Exercise 2. *A set is said to be countable if it is either finite or countably infinite. Show that any subset of a countable set is again countable.*

We will prove that any subset B of a countably infinite subset A is countable. To complete the proof, the reader should verify that any subset of a finite set is finite.

To show B is countable, it suffices to assume B is not finite and show that it must then be countably infinite. Since A is countably infinite, we have a one to one correspondence consisting of the pairs $(1, a_1), (2, a_2), \ldots (i, a_i) \ldots$ between $\mathbb{N}$ and A. We will now use induction to define an element $b_i \in B$ and a infinite set Q_i for each $i \in \mathbb{N}$.

Let $Q_1 = \{j \in \mathbb{N} | a_j \in B\}$. Observe that since A is infinite, Q_1 is also infinite. By the first problem, Q_1 has a smallest element m. Let $b_1 = a_m$ and $Q_2 = Q_1 - \{m\}$. Since Q_1 is infinite, Q_2 is also infinite.

Now assume that we have defined elements $b_1, \ldots b_n \in B$ and sets $Q_1, Q_2, \ldots Q_{n+1}$. Let m be the smallest element of Q_{n+1} and define $b_{n+1} = a_m$ and $Q_{n+2} = Q_{n+1} - \{m\}$. Note that Q_{n+2} is infinite. It now follows by induction that we have a b_i and Q_i with the required properties for all $i \in \mathbb{N}$.

As an exercise, the reader can verify that the pairs $(1, b_1), (2, b_2) \ldots$ give a one to one correspondence between $\mathbb{N}$ and B.