(x_t^-), g_t(x_t^+)],$$

where $g_t(x_t^-) = \sup\{g_t(y) : y < x_t^-\}, g_t(x_t^+) = \inf\{g_t(y); y > x_t^+\}$, and similarly for $D_t^{\pm}$. Note that $g_t(x_t^-) \leq U_t \leq g_t(x_t^+)$.

Lemma 9.1. *Suppose g_t satisfies (49), $r > 0$, and $t \mapsto a(t)$ is a strictly increasing C^1 function. Define $\tilde{g}_t$ by*
$$\tilde{g}_t(z) = r\, g_{a(t)/2}(z/r).$$
Then $\tilde{g}_t$ satisfies
$$\partial_t \tilde{g}_t(z) = \frac{r^2 \dot{a}(t)}{\tilde{g}_t(z) - \tilde{U}_t}, \qquad \text{where} \quad \tilde{U}_t = r\, U_{a(t)/2}.$$
In particular, if $U_t = \sqrt{\kappa}\, B_t$ where B_t is a standard Brownian motion, then

- *If $a(t) = t/r^2$, then $\tilde{U}_t$ is a Brownian motion with variance parameter κ.*
- *If $r = 1$ and $a(t) = (2/\kappa)\, t$, then $\tilde{U}_t$ is a standard Brownian motion.*

Proof. The first assertion follows from chain rule and the second uses standard scaling properties of Brownian motion. □

Proposition 9.2. *Suppose $g_t = u_t + iv_t$ satisfies (49).*

1. *For all $z = x + iy \in \mathbb{H}$ and all $t < T_z$,*
$$v_t(z) \geq \sqrt{y^2 - 4t}.$$
In particular $T_z \geq y^2/4$.

2. *If $r > 0$, $z \in \mathbb{H}$ and $\|U\|_\infty \leq r$, then*
$$|g_t(z)|^2 \geq |z|^2 - 8t, \quad \text{if} \quad 0 \leq t \leq \frac{|z|^2 - 4r^2}{8}.$$
In particular, $|g_t(z)| \geq r/\sqrt{2}$ for $|z|^2 \geq 8t + 4r^2$, and
$$K_t \subset \{z : |z| < 2\sqrt{2t + r^2},\ \operatorname{Im}(z) \leq 2\sqrt{t}\}. \tag{50}$$

Proof.

1. Let $Y_t = \operatorname{Im}[g_t(z)]$ and note that
$$\partial_t Y_t^2 = -\frac{4Y_t^2}{|Z_t|^2} \geq -4.$$
Therefore, $Y_t^2 \geq y^2 - 4t$.

2. Let $Q_t = |g_t(z)|$ and $\sigma = \inf\{t : Q_t = 2r\}$. Then for $t \leq \sigma$, $Q_t - r \geq Q_t/2$, and hence,
$$\partial_t Q_t \geq -\frac{4}{Q_t}, \quad \partial_t [Q_t^2] \geq -8.$$
and hence
$$Q_t^2 \geq Q_0^2 - 8t, \quad t \leq \sigma,$$
and $\sigma \geq (|z|^2 - 4r^2)/8$.

□

With a little more care in the proof, we could improve the constants in (50), but this result suffices for our purposes.

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10 Multiply connected domains

We will classify finitely connected domains up to conformal equivalence. Suppose $D \subset \hat{\mathbb{C}}$ is a domain whose complement consists of a finite number of connected components $\partial_0, \partial_1, \ldots, \partial_k$, each consisting of more than a single point. Let $\mathcal{O}_k$ denote the set of such domains with $\partial_0 = \hat{\mathbb{C}} \setminus \mathbb{H}$, that is, $D \subset \mathbb{H}$ of the form

$$D = \mathbb{H} \setminus (\partial_1 \cup \cdots \cup \partial_k),$$

where $\partial_1, \ldots, \partial_k$ are disjoint compact sets with $\mathrm{dist}(\partial_j, \mathbb{R}) > 0$. It suffices to classify domains in $\mathcal{O}_k$ since we can map $\hat{\mathbb{C}} \setminus \partial_0$ to the upper half plane. We let $\mathcal{O}^*$ denote the set of such domains such that each ∂_j is a horizontal line segment,

$$\partial_j = [x_{j,-} + iy_j, x_{j,+} + iy_j].$$

Theorem 15. *Every domain $D \in \mathcal{O}_k$ is conformally equivalent to a domain $D^* \in \mathcal{O}_k^*$. Moreover, there exists a unique conformal map $f : D \to D^*$ to a domain $D^* \in \mathcal{O}$ that*

$$f(z) = z + O(|z|^{-1}), \quad |z| \to \infty.$$

Remarks.

