Teaching Portfolio

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Statement of Teaching Philosophy

“Mathematics is the art of explanation. If you deny students the opportunity to engage in this activity – to pose their own problems, make their own conjectures and discoveries, to be wrong, to be creatively frustrated, to have an inspiration, and to cobble together their own explanations and proofs – you deny them mathematics itself.” – Paul Lockhart

My goal as a teacher is to give my students tools to express themselves through the creative medium of mathematics, despite the fact that it often has a reputation as a sterile activity that solely involves memorizing formulas and answering repetitive questions. I want students of all backgrounds to understand and engage with mathematics using their own perspective and to develop their own mathematical voice, that is, their awareness of and ability to articulate their own points of view and problem-solving tendencies.

When I teach calculus to first-year university students, one place where I encourage this development is on the daily homework assignments, where I assign