1. (a) First we give a proof by induction. If $k = 1$, then

$$\frac{3^k - 1}{2} = 1 = 1 + \dots 3^{k-1}.$$

So the statement holds for $k = 1$. Now assume the statement is true for some natural number $k \geq 1$. It follows that

$$1 + 3 + \dots 3^{k-1} + 3^k = \frac{3^k - 1}{2} + 3^k.$$

Some easy algebra shows us that

$$\begin{aligned} \frac{3^k - 1}{2} + 3^k &= \frac{3^k - 1}{2} + \frac{2 \cdot 3^k}{2} \\ &= \frac{3 \cdot 3^k - 1}{2} \\ &= \frac{3^{k+1} - 1}{2} \end{aligned}$$

Putting this together, we have

$$1 + 3 + +3^k = \frac{3^{k+1} - 1}{2}.$$

So by assuming the statement true for k, we were able to show it is true for $k + 1$. Since we also showed the statement is true for $k = 1$, induction tells us the statement is true for all $k \in \mathbb{N}$.

(b) Now we will give a proof without using induction. Some easy algebra shows us that

$$\begin{aligned} (1 + \dots 3^{k-1}) &= \frac{(3-1)(1 + \dots + 3^{k-1})}{2} \\ &= \frac{(3 + \dots + 3^k) + (-1 - \dots - 3^{k-1})}{2} \\ &= \frac{3^k - 1}{2} \end{aligned}$$

2. We will give a proof by induction. For $n = 22$, if we let $k = 0$ and $l = 2$, then

$$3k + 11l = 3 \cdot 0 + 11 \cdot 2 = 22.$$

So the statement holds for $n = 22$. Now assume the statement is true for some natural number $n \geq 22$. By assumption, there exist k_n and l_n so that

$$n = 3k_n + 11k_n.$$

We will consider two cases. If $l_n > 0$, let $k_{n+1} = k_n + 4$ and $l_{n+1} = l_n - 1$. Observe that k_{n+1} and l_{n+1} are both whole numbers and

$$3k_{n+1} + 11l_{n+1} = 3(k_n + 4) + 11(l_n - 1) = 3k_n + 11l_n + 12 - 11 = n + 1.$$

Therefore, the statement is true for $n + 1$.

If l_n is not greater than 0, since it is a whole number, it must be zero. It follows that $n = 3k_n$. Since $n \geq 22$, we must have that $k_n \geq 8$. In particular, $k_n - 7$ is a whole number. If we set $k_{n+1} = k_n - 7$ and $l_{n+1} = l_n + 2$, then k_{n+1} and l_{n+1} are whole numbers and

$$3k_{n+1} + 11l_{n+1} = 3(k_n - 7) + 11(l_n + 2) = 3k_n + 11l_n + 22 - 21 = n + 1.$$

Therefore, the statement is true for $n + 1$.

So assuming the statement is true for n, we were able to show it is true for $n + 1$. Since we also showed the statement is true for $n = 22$, induction tells us the statement is true for all natural numbers $n \geq 22$.

3. In exercise 4, we will show (without using the result of exercise 3!) that if $p = 2k$ is a natural number and k is odd (in the language of modular arithmetic this can be written $p \equiv 2 \mod 4$), then there are no $m, n \in \mathbb{N}$ satisfying $m^2 - n^2 = p$. Since $14 = 2 \cdot 7$ and 7 is odd, exercise 3 follows from exercise 4.

4. We will show that if $m, n \in \mathbb{N}$, then $m^2 - n^2$ is either an odd number or multiple of 4. It will immediately follow that if p is an even number that is not a multiple of 4 (that is, $p = 2k$ where k is an odd number), that there are no $m, n \in \mathbb{N}$ with $p = m^2 - n^2$. Since there are infinitely many p of this form, we will be done.

   We will break this into two cases. If both m and n are odd or both m and n are even, then $m + n$ and $m - n$ are both even. So we can write $m + n = 2k_1$ and $m - n = 2k_2$ for some $k_1, k_2 \in \mathbb{Z}$. As a result,

$$m^2 - n^2 = (m + n)(m - n) = 4k_1k_2$$

   and $m^2 - n^2$ is a multiple 4.

   If one of m, n is even and the other is odd, then $m + n$ and $m - n$ are both odd. So we can write $m + n = 2k_1 + 1$ and $m - n = 2k_2 + 1$ for some $k_1, k_2 \in \mathbb{Z}$. As a result,

$$m^2 - n^2 = (m + n)(m - n) = 4k_1k_2 + 2k_1 + 2k_2 + 1$$

and $m^2 - n^2$ is an odd number.

Since any pair of natural numbers is covered by one of these cases, we have shown that if $m, n \in \mathbb{N}$, then $m^2 - n^2$ is either a multiple of 4 or an odd number.