

Math 16100, Section 50

Part II: C Inherits a Topology

Definition A point set which contains all of its limit points is *closed*.

Definition If M is a point set, then \overline{M} denotes the *closure* of M , which is the set consisting of M and all of its limit points. Also let M^d denote the *derived set* of M , which consists of all the limit points of M .

Theorem 11 A point set M is closed if and only if $M = \overline{M}$.

Note We often abbreviate the phrase “if and only if” as “iff.”

Theorem 12 If M is a point set, then $\overline{M} = \overline{\overline{M}}$.

Corollary 12.1 If M is a point set, then \overline{M} is closed.

Theorem 13 The sets C and \emptyset are closed.

Definition Given two sets A, B with $B \subset A$, the set $A \setminus B$ is called the *complement* of B with respect to A , or simply the complement of B rel A , and it consists of all the elements of A but not B . For any point set M , we assume that the phrase *complement of M* means $C \setminus M$.

Proposition (De Morgan’s Law) If \mathcal{A} is a collection of sets $A \subset C$, then

$$C \setminus \left(\bigcup_{A \in \mathcal{A}} A \right) = \bigcap_{A \in \mathcal{A}} (C \setminus A) \text{ and } \bigcup_{A \in \mathcal{A}} (C \setminus A) = C \setminus \left(\bigcap_{A \in \mathcal{A}} A \right).$$

Definition If the complement of a point set M is closed, then M is *open*.

Theorem 14 The sets C and \emptyset are open.

Theorem 15 A set A of points is open iff for all $x \in A$, there exists a region R such that $x \in R \subset A$.

Corollary 15.1 Every region is open and its complement is closed.

Corollary 15.2 Every nonempty open set is the union of some set of regions.

Theorem 16 The intersection of any set of closed sets is closed, and the intersection of a finite number of open sets is open.

Question Is it necessarily the case that the intersection of an infinite number of open sets is open?

Theorem 17 The union of any number of open sets is open, and the union of a finite number of closed sets is closed.

Question Is it necessarily the case that the union of an infinite number of closed sets is closed?

Definition Let A be any set. A *topology* on A is a set T of subsets of A satisfying the following conditions:

1. A and \emptyset are elements of T .
2. The union of the sets in any subset of T is again an element of T .
3. The intersection of any pair of sets in T is an element of T .

The elements of T are called open sets in A . The complements of open sets are called closed sets.

Corollary 17.1 Every finite point set is closed.

Corollary 17.2 A nonempty set is open iff it is the union of regions.

Corollary 17.3 If ab is a region, then the sets $\{x \mid x < a\}$ and $\{x \mid b < x\}$ are open. Accordingly, $\text{ext}(ab)$ is open.

Definition If A, B are nonempty ($A \neq \emptyset \neq B$), disjoint point sets each containing no point in the closure of the other ($\overline{A} \cap B = \emptyset = A \cap \overline{B}$), then they are *separated*. If a point set M is the union of separated sets A, B , then we write $M = A \cup B \text{ sep}$.

Theorem 18 If U is an open set, and $U = A \cup B \text{ sep}$, then A and B are open. Also, if $F = D \cup E \text{ sep}$ and F is closed, then D and E are closed.

Definition A point set is *connected* if it is not the union of any pair of separated sets.

Theorem 19 Any degenerate point set is connected.

Theorem 20 If M is a nondegenerate connected point set and $x \in M$, then $x \text{ lp } M$.

Lemma E If $M = A \cup B \text{ sep}$ and S is a connected subset of M , then either $S \subset A$ or $S \subset B$.

Theorem 21 If M is a connected point set and A is a point set for which $M \subset A \subset \overline{M}$, then A is connected.

Axiom 3 The only point sets which are both closed and open (i.e. *clopen*) are C and \emptyset .

Theorem 22 The set C is connected.

Proposition (Betweenness) All regions are nonempty, i.e. if $a < b$, there exists $c \in C$ such that $a < c < b$.

Lemma F If ab is a region, then $a \text{ lp } ab$ and $b \text{ lp } ab$.

Theorem 23 Every region is connected.

Theorem 24 Every point of a region is a limit point of that region.

Corollary 24.1 Every region contains infinitely many points.

Corollary 24.2 Every point in C is a limit point of C .

Corollary 24.3 If $C \neq \emptyset$, then C is infinite.