Mathematics 16100
Problem Set #8
Due November 28, 2006

Read the notes up to Corollary 25.1 before doing this problem set.

1. In this problem, we will construct a pair $(C, <)$ satisfying Axioms 1, 2 and 3. Suppose $(C', <)$ is a pair satisfying Axioms 1 and 2 and the Betweenness Proposition, e.g. we can take C' to be the set of all rational numbers, or the set of all rational numbers expressible with denominators which are powers of 10, etc. We define a Dedekind cut in this situation to be a nonempty subset $A \subset C'$ satisfying:

   1. If $x \in A$, $y \in C'$ and $y < x$, then $y \in A$.
   2. There exists some $z \in C'$ such that $x < z$ whenever $x \in A$.
   3. A does not contain a largest element.

   Let C be the set of all Dedekind cuts in C'. For two such Dedekind cuts A, B, we say that $A < B$ if $A \subset B$ and $A \neq B$ (i.e. A is a *proper* subset of B). (NB: Do not apply any results from the notes which are dependent on Axiom 3 to C, since we are trying to *prove* that C satisfies Axiom 3.)

   (a) Prove that any nonempty set D in C which is bounded above has a least upper bound. (Note that this property is used in Spivak as one of the axioms characterizing the real numbers.)

   (b) Prove that any nonempty subset $D \subset C$ which is bounded below has a greatest lower bound.

   (c) Suppose $E \subset C$ is a nonempty proper subset of C which is both closed and open. Fix any $x \in E$, and define $U \subset E$ to be the union of all regions $R \subset E$ containing x. Show that U is nonempty, open, and closed.

   (d) With U as in (c), show that U is either bounded above or bounded below.

   (e) Again with U as in (c), let p be either a least upper bound or greatest lower bound of U. Show that p is a limit point for both U and its complement. Derive a contradiction.