- If $D \in \mathcal{O}_k$ and $f : D \to D^* \in \mathcal{O}^*$ is a conformal transformation sending the real line to the real line with $f(\infty) = \infty$, then f can be extended to

  $$\mathbb{R} \cup D \cup \{\overline{z} : z \in D\}.$$

  by Schwarz reflection. By considering the function

  $$g(z) = \frac{1}{f(1/z)},$$

  which is holomorphic in a neighborhood of the origin sending a neighbhorhood of the real line to the real line, we can see that f has an expansion at infinity,

  $$f(z) = b_{-1}z + b_0 + b_1 z^{-1} + \cdots$$

  where $b_{-1} > 0$ and $b_j \in \mathbb{R}$ for $j \geq 0$. If we set $\hat{f}(z) = [f(z) - b_0]/b_{-1}$, then

  $$\hat{f}(z) = z + O(|z|^{-1}).$$

- The "dimension" of the set of conformally equivalent domains is $3k-2$. There are $3k$ choices for the $\{x_{j,-}, x_{j,+}, y_j\}$ but the equivalence classes are invariant under the map $z \mapsto rz+t$.

Proof. Let D be given and suppose $D^* \in \mathcal{O}^*$ given by $\{(x_{j,-}, x_{j,+}, y_j)\}$. Let $y(D) = \max\{\mathrm{Im}(z) : z \in \partial D\}, y(D^*) = \max\{\mathrm{Im}(z) : z \in \partial D^*\} = \max\{y_j\}, R = R(D) = \sup\{|z| : z \in \mathbb{H} \setminus D\}$. Suppose that $f : D \to D^*$ is a conformal transformation with $f(\infty) = \infty, f'(\infty) = 1$. Let us write $f(z) = u(z) + iv(z)$. Note that v is a harmonic function on D with boundary value 0 on $\mathbb{R}$ and y_j on ∂_j and satisfying

$$v(z) = \mathrm{Im}(z) + O(1), \quad \mathrm{Im}(z) \to \infty.$$

Since $h(z) := v(z) - \mathrm{Im}(z)$ is a bounded harmonic function, the function v is given on D by

$$v(z) = v_D(z) + \mathbb{E}^z\left[v(B_{\tau_D})\right] = \mathrm{Im}(z) + \mathbb{E}^z\left[h(B_{\tau_D})\right]$$

where v_D is the function from Lemma 8.1. Note that

$$|v(z) - \mathrm{Im}(z)| \le \mathbb{P}^z\{B_{\tau_D} \notin \mathbb{R}\}\,[y(D) + y(D^*)].$$

Using explicit forms of Poisson kernels, we can see that $\mathbb{P}^z\{B_{\tau_D} \notin \mathbb{R}\} \le cR/|z|$, and hence

$$|h(z)| \le \frac{c}{|z|}.$$

Here, and for the rest of this proof, we allow constants to depend on D. Using derivative estimates for harmonic function, we see that for $|z| \ge 2R$,

$$|\nabla h(z)| \le \frac{c}{|z|^2}. \tag{51}$$

We have found the candidate for v without making any restriction on the target domain D^*. If $f = u + iv$ is a holomorphic extension, then we know that it is given for $y > y(D)$ by

$$u(iy) = -\int_y^\infty \partial_x v(it)\,dt. \tag{52}$$

The existence of the integral follows from (51) as well as the estimate $|u(iy)| \le c/y$.

We will now give a criterion under which we can find a conjugate harmonic function u such that $f = u + iv$ is holomorphic. For each $j = 1, \ldots, k$, let F_j be a confomal map

$$F_j : \{|z| > 1\} \to \mathbb{C} \setminus \partial_j$$

and let $\gamma_r(t) = \gamma_{r,j}(t) = F(e^{r+it}), 0 \le t \le 2\pi$. For sufficiently small r, the curve $\gamma_{r,j}$ separates ∂_j from the other parts of ∂D. We only consider such r here. Let $\phi_j(z) = v(F_j(z))$ which is a harmonic function on $\{1 < |z| < e^r\}$. The condition that we need satisfied is

$$\int_{\gamma_{r,j}} \partial_n v(z) = 0, \tag{53}$$

which is the same as

$$\int_{C_{-r}} \partial_n \phi_j(z)\,|dz| = 0. \tag{54}$$

Since ϕ is harmonic on $\{1 < |z| < e^r\}$, Proposition 2.12 shows that

$$M_r = \frac{1}{2\pi}\int_0^{2\pi} \phi_j(e^{r+i\theta})\, d\theta,$$

is a linear function of r and (54) holds if and only if M_r is constant. Note that if $O_r = \{1 < |z| < e^r\}$ and $\mathcal{E}^{\#}_{O_r}(C_0, dw)$ denotes the probability measure on C_r,

$$\mathcal{E}^{\#}_{O_r}(C_0, \cdot) = \frac{\mathcal{E}_{O_r}(C_0, \cdot)}{\mathcal{E}_{O_r}(C_0, C_r)},$$

then

$$M_r = \int_{C_r} \phi_j(w)\, \mathcal{E}^{\#}_{O_r}(C_0, dw).$$

Using conformal invariance, we see that this translates to the condition

$$\frac{1}{\mathcal{E}_D(\partial_j, \gamma_{r,j})}\int_{\gamma_{r,j}} v(w)\, d\mathcal{E}_D(\partial_j, dw) = v(\partial_j) = y_j.$$

Using Proposition 2.12, we can see that if γ is any simple curve γ that wraps around ∂_j but no other part of the boundary,

$$\frac{1}{\mathcal{E}_D(\partial_j, \gamma)}\int_{\gamma} v(w)\, d\mathcal{E}_D(\partial_j, dw) = y_j. \tag{55}$$

In fact, if this holds for one such γ, then it holds for all such γ. This is the necesssary and sufficient condition so that u as defined in (52) gives a holomorphic function.

