Mathematics 16100
Final Exam

Be prepared to present proofs of any of the following results. Do not discuss these problems or your experience with the exam with anyone other than the four instructors between your exam and 5 pm on December 5. You may bring your journal, homework assignments and any notes you would like to your exam.

1. **Lemma A**: If A is a nonempty finite subset of C, then A has a first and last point.
2. **Theorem 10**: No infinite sequence has more than one sequential limit point.
3. **Theorem 12**: If M is a point set, then $\overline{M} = \overline{\overline{M}}$.
4. Suppose $x_1, x_2, x_3, \ldots$ is a sequence in C with a sequential limit point x. Show that the only possible limit point of the set $A = \{x_1, x_2, x_3, \ldots\}$ is x. Show that x is a limit point of A iff A is infinite.
5. **Theorem 18 (part 1)**: If $U \subset C$ is an open set, and $U = A \cup B\,\text{sep}$, then A and B are open.
6. From homework #5: Consider a pair $(C, <)$ satisfying Axioms 1 and 2, with the additional assumption that C is countable. Prove that any limit point x of a subset $A \subset C$ is the sequential limit point of a sequence $x_1, x_2, x_3, \ldots$ with $x_k \in A$ for all $k \in \mathbb{N}$. You may wish to use the following outline.
   (a) First, show that is suffices to assume C (and thus $C\backslash\{x\}$) is countably infinite.
   (b) Let $B = \{(n, c_n) | n \in \mathbb{N}\}$ be a one-to-one correspondence between $\mathbb{N}$ and $C\backslash\{x\}$. For each $n \in \mathbb{N}$, find a region R_n containing x, but not containing any of $c_1, \ldots, c_n$.
   (c) Show that $R_n \cap (A\backslash\{x\})$ is nonempty, and thus that you can choose some $x_n \in R_n \cap (A\backslash\{x\})$.
   (d) Let $R = ab$ be any region containing x. Show that there exists some $N \in \mathbb{N}$ so that $a, b \in \{c_1, \ldots, c_N\}$, and that $R_n \subset R$ for every $n \geq N$. Conclude that $x_n \in R$ whenever $n \geq N$.
   (e) Conclude that the sequence $x_1, x_2, x_3, \ldots$ has x as a sequential limit point.

   Alternatively, you may prefer this outline.
   (a) First, show that is suffices to assume C (and thus $C\backslash\{x\}$) is countably infinite.
   (b) Show that the set of regions containing x is countably infinite. Enumerate them as $R_1, R_2, R_3, \ldots$.
   (c) Show that we can choose some $x_n \in A \cap R_1 \cap R_2 \cap \ldots \cap R_n$.
   (d) Show that the sequence $x_1, x_2, x_3, \ldots$ has x as a sequential limit point.
7. Let C be the set of all sequences $x_1, x_2, \ldots$ where
   (i) $x_i \in \{0, 1\}$ for all $i \in \mathbb{N}$,
   (ii) $\{i \in \mathbb{N} | x_i = 0\}$ is infinite, and
   (iii) $\{i \in \mathbb{N} | x_i = 1\}$ is infinite.

   Let $X = (x_1, x_2, \ldots)$ and $Y = (y_1, y_2, \ldots)$ be distinct elements of C; define $X < Y$ if $x_n = 0$ and $y_n = 1$, where n is the smallest natural number such that $x_n \neq y_n$. Show that $(C, <)$ satisfies Axioms 1 and 2, but not Axiom 3.
8. **Theorem 25**: If $C = A \cup B$ and $A < B$, then either A has a last point or B has a first point.