Math 16100, Section 50
Part III: Compactness, Functions, Continuity, and more

Definition If A and B are nonempty point sets and if, for every $a \in A$ and $b \in B$, the relation $a < b$ holds, then we write $A < B$. We often write $x < A$ if $\{x\} < A$, and $A < x$ if $A < \{x\}$.

Lemma G If R is a region and $x \notin R$, then $x < R$ or $R < x$.

Theorem 25 If $C = A \cup B$ and $A < B$, then either A has a last point or B has a first point.

Definition If A is a point set and p is a point such that $A \backslash \{p\} < p$, then p is called an *upper bound* of A and A is said to have p as an upper bound. (*Lower bound* is defined similarly). If p is an upper bound of A and there does not exist $x < p$ such that x is an upper bound of A, then p is the *least upper bound* (or *supremum*) of A; in symbols p lub A. If p is a lower bound of A and there is no lower bound x with $p < x$, then p is said to be the *greatest lower bound* or *infimum* of A; this is written p glb A.

Definition A set is said to be *bounded* if it has both an upper bound and a lower bound. Equivalently, it is bounded if there exists a region containing it.

Corollary 25.1 Every nonempty bounded set has both a least upper bound and a greatest lower bound.

Corollary 25.2 Every nonempty closed and bounded set has both a first point and a last point.

Definition A set G of point sets is said to *cover* a point set A if every point of A is in some element of G. We call G a *cover* of M in this situation. G is an *open cover* if all of its elements are open.

Definition If $x, y \in C$, then a finite collection of regions $R_1, R_2, ..., R_n$ will be said to form a *simple chain of regions from x to y* if (1) R_i contains x if and only if $i = 1$; (2) R_i contains y if and only if $i = n$; (3) $R_i \cap R_j \neq \phi, i < j$ if and only if $j = i + 1$. The regions R_i are called links of the chain.

Theorem 26 If $a < x < b$ and $R_1, R_2, ..., R_n$ form a simple chain from a to b then there exists $i, 1 \leq i \leq n$, such that $R_1, R_2, ..., R_i$ form a simple chain from a to x.

Theorem 27 If ab is a region and G is a collection of regions covering $\overline{ab}$ then G has a finite subset which forms a simple chain of regions from a to b.

Corollary 27.1 If ab is a region and G is a collection of regions covering $\overline{ab}$, then some finite sub-collection of G covers $\overline{ab}$.

Corollary 27.2 If ab is a region and G is a collection of open sets covering $\overline{ab}$, then some finite sub-collection of G covers $\overline{ab}$.

Definition A point set M is said to be *compact* if, for every open cover G of M, there is a finite subset $G' \subset G$ which is also an open cover of M.

Theorem 28 (Heine-Borel) M is a closed bounded point set iff M is compact.

Theorem 29 (Bolzano-Weierstrass) Every bounded infinite point set has at least one limit point.

Theorem 30 If $M_1, M_2, ..., M_n, ...$ is an infinite sequence of nonempty compact sets such that, for each n, $M_{n+1} \subset M_n$, then $\cap_{n=1}^{\infty} M_n$ is a nonempty compact set.

Definition If A and B are sets, then a *function from A to B* (or a *mapping from A to B*) is a set of ordered pairs (a, b) such that $a \in A$, $b \in B$ and each $a \in A$ occurs in exactly one pair $(a, b) \in f$. If for each $b \in B$ there exists an $a \in A$ such that (a, b) belongs to f, then we say the function is *onto* B. To designate a function from A to B, we use the notation $f : A \to B$. If (a, b) is a specific pair in the set defining the function f, we write $f(a) = b$. Generally, if $M \subset A$ then we define the image $f(M)$ of M in B under f to be $f(M) = \{f(x)|x \in M\}$. In particular, if f is onto then $f(A) = B$. We say f is *one-to-one* if, for all $a_1, a_2 \in A$, we have $f(a_1) = f(a_2)$ implies $a_1 = a_2$. If f is a function from A to B and $N \subset B$, we define $f^{-1}(N) = \{a \in A|f(a) \in N\}$. $f^{-1}(N)$ is called the inverse image of N under f.

Question Does a one-to-one function $f : A \to B$ necessarily determine a one-to-one correspondence?

Theorem 31 If $f : A \to B$, $X \subset B$ and $Y \subset B$, then (1) $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$ and (2) $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$.

Theorem 32 If $f : A \to B$, $X \subset A$ and $Y \subset A$ then $f(X \cup Y) = f(X) \cup f(Y)$.

Question Is it necessarily true that $f(X \cap Y) \subset f(X) \cap f(Y)$?

Question Is it necessarily true that $f(X \cap Y) \supset f(X) \cap f(Y)$?

Definition Suppose A and B are point sets. A function $f : A \to B$ is said to be *continuous* if, for each $x \in A$ and region R containing $f(x)$, there exists a region S containing x such that $f(A \cap S) \subset R$.

Lemma H If f is a function $f : A \to B$, then $f(f^{-1}(B)) \subset B$. $f(f^{-1}(B)) = B$ if and only if $f(A) = B$.

Theorem 33 (Intermediate Value Theorem) If A is connected and $f : A \to C$ is continuous, then $f(A)$ is connected.

Question How is this version of the Intermediate Value Theorem related to the version one usually encounters in a calculus textbook?

Definition If A and B are point sets with $A \subset B$, then A is said to be *closed relative to B* if $\overline{A} \cap B \subset A$. A is said to be *open relative to B* if $B - A$ is closed relative to B.

Lemma I Suppose A and B are point sets with $A \subset B$. A necessary and sufficient condition that A is open relative to B is that, for all $x \in A$, there exists a region R containing x such that $R \cap B \subset A$.

Theorem 34 If A, B are point sets and $f : A \to B$ is a function, then a necessary and sufficient condition that f be continuous is that, for every U which is open relative to B, we have that $f^{-1}(U)$ is open relative to A.

Theorem 35 If $f : A \to B$ is a continuous mapping of point sets and A is closed and bounded, then $f(A)$ is closed and bounded.

Corollary 35.1 (Extreme Value Theorem) If $f : A \to B$ is a continuous mapping of point sets and A is compact, then $f(A)$ has a first and last point.

Definition If M is a point set, $f : M \to C$, and $x \in M$ satisfies $f(x) = x$, then x is called a *fixed point* of the function f.

Theorem 36 If R is a region and $f : \overline{R} \to \overline{R}$ is a continuous mapping, then f has a fixed point.