Mathematics 16100
Problem Set #7
Due November 21, 2006

1. Assume $(C, <)$ satisfies Axioms 1 and 2.

   (a) If A is a subset of C, then we say A is disconnected if it is not connected. Now take $C = \mathbb{Q}$. Show that $\mathbb{Q}$ is disconnected.

   (b) Show that any two distinct rational numbers a, b can be separated by open sets, i.e. there are open subsets $U, V \subset \mathbb{Q}$ such that $a \in U$, $b \in V$, $\mathbb{Q} = U \cup V$ and $U \cap V = \emptyset$.

   (c) A set $A \subset C$ is said to be totally disconnected if the only connected subsets it contains are degenerate. Show that $\mathbb{Q}$ is totally disconnected.

2. Let A be any set. A relation on A is a set R of pairs from A; we sometimes write $x \sim y$ if $(x, y) \in R$. R is said to be reflexive if $(x, x) \in R$ whenever $x \in R$. R is said to be symmetric if $(x, y) \in R$ implies $(y, x) \in R$. R is said to by transitive if $(x, y), (y, z) \in R$ implies $(x, z) \in R$. We say that R is an equivalence relation if it is reflexive, symmetric and transitive.

   (a) Suppose A is a nonempty set with an equivalence relation R. Show that A can be written as the disjoint union of a set $\mathcal{B}$ of nonempty subsets of A with the property that, for any $B \in \mathcal{B}$, we have that $x, y \in B$ implies $x \sim y$, and $x \sim y$ implies that x and y both belong to some element of $\mathcal{B}$. (Note: the elements of $\mathcal{B}$ are referred to as the equivalence classes of R.)

   (b) Let $(C, <)$ satisfy Axioms 1 and 2, and let $A \subset C$ be an open subset. Define a relation on A by $x \sim y$ if there exists a region $ab \subset A$ with $x, y \in ab$. Show that this is an equivalence relation.

   (c) Show that the equivalence classes of the relation defined in part (b) are open sets.