This condition can be expressed in terms of a process that is called *excursion reflected Brownian motion*. This is a process X_t whose state space is $D \cup \{\partial_0, \partial_1, \ldots, \partial_k\}$. In other words, we identify the "holes" ∂_j into single points. To describe the process we give the properties.

- When the process reaches ∂_0 the process stops.
- When the process is in D it acts like usual Brownian motion.
- Suppose $j \geq 1$ and γ is a curve as above that separates ∂_j from the rest of the boundary. Let $T = \inf\{t : B_t = \partial_j\}$ and $S = \inf\{t \geq T : B_t \in \gamma\}$. Then for any z, the conditional distribution on B_S given $T < \infty$ is that of normalized excursion measure $\mathcal{E}^{\#}_C(\partial_j, \cdot)$.

It is not difficult to define Brownian motion "excursion reflected" off of the unit circle. Roughly speaking, everytime the process hits the boundary it chooses an angle randomly. (Since the number of visits to the boundary is uncountable, one needs to take a little care here, but it is not a problem.) For other domains, we can use conformal invariance to define the process near holes, and away from holes it acts like Brownian motion. The equation (55) can be viewed as a mean value property for the function v with respect to excursion reflected Brownian motion, that is, the required condition on v is that v is excursion reflected harmonic.

We now ask: can we find y_j so that v is excursion reflected harmonic? Let us view the excursion reflected Brownian motion at the times it visits the boundary points $\{\partial_0, \partial_1, \ldots, \partial_k\}$. This gives a discrete time, discrete space Markov chain Y_n with absorbing state ∂_0. The transition probabilities $q(j,l)$ are given by

$$\mathbb{P}\{Y_{n+1} = l \mid Y_n = j\} = \frac{\mathcal{E}_D(\partial_j, \partial_l)}{\mathcal{E}_D(\partial_j, \partial D \setminus \partial_j)}.$$

Let N be sufficiently large so that $\mathbb{H} \cap \{\mathrm{Im}(w) \geq N\} \subset D$. Let X_t be the excursion reflected Brownian motion starting at ∂_j, let $A_j = \{\partial_0, \ldots, \partial_k\} \setminus \{\partial_j\}$, and let

$$T_N = T_{N,j} = \inf\{t : \mathrm{Im}(X_t) = N \text{ or } X_t \in A_j\}.$$

If v is excursion reflected harmonic, then

$$y_j = \mathbb{E}\left[v(X_{T_N})\right] = \mathbb{P}\{\mathrm{Im}(X_T) = N\}\, \mathbb{E}[v(X_T) \mid \mathrm{Im}(X_T) = N\} + \sum_{l \neq j} \mathbb{P}\{X_T = \partial_l\}\, y_l.$$

Note that

$$\lim_{N \to \infty} \sum_{l \neq j} \mathbb{P}\{X_T = \partial_l\}\, y_l = \sum_{l \neq j} q(j,l)\, y_l.$$

Since $v(z) = \mathrm{Im}(z) + O(1)$ as $z \to \infty$,

$$\lim_{N \to \infty} \mathbb{P}\{\mathrm{Im}(X_T) = N\}\, \mathbb{E}[v(X_T) \mid \mathrm{Im}(X_T) = N\} = \lim_{N \to \infty} N \mathbb{P}\{\mathrm{Im}(X_T) = N\}.$$

Also,

$$\lim_{N \to \infty} N\, \mathbb{P}\{\mathrm{Im}(X_T) = N\} = \frac{\mathcal{E}_D(\infty, \partial_j)}{\mathcal{E}_D(\partial_j, \partial D \setminus \partial_j)},$$

where

$$\mathcal{E}_D(\infty, \partial_j) = \lim_{N \to \infty} N \mathcal{E}_D(I_N, p_j) = \alpha_j := \int_{-\infty}^{\infty} \mathrm{hm}_D(x + ib, \partial_j)\, dx.$$

The integral on the right is the same for all $b > y(D)$.

