- A lot of people are doing proofs by contradiction when they do not need to. For instance, suppose you have two sets A and B and you want to show $A \subset B$. The approach a lot of you took was to assume that there was some $x \in A$ with $x \notin B$. You would then prove that $x \in B$ and say that this was a contradiction. While this is not incorrect, it is poor style to give a proof by contradiction that contains a direct proof in it.

Here's a solution to # 1 on the homework.

Exercise 1. *Let C be the set of rational numbers, and give $<$ its usual meaning in this context. What is the set of limit points of the set*

$$A = \{1/m + 1/n | m, n \in \mathbb{N}\}?$$

First we will prove a lemma

Lemma 1. *If a is a positive rational number, then $0 < 1/n < a$ for some $n \in \mathbb{N}$.*

Proof. We can write $a = b/c$ for some $b, c \in \mathbb{N}$. Observe that

$$0 < \frac{1}{c+1} < 1/c \leq b/c = a.$$

So if we let $n = c + 1 \in \mathbb{N}$, we have $0 < 1/n < a$. □

Now we will prove 0 is a limit point of A. Let $R = ab$ be any region containing 0. Since $0 \in R$, we have $b < 0 < a$. By the lemma, for some $n \in \mathbb{N}$, we have $0 < 1/n < a$. It follows that $\frac{1}{2n} + \frac{1}{2n} \in R \cap (A - \{0\})$. So any region containing 0 has non-empty intersection with $A - \{0\}$. We conclude 0 is a limit point of A.

Next we will prove $1/m$ is a limit point of A for all $m \in \mathbb{N}$. Let $R = ab$ be any region containing $1/m$. Since $1/m \in R$, we have $a < 1/m < b$. It follows we can write $b = 1/m + c$ for some positive rational number c ($c = b - 1/m$). By the lemma, for some $n \in \mathbb{N}$, we have $0 < 1/n < c$. It follows that $\frac{1}{n} + \frac{1}{m} \in R \cap (A - \{1/m\})$. So any region containing $1/m$ has non-empty intersection with $A - \{1/m\}$. We conclude $1/m$ is a limit point of A.

So far, we have shown that every member of the set $B = \{0\} \cup \{1/n | n \in \mathbb{N}\}$ is a limit point of A. We will now show that no other points are limit points. First note that if $x \in A$, then $0 < x \leq 2$. Let q be a negative rational number. Both $q/2$ and $3q/2$ will be negative rational numbers and $3q/2 < q < q/2$. So if $a = 3q/2$ and $b = q/2$, then $q \in ab$ and $ab \cap (A - \{q\}) = \emptyset$ (since no member of A is less than 0). It follows that no rational number less than 0 a limit point of A.

Now let q be a rational number greater than 2. By the lemma, for some $n \in N$ we have $0 < 1/n < q - 2$. So if $a = 2 + 1/n$ and $b = q + 1/n$, then $q \in ab$ and $ab \cap (A - \{q\}) = \emptyset$ (since no member of A is greater than 2). It follows that no rational number greater than 2 is a limit point of A.

Let $a = 1.9$ and $b = 2.1$. Observe that $2 \in ab$ and $ab \cap A - \{2\} = \emptyset$. It follows that 2 is not a limit point.

Lemma 2. *Let $a = 1 + 1/n$ for any $n \in \mathbb{N}$ and $b = 3$. The set $A \cap ab$ is finite.*

Proof. If $1/k + 1/m \in A \cap ab$, then one of k and m is 1 and the other is less than n. There are only finitely many pairs of elements $k, m \in \mathbb{N}$ satisfying these conditions. □

Now let q be a rational number greater than 1 and less than 2. By lemma 1, there is some $1 \neq n \in \mathbb{N}$ with $q > 1 + 1/n$. Let $a = 1 + 1/n$ and $b = 3$. By lemma 2, $(ab \cap A) \cup \{q\}$ is a finite set. Theorem 2 of our class notes allows us label these points $a_1, \ldots a_s$ so that $a_1 < a_2 < \ldots a_s$. Observe that $q = a_i$ for some $i \neq 1, n$. Now $q \in a_{i-1}a_{i+1}$ and $a_{i-1}a_{i+1} \cap (A - \{q\}) = \emptyset$. It follows that q is not a limit point of A.

An exercise for the reader is to construct a similar proof that no rational number q between 0 and 1 not of the form $1/n$ for some $n \in \mathbb{N}$ is a limit point of A. The following lemma might prove useful.

Lemma 3. *Let $n, m \in \mathbb{N}$ satisfy $\frac{1}{n+1} < \frac{1}{n+1} + \frac{1}{m} < \frac{1}{n}$. If $a = \frac{1}{n+1} + \frac{1}{m}$ and $b = 1/n$, then $ab \cap A$ is a finite set.*