Mathematics 16100
Problem Set #5
Due November 2, 2006

1. Consider a pair $(C, <)$ satisfying Axioms 1 and 2, with the additional assumption that C is countable. Prove that any limit point x of a subset $A \subset C$ is the sequential limit point of a sequence $x_1, x_2, x_3, \ldots$ with $x_k \in A$ for all $k \in \mathbb{N}$.

2. We define a *Dedekind cut* to be a subset $A \subset \mathbb{Q}$ satisfying the following three conditions:

   1. If $x \in A$, $y \in \mathbb{Q}$ and $y < x$, then $y \in A$.
   2. There exists some $z \in \mathbb{Q}$ such that $x < z$ whenever $x \in A$.
   3. A does not contain a largest element.

   If you have some prior conception of what a real number is, notice that any real number r determines a Dedekind cut by $A = \{x \in \mathbb{Q} | x < r\}$. For the purposes of this problem, we define the set of real numbers $\mathbb{R}$ to be the set of all Dedekind cuts. We identify $\mathbb{Q}$ with a subset of $\mathbb{R}$ by identifying $s \in \mathbb{Q}$ with $A = \{t \in \mathbb{Q} | t < s\}$. For all $A, B \in \mathbb{R}$, define $A < B$ if $A \subset B$ and $A \neq B$.

   (a) Show that $(\mathbb{R}, <)$ satisfies Axioms 1 and 2. (Just to be clear, $\mathbb{R}$ plays the role of C in Axioms 1 and 2.)

   (b) Let $x_1, x_2, x_3, \ldots$ be an infinite sequence in $\mathbb{R}$. It is said to be strictly increasing if $x_i < x_{i+1}$ for all $i \in \mathbb{N}$. It is said to be bounded above if there is some $y \in \mathbb{R}$ such that $x_i < y$ for all $i \in \mathbb{N}$. Prove that any strictly increasing sequence of real numbers which is bounded above has a sequential limit point.