The parameters $q(j,l)$ and α_j can be determined from the domain D. What we have seen is that we need y_j to satisfies

$$y_j = \alpha_j + \sum_{l \neq j} q(j,l)\, \alpha_l,$$

where $y_0 = 0$. This has a unique solution that can be given in terms of the finite Markov chain,

$$y_j = \mathbb{E}^{\partial_j}\left[\sum_{n=0}^{\infty} \alpha_{Y_n}\right] = \sum_{l=1}^{k} \alpha_l\, g(j,l).$$

Here

$$g(j,l) = \sum_{n=0}^{\infty} \mathbb{P}\{Y_n = y_l \mid Y_0 = y_j\}$$

is the Green's function for the discrete Markov chain with absorbing state ∂_0.

At this point we have shown that for every $D \in \mathcal{O}$, there is a unique choice of $\{y_j\}$ for which we can find a function v on $\mathbb{H}$ with the following properties:

- v is positive on D with
$$v(z) = \mathrm{Im}(z) + O(1), \quad z \to \infty.$$
- $v \equiv 0$ on $\mathbb{R}$.
- v is continuous on $\mathbb{H}$ with $v(z) = y_j$ for $z \in \partial_j, j \geq 1$.
- v is excursion reflected harmonic. Equivalently for every closed curve γ in v,
$$\int_\gamma \partial_n v(w)\,|dw| = 0. \tag{56}$$

Let $z_0 \in D$. We define u by
$$u(z) = \int_\gamma \partial_n v(w)\,|dw|.$$
Here γ is any curve from z_0 to z. Since v satisfies (56) the value is independent of the curve γ. This gives a holomorphic function $f : D \to D^*$. We need to show that f is one-to-one and onto. As in the proof of the Riemann mapping theorem, we consider the level sets of v. Let
$$V_s = \{z : v(z) = s\}, \quad V_s^+ = \{z : v(s) > s\}, \quad V_s^- = \{z : v(s) < s\}.$$
Let $\mathbb{H}_b = \{z \in \mathbb{H} : \mathrm{Im}(z) < b\}$. There exists c such that $v(z) \leq \mathrm{Im}(z) + c$. We claim that V_s^+ is connected. Indeed there is one connected component, say U, of V_s^+ that contains $\mathbb{H} \setminus \mathbb{H}_{s+c}$ for some c. Since v is bounded on $\mathbb{H}_{s+c}$, we can see that v is bounded on any other component U_1, and its maximum value on that component must be bounded by the maximum of v on ∂U_1 which is s. For V_s^-, note that there exists $c_D < \infty$ such that for all z,
$$v(z) \leq \mathrm{Im}(z) + \|y_j\|_\infty \, \mathbb{P}^z\{B_\tau \notin \mathbb{R}\} \leq c_D \, \mathrm{Im}(z).$$
In particular, $\mathbb{H}_\epsilon \subset V_s^-$ for all ϵ sufficiently small, and there is a unique component of V_s^- that contains $\mathbb{R}$ on its boundary. Suppose there were another component of V_s^-, say U'. Then the value of v on $\partial[D \cap U']$ is either s or in $\{y_1, \ldots, y_k\}$. Since v is a bounded harmonic function on $D \cap U'$,
$$v(z) = \mathbb{E}^z\left[v(B_{\tau'})\right], \quad \tau' = \tau_{D \cap U'}.$$
Since $v(z) < s$, there must be at least one j such that $\partial_j \subset U'$. Let us choose ∂_j such that y_j is minimal (if there is a tie, choose any one). Then by the maximal (minimal) principle, $v(z) \geq y_j$ for all $z \in U'$. However, if γ is a curve in D surrounding ∂_j and close enough to ∂_j, we know that the average value of v on γ (with respect to normalized excursion measure) is y_j. This means either v is constant (which is clearly not true in our case), or v takes on values smaller than v_j. This is a contradiction, and hence U' does not exist.

We fix D and allow constants to depend on D. Let $z = x + iy$ and consider $|z|$ large. Note that
$$v(z) = y + \mathbb{E}^z[h(B_\tau)],$$
where $\phi(z) = \mathbb{E}^z[h(B_\tau)]$ and h is a bounded function that vanishes on the real line. Using the form of the Poisson kernel, we can see that
$$|\phi(x + iy)| \leq \frac{c\,y}{|z|^2}.$$

Using the fact that ϕ is a harmonic function in the disk of radius $|z|/2$ about z (if x is large, but y is not, use Schwarz reflection about the origin to extend the function), we see that

$$|\nabla\phi(z)| \leq \frac{c\,y}{|z|^3},$$

and hence

$$|\partial_x v(x+iy)| + |\partial_y v(x+iy) - 1| \leq \frac{c\,y}{|z|^3}.$$

We will define (at least for large y)

$$u(iy) = -\int_y^\infty \partial_x v(iy')\,dy'.$$

Since $|\partial_x v(y')| = O(|y|^{-2})$, we see that $u(iy)$ is well defined and $|u(iy)| = O(|y|^{-1})$. We then define

$$u(x+iy) = u(iy) + \int_0^x \partial_y v(x'+iy)\,dx',$$

and, similarly we see that

$$u(x+iy) = x + O(|y|^{-1}).$$

From this we can see that

$$\lim_{y\to\infty} u(x+iy) = x,$$

and this allows us to see that we can also write

$$u(x+iy) = x - \int_y^\infty \partial_x v(x+iy')\,dy'.$$

This allows to improve the error to

$$u(z) = \mathrm{Re}(z) + O(|z|^{-1}),$$

and

$$f(z) = z + O(|z|^{-1}).$$

This guarantees injectivity outside a compact subset. Inside we do an argument as in the Riemann mapping theorem.

