Mathematics 16100
Problem Set #2
Due October 12, 2006

1. In each part of this problem, we will describe a set C and a relation $<$ between elements of C. In each case, determine which among Axioms 1.1, 1.2, 1.3 and 2 are satisfied. Explain your answer in each case.

   (a) Let C be the set of all natural numbers. If $m, n \in C$ define $m < n$ if m divides n, but is not equal to it.

   (b) C is the set of all nonnegative powers of 5. The relation $<$ is the same as in part (a).

   (c) Let C be the set of all finite sets of natural numbers. If $A, B \in C$, define $A < B$ if A is a subset of B, but not equal to B.

   (d) Let C be the set of all subsets $A \subset \mathbb{N}$ such that both A and $\mathbb{N} \backslash A$ are infinite. If $A, B \in C$, define $A < B$ if A is a subset of B.

   (e) C is the set of all linear functions $f(x) = ax + b$ with a, b both integers. Define $f(x) < g(x)$ if there exists some $n \in \mathbb{N}$ with $f(m) < g(m)$ whenever $m \geq n$.

2. A set A is said to be countably infinite if there is a one-to-one correspondence between $\mathbb{N}$ and A. In essence, this means that all the elements of A can be arranged into a list.

   (a) Prove that the set of integers $\mathbb{Z}$ is countable.

   (b) Prove that the set of pairs of integers is countable.

   (c) Prove that the set $\mathbb{Q}$ of rational numbers is countable.

Note: If you want a formal definition of the integers based on the natural numbers, consult the "Construction" section of the Wikipedia article on "Integer." Similarly, a formal construction of the rational numbers from the integers can be found in the Wikipedia article on "Rational number." However, for the purposes of this assignment, you can take standard properties of integers and rational numbers as known.