Math 161
Section 50
Course Notes: Topology of the Continuum

Course Goals and Structure

The goals of this course are threefold:

1. to develop an understanding of the structure of the continuum of real numbers which is essential to analysis,

2. to introduce students to methods of mathematical proof, and

3. to develop students' ability to formulate, understand and judge mathematical proofs.

We will use an axiomatic approach to develop our understanding of the continuum, i.e. our understanding will unfold from a few basic principles. The class notes will describe the principle implications of the starting principles, without giving the proofs which describe how the basic principles lead to these conclusions. It will be up to the students to provide this connective tissue. At first, we will focus primarily on the correctness of arguments offered, but as time goes on it is hoped that we will also begin to consider the clarity and elegance of arguments offered.

The mathematical subject matter during this quarter is the topology of the real numbers. We start, in essence, with a set C on which we have a basic relation $<$. Based on a few assumptions, we develop notions of limit and limit point in C, and eventually reach basic results such as the Bolzano-Weierstrass and Heine-Borel theorems.

Undefined Terms and Assumptions

Any discussion must conducted using terms which are already understood. In our case, these will include

- the mathematical notion of a set, as well as standard operations on sets such as union, intersection, and so on

- the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ as well as the larger set $\mathbb{W} = \{0, 1, 2, \ldots\}$ of whole numbers, along with their usual orderings and arithmetic operations, and

- basic notions of logic.

We will use notation standard in logic: $\wedge$ meaning *and,* $\vee$ meaning *or,* $\exists$ meaning *there exists,* $\forall$ meaning *for all,* $\Longrightarrow$ meaning *implies* or as a substitute for *if... then...*, and $\Longleftrightarrow$ meaning *if and only if* or as a substitute for *... is a necessary and sufficient condition for...*, etc.

Credits

These notes are based on notes used by Diane Herrmann and Paul Sally in a similar course they taught in 2004-5, as well as a revision and expansion of those notes by a student in the course, Piotr Behr.

To begin, fix a set denoted by C. We shall sometimes refer to elements of C as "points."

Axiom 1 C is simply ordered by the relation $<$. Thus:

1. If x and y are distinct points in C, then either $x < y$ or $y < x$.
2. Given elements x, y of C, if $x < y$, then x and y are distinct points.
3. Given elements x, y, z of C, if $x < y$ and $y < z$, then $x < z$ (transitivity).

Definition If A is a set and x is *an element of* A or *in* A, we denote this relation by $x \in A$.

Definition Two sets A and B are equal if they have the same elements, i.e. if $x \in A$ implies $x \in B$, and $x \in B$ implies $x \in A$.

Definition The *section of the natural numbers determined by* $k \in \mathbb{N}$ (denoted $\mathbb{N}_k$) is the set of all natural numbers n such that $n < k$.

Definition If C_1 and C_2 are sets, then a *one-to-one correspondence* between C_1 and C_2 is a set $C_{1,2}$ of pairs (x_1, x_2) where $x_i \in C_i$ ($i = 1$ or $i = 2$) and every element of C_i is part of one and only one element of $C_{1,2}$.

Definition The set with no elements is called the *empty* or *null set* and denoted by $\emptyset$.

Definition A set A is called *finite* if there is a number k for which A and $\mathbb{N}_k$ have a one-to-one correspondence. Since each pair in this correspondence is of the form (x, n), where $x \in A$ and $n \in \mathbb{N}_k$, we denote each element of A in each pair as follows: x_n. The number $k - 1$ is called the cardinality of A, and we sometimes write $|A| = k - 1$.

Definition If A is a set of points of C, then a point $a \in A$ is the *first point* of A if, for every element $x \in A$, either $a < x$ or $a = x$. Analogously, a point $b \in A$ is the *last point* of A if, for every $x \in A$, either $x < b$ or $x = b$.

Theorem 1 If x and y are both points in C, then $x < y$ and $y < x$ do not both hold.

Lemma A If A is a nonempty finite subset of C, then A has a first and last point.

Theorem 2 If A is a set of n distinct points in C, then symbols $a_1, a_2, \ldots, a_n$ may be assigned to each point of A so that $a_1 < a_2 < \cdots < a_n$ (that is, $a_i < a_{i+1},\ i \in \{1, 2, \ldots, n-1\}$).

Definition If $x, y, z \in C$ and we have $x < y$ and $y < z$, then y is *between* x and z.

Corollary 2.1 Of three distinct points, one must be between the other two.

Axiom 2 C has no first or last point.