□

The following is proved similarly. We only sketch the proof. Let $\mathcal{O}'_k$ denote the set of domains $U \subset \mathbb{D}$ of the form

$$U = \mathbb{D} \setminus (\partial'_1 \cup \cdots \cup \partial'_k)$$

where the ∂_j are disjoint circular arcs of the form

$$\partial_j = \{e^{-r_j+i\theta} : \theta_{j,-} \leq \theta \leq \theta_{j,+}\},$$

with $r_j > 0$ and $0 \leq \theta_{j,-} < 2\pi, \theta_{j,-} < \theta_{j,+} < \theta_{j,-} + 2\pi$.

Proposition 10.1. *Suppose $D = \mathcal{O}_k$ and $\zeta \in D$. Then there exists a unique $U \in \mathcal{O}_k'$ such that there exists a (unique) conformal map $f : D \to U$ with $f(\partial_0) = \partial\mathbb{D}$, $f(\zeta) = 0$, $f'(\zeta) > 0$.*

Proof. Without loss of generality we will assume that $\zeta = 0$. Suppose such a domain U and function f exists with parameters $(r_j, \theta_{j,-}, \theta_{j,+})$. Consider the harmonic function on $D \setminus \{\zeta\}$ given by

$$h(z) = -\log|f(z)|.$$

If h is known, then $\theta(z) := \arg f(z)$ can be determined from the Cauchy-Riemann equations. As in Theorem 15, in order for this to be well defined, we need the condition that $h(z)$ must be a harmonic function for excursion reflected Brownian motion. Note that as $z \to 0$,

$$h(z) = -\log|z| - \log|f'(0)| + o(1).$$

Consider excursion reflected Brownian motion started on ∂_j stopped when it reaches the either $\partial D \setminus \partial_j$ or $\mathcal{B}_s := e^{-s}\mathbb{D}$. For fixed s, h is a bounded excursion reflected Brownian motion on $D \setminus \mathcal{B}_s$. Arguing as above and letting $s \to \infty$, we see that

$$r_j = G_D^{ER}(z, 0) + \sum_{l \neq j} q(j, l) r_l,$$

where $G_D^{ER}(z, 0)$ is the Green's function for excursion reflected Brownian motion defined in the natural way, and $q(j, l)$ are the probabilities as before. This determines the parameters r_j and hence it determines $|f(z)|$. The function $f(z)$ is obtained by fixing some $z \in D \setminus \{0\}$ and arbitrarily choosing $\theta(z)$. This will define f up to a rotation and exactly one of those rotations will have $f'(0) > 0$.

□

11 Poisson kernels and Green's functions for standard domains

Here we will give the Poisson kernels and Green's functions for a number of standard domains. Recall that we have chosen the constants in our definitions so that

$$H_{\mathbb{D}}(0, e^{i\theta}) = \frac{1}{2}, \quad G_{\mathbb{D}}(0, z) = -\log|z|.$$

If $n_z = n_{z,D}$ represents the inward normal at z, then if $z \in D$ and $w, \zeta \in \partial D$,

$$H_{\mathbb{D}}(z, w) = \frac{1}{2}\,\partial_{n_w} G_D(z, w),$$

$$H_{\partial D}(\zeta, w) = \partial_{n_\zeta} H_D(\zeta, w) = \partial_{n_w} H_D(w, \zeta) = \frac{1}{2}\,\partial_{n_\zeta}\partial_{n_w}\, G_D(\zeta, w).$$

We will do straightforward computations using the scaling rules if $f : D \to f(D)$ is a conformal transformation, then

$$G_D(z, z') = G_{f(D)}(f(z), f(z')), \quad H_D(z, w) = |f'(w)|\, H_{f(D)}(f(z), f(w)),$$

$$H_{f(D)}(f(z), f(w)) = |f'(\zeta)|\,|f'(w)|\, H_{f(D)}(f(\zeta)), f(w)).$$

11.1 Disk

Proposition 11.1. *For the unit disk* $\mathbb{D} = \{|z| < 1\}$,

$$H_{\mathbb{D}}(w, e^{i\theta}) = H_{\mathbb{D}}(we^{-i\theta}, 1) = \frac{1}{2}\,\frac{1-|w|^2}{|e^{i\theta}-w|^2}$$

$$G_{\mathbb{D}}(z, w) = -\log\left|\frac{z-w}{1-z\overline{w}}\right|,$$

$$H_{\partial\mathbb{D}}(e^{i2\theta}, e^{i2\psi}) = \frac{1}{4\,\sin^2(\theta-\psi)}.$$

Proof. Using the Möbius transformation

$$T_w(z) = \frac{z-w}{1-z\overline{w}}, \quad T_w'(z) = \frac{1-|w|^2}{(1-z\overline{w})^2}$$

$$G_{\mathbb{D}}(z, w) = G_{\mathbb{D}}(T_w(z), 0) = -\log\left|\frac{z-w}{1-z\overline{w}}\right|,$$

$$H_{\mathbb{D}}(w, 1) = |T_w'(1)|\, H_{\mathbb{D}}(T_w(w), T_w(1)) = \frac{1}{2}\,\frac{1-|w|^2}{|1-\overline{w}|^2} = \frac{1}{2}\,\frac{1-|w|^2}{|1-w|^2}.$$

$$H_{\mathbb{D}}(w, e^{i\theta}) = H_{\mathbb{D}}(we^{-i\theta}, 1) = \frac{1}{2}\,\frac{1-|w|^2}{|e^{i\theta}-w|^2}$$

$$\begin{aligned}
H_{\partial\mathbb{D}}(e^{i\theta}, 1) &= \lim_{r\downarrow 0} r^{-1}\, H_{\mathbb{D}}((1-r)e^{i\theta}, 1) \\
&= \lim_{r\downarrow 0} \frac{1}{r}\left[\frac{1}{2}\,\frac{1-(1-r)^2}{|1-(1-r)e^{i\theta}|^2}\right] \\
&= \frac{1}{|1-e^{i\theta}|^2} = \frac{1}{4\sin^2(\theta/2)},
\end{aligned}$$

and

$$H_{\partial\mathbb{D}}(e^{i2\theta}, e^{2i\psi}) = H_{\partial\mathbb{D}}(e^{i2(\theta-\psi)}) = \frac{1}{4\,\sin^2(\theta-\psi)}.$$

□

We start with the upper half plane,

$$H_{\mathbb{H}}(x+iy, x') = \frac{y}{(x-x')^2+y^2}, \quad H_{\mathbb{H}}(x, x') = (x-x')^{-2}.$$

11.2 Rectangle

Let $\mathcal{R}_L$ be the rectangle $\{x+iy : 0 < x < L, 0 < y < \pi\}$ and let ∂_L denote the vertical line segment $[L, L+i\pi]$.

Using separation of variables, we can see that

$$f(x+iy) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sinh(nx)\, \sin(ny)\, \sin(ny'n)}{\sinh(nL)},$$

is a harmonic function in $\mathcal{R}_L$ that vanishes on $\partial\mathcal{R} \setminus \partial_L$ and equals (in the sense of distributions) $\delta_{y'}$ on ∂_L. Therefore,

$$H_{\mathcal{R}_L}(x+iy, L+iy') = 2\sum_{n=1}^{\infty} \frac{\sinh(nx)\, \sin(ny)\, \sin(ny')}{\sinh(nL)},$$

$$H_{\mathcal{R}_L}(iy, L+iy') = 2\sum_{n=1}^{\infty} \frac{n\, \sin(ny)\, \sin(ny')}{\sinh(nL)}.$$

The sums are absolutely convergent for $|x| < L$. Note that for $0 < x < 1$,

$$H_{\mathcal{R}_L}(x+iy, L+iy') = 2\sinh x\, \sin y\, \sin y'\, [1 + O(e^{-L})].$$

$$H_{\mathcal{R}_L}(iy, L+iy') = 2\sin y\, \sin y'\, [1 + O(e^{-L})].$$

11.3 Upper half plane

Proposition 11.2. *For the upper half plane* $\mathbb{H} = \{x+iy : Y > 0\}$,

$$G_{\mathbb{H}}(z,w) = \log|z - \overline{w}| - \log|z-w|$$

$$H_{\mathbb{H}}(x+iy, x') = \frac{y}{(x-x')^2 + y^2}, \quad H_{\mathbb{H}}(x, x') = \frac{1}{(x-x')^2},$$

Proof. The map

$$f(z) = \frac{z-i}{z+i}, \quad f'(z) = \frac{2i}{(z+i)^2},$$

takes the upper half plane $\mathbb{H}$ onto $\mathbb{D}$ with $f(i) = 0$. Therefore,

$$G_{\mathbb{H}}(z,i) = G_{\mathbb{D}}(f(z), f(i)) = -\log\left|\frac{z-i}{z+i}\right| = \log\frac{|z+i|}{|z-i|},$$

$$G_{\mathbb{H}}(z, x+iy) = G_{\mathbb{H}}\left(\frac{z-x}{y}, i\right) = \log\frac{|\frac{z-x}{y} + i|}{|\frac{z-x}{y} - i|} = \log\frac{|z - \overline{(x+iy)}|}{|z-(x+iy)|},$$

$$H_{\mathbb{H}}(i,x) = |f'(x)|\, H_{\mathbb{H}}(0, f(x)) = \frac{1}{2}\, \frac{2}{|x+i|^2} = \frac{1}{x^2+1},$$

$$H_{\mathbb{H}}(x+iy, x') = y^{-1}\, H_{\mathbb{H}}\left(i, \frac{x'-x}{y}\right) = \frac{y}{(x-x')^2 + y^2}.$$

$$H_{\mathbb{H}}(x,x') = \lim_{y\downarrow 0} y^{-1}\, H_{\mathbb{H}}(x,x'+iy) = \frac{1}{(x-x')^2}.$$