Definition If $a < b$, then the set of points between a and b is called a *region,* denoted by ab.

Theorem 3 If $p \in C$, then there exists a region ab such that $p \in ab$.

Definition A set is *nondegenerate* if it contains more than one element.

Definition If A, B are sets, then the set of elements in both A, B is called the *intersection* of A and B, and it is denoted $A \cap B$.

Definition We use the symbol $\setminus$ to denote the subtraction of sets; if A, B are sets, then $A \setminus B$ is the set of all elements which are in A and not in B.

Definition Let A be a nonempty subset of C. A point p is called a *limit point* of A if every region R containing p has nonempty intersection with $A \setminus \{p\}$. This can be expressed in symbols as follows: $p \operatorname{lp} A$ if for every region R with $p \in R$, $R \cap (A \setminus \{p\}) \neq \emptyset$.

Definition If A, B are sets and for each point $x \in A$ we have $x \in B$, then A is a *subset* of B, and this is expressed in symbols as $A \subset B$.

Theorem 4 If $x \operatorname{lp} A$ and $A \subset B$, then $x \operatorname{lp} B$.

Definition The following notation is used to indicate the set of all elements x which has some property P: $\{x \mid x \text{ has property } P\}$. Often, when we are referring to a subset of C, we will omit $x \in C$ from this notation, e.g. $\{x < a\}$ means $\{x \in C | x < a\}$.

Definition If A, B are sets, then the set of all elements which are in either set is called the *union* of A and B, and it is denoted $A \cup B$.

Lemma B If ab is a region in C, then $C = \{x \mid x < a\} \cup \{a\} \cup ab \cup \{b\} \cup \{x \mid b < x\}$.

Definition If ab is a region, then $C \setminus (ab \cup \{a\} \cup \{b\})$ is called the *exterior* of ab, and it is denoted $\operatorname{ext}(ab)$.

Lemma C If ab is a region in C, then $C = \operatorname{ext}(ab) \cup \{a\} \cup \{b\} \cup ab$.

Lemma D No point in the exterior of a region is a limit point of that region, and no point of that region is a limit point of the exterior of that region.

Theorem 5 If two regions have a point x in common, their intersection is a region containing x.

Corollary 5.1 If $R_1, \ldots, R_n$ are arbitrary regions in C having a point x in common, then their intersection $\bigcap_{k=1}^{n} R_k = R_1 \cap \ldots \cap R_n$ is a region containing x.

Theorem 6 If $p \operatorname{lp} A \cup B$, then $p \operatorname{lp} A$ or $p \operatorname{lp} B$ (or both).

Corollary 6.1 A point which is a limit point of a finite union of sets is a limit point of at least one set which is a member of that union.

Corollary 6.2 Let A, B be sets and x a point. Then $x \operatorname{lp} A \cup B \iff x \operatorname{lp} A \vee x \operatorname{lp} B$.

Definition If A, B are sets and $A \cap B = \emptyset$, then they are *pairwise disjoint* (or simply *disjoint*).

Theorem 7 If $p, q \in C$ and $p < q$, then there exist disjoint regions containing p and q, respectively.

Corollary 7.1 No set consisting of one point has a limit point.

Theorem 8 No finite point set has a limit point.

Corollary 8.1 If A is a finite point set and $x \in A$, then there exists a region $R \ni x$ which contains no point of A other than x.

Theorem 9 If $p \operatorname{lp} A$ and R is a region containing p, then the set $R \cap A$ is infinite.

Definition An *infinite sequence* is a set S of pairs of the form (n, x) where $n \in \mathbb{N}$, and every $n \in \mathbb{N}$ appears in exactly one element of S. A *finite sequence* is similar, except that only the elements $n \in \mathbb{N}_k$ are used. We often refer to a sequence by simply writing down the list $x_1, x_2, x_3, \ldots$ of elements which occur as the second components of the pairs $(1, x_1), (2, x_2), (3, x_3), \ldots$

Question Would it be equivalent to define a sequence as a one-to-one correspondence between $\mathbb{N}$ (or $\mathbb{N}_k$) and a set A?

Definition If $S = \{x_1, x_2, \ldots, x_n, \ldots\}$ is an infinite sequence of points in C, then a point p is called a *sequential limit point* of S if for each region $R \ni p$ there exists some $n(R) \in \mathbb{N}$ such that $n > n(R) \implies x_n \in R$. This is denoted in symbols as $p \operatorname{slp} S$.

Theorem 10 No infinite sequence has more than one sequential limit point.