Another way to see that the Green's function is

$$G_{\mathbb{H}}(z,w) = \log|z-\overline{w}| - \log|z-w|.$$

is to note that for fixed z, the right-hand side is a harmonic function of w that vanishes on the real line and looks like $-\log|z-w| + O(1)$ as $w \to z$.

□

If $z = re^{i\theta}$, then

$$G_{\mathbb{H}}(re^{i\theta}, i) = \frac{1}{2}\,\log\left|\frac{r^2\cos^2\theta + (r\sin\theta+1)^2}{r^2\cos^2\theta + (r\sin\theta-1)^2}\right|.$$

As $r \to \infty$,

$$G_{\mathbb{H}}(re^{i\theta}, i) = \frac{1}{2}\,\log\left|1 + \frac{4r\,\sin\theta}{r^2\cos^2\theta + (r\sin\theta-1)^2}\right| = 2r^{-1}\sin\theta\,[1+O(r^{-1})]. \tag{57}$$

11.4 Half Disk

Proposition 11.3. *Let $\mathbb{D}_+ = \mathbb{H}\cap\mathbb{D}$ be the upper half disk. Then,*

$$H_{\partial\mathbb{D}_+}(e^{i\theta}, e^{i\psi}) = \frac{\sin\theta\,\sin\psi}{(\cos\theta-\cos\psi)^2},$$

$$H_{\partial\mathbb{D}_+} = (x, e^{i\theta})\frac{2(1-x^2)}{[x^2-2x\cos\theta+1]^2}\,\sin\theta.$$

In particular, $H_{\partial\mathbb{D}_+}(0,e^{i\theta}) = 2\,\sin\theta$. More generally, for z near the origin,

$$H_{\mathbb{D}_+}(z, e^{i\theta}) = 2\,\mathrm{Im}(z)\,\sin\theta\,\left[1+O(|z|)\right]. \tag{58}$$

Proof. The function

$$f(z) = \frac{2z}{z^2+1}, \qquad f'(z) = \frac{2\,(1-z^2)}{(z^2+1)^2},$$

is a conformal transformation of $\mathbb{D}_+$ onto $\mathbb{H}$ with $f(0)=0, f(i)=\infty, f(1)=1, f(-1)=-1$ and

$$f(e^{i\theta}) = \frac{2e^{i\theta}}{e^{2i\theta}+1} = \frac{2}{e^{i\theta}+e^{-i\theta}} = \frac{1}{\cos\theta}.$$

Note that

$$|f'(e^{i\theta})| = \left|\frac{2(1-e^{2i\theta})}{(e^{2i\theta}+1)^2}\right| = \frac{\sin\theta}{\cos^2\theta}.$$

Therefore,

$$\begin{aligned} H_{\partial\mathbb{D}_+}(e^{i\theta}, e^{i\psi}) &= |f'(e^{i\theta})|\,|f'(e^{i\psi})|\,H_{\mathbb{H}}(f(e^{i\theta}), f(e^{i\psi})) \\ &= \frac{\sin\theta\,\sin\psi}{\cos^2\theta\,\cos^2\psi}\,\left[\frac{1}{\cos\theta} - \frac{1}{\cos\psi}\right]^{-2} \\ &= \frac{\sin\theta\,\sin\psi}{(\cos\theta-\cos\psi)^2}. \end{aligned}$$

$$H_{\partial\mathbb{D}_+}(x, e^{i\theta}) = \frac{2(1-x^2)}{(x^2+1)^2}\,\frac{\sin\theta}{\cos^2\theta}\left[\frac{2x}{x^2+1} - \frac{1}{\cos\theta}\right]^{-2} = \frac{2(1-x^2)}{[x^2-2x\cos\theta+1]^2}\,\sin\theta.$$

One could derive (58) from the exact formulas. Alternatively, note that the function $h(z) = H_{\mathbb{D}_+}(z, e^{i\theta})$ can be extended by Schwarz reflection to a harmonic function on $\mathbb{D}$ that vanishes on the real line. Note that $\partial_y h(0) = 2\sin\theta$ and $\partial_x h$ vanishes on the real line. Also, $|h(z)| \leq c\sin\theta$ for $|z| \leq 1/2$, and hence we can see that that for $|z| \leq 1/4$ all the second partials are bounded by a universal constant times $\sin\theta$. Hence

$$\partial_y h(x) = 2\,\sin\theta\,[1 + O(|x|)],$$

and

$$h(x+iy) = y\,\partial_y h(x) + O((\sin\theta)\,y^2) = 2\,y\sin\theta\,[1 + O(|x|) + O(y)].$$

□

11.5 Infinite Strip

Proposition 11.4. *Let $S_r = \{x + iy \in \mathbb{H} : y < r\}$ denote the half-infinite strip. Then*

$$H_{\partial S_r}(0, x) = \frac{\pi^2}{4r^2}\left[\sinh\left(\frac{\pi x}{2r}\right)\right]^{-2},$$

$$H_{\partial S_r}(0, x+ir) = \frac{\pi^2}{4r^2}\left[\cosh\left(\frac{\pi x}{2r}\right)\right]^{-2},$$

Proof. Note that $f(z) = e^z$ is a conformal transformation of S_π onto $\mathbb{H}$ and hence

$$\begin{aligned} H_{\partial S_\pi}(0, x) &= |f'(0)|\,|f'(x)|\,H_{\partial\mathbb{H}}(f(0), f(x+i\pi)) \\ &= e^x\,H_{\mathbb{H}}(1, e^x) \\ &= \frac{e^x}{(e^x-1)^2} = \frac{1}{(e^{x/2}-e^{-x/2})^2} = \frac{1}{4\sinh^2(x/2)}. \end{aligned}$$

$$\begin{aligned} H_{\partial S_\pi}(0, x+i\pi) &= |f'(0)|\,|f'(x)|\,H_{\partial\mathbb{H}}(f(0), f(x+i\pi)) \\ &= e^x\,H_{\partial\mathbb{H}}(1, -e^x) \\ &= \frac{e^x}{(e^x+1)^2} = \frac{1}{4\cosh^2(x/2)}. \end{aligned}$$

By using the conformal transformation $z \mapsto (\pi/r)z$, we get

$$H_{\partial S_r}(0, x) = (\pi/r)^2\,H_{\partial S_\pi}(0, \pi x/r) = \frac{\pi^2}{4r^2}\left[\sinh\left(\frac{\pi x}{2r}\right)\right]^{-2},$$

$$H_{\partial S_r}(0, x+ir) = (\pi/r)^2\,H_{\partial S_\pi}(0, \pi x/r + i\pi) = \frac{\pi^2}{4r^2}\left[\cosh\left(\frac{\pi x}{2r}\right)\right]^{-2}.$$

□

We will compute this another way using Fourier series. This will be useful when comparing to simple random walk. We will let $S = S_1$, and we first consider the domain $R_m = \{x + iy : 0 < x < 2m, 0 < y < 1\}$. Consider

$$F(x + iy) = \sum_{j=1}^{\infty} b_j \sin(j\pi x/2m) \sinh(j\pi y/2m).$$

For any choice of constants (decaying sufficiently fast), this gives a harmonic function. If we choose

$$b_j = \frac{\sin(mj\pi/2m)}{2m \sinh(j\pi/2m)},$$

we see that the boundary condition on ∂R_m is the delta function at $m + ni$. Therefore,

$$\frac{1}{\pi} H_{R_m}((m+u) + iy, m+i) = \sum_{j=1}^{\infty} \frac{\sin(mj\pi/2m)}{2m \sinh(j\pi/2m)} \sin(j\pi(m+u)/2m) \sinh(j\pi y/2m).$$

Note that

$$\sin(mj\pi/2) \sin(j\pi(m+u)/2mm) = \begin{cases} \cos(j\pi u/2m), & j \text{ odd}, \\ 0 & j \text{ even}, \end{cases}$$

Therefore,

$$\frac{1}{\pi} H_{R_m}((m+u) + iy, m+in) = \sum_{j=1}^{\infty} \frac{\cos((2j-1)\pi u/2m) \sinh((2j-1)\pi y/2m)}{2m \sinh((2j-1)\pi/2m)}.$$

This is a Riemann sum approximation of an integral and hence we get

$$\lim_{m\to\infty} H_{R_m}((m+u) + iy, m+i) = \pi \int_0^{\infty} \frac{\cos(t\pi u) \sinh(t\pi y)}{\sinh(\pi t)} \, dt = \int_0^{\infty} \frac{\cos(su) \sinh(sy)}{\sinh s} \, ds,$$

$$\begin{aligned} H_{\partial S}(u, i) &= \partial_y H_S(z, i) \,|_{z=u} \\ &= \int_0^{\infty} \frac{s \cos(su)}{\sinh s} \, ds \\ &= \partial_u \left[\int_0^{\infty} \frac{\sin(su)}{\sinh s} \, ds \right] \\ &= \partial_u \left[\frac{\pi}{2} \tanh(u\pi/2) \right] \\ &= \frac{\pi^2}{4 \cosh^2(u\pi/2)}. \end{aligned}$$

The penultimate inequality is identity 711 in the CRC table of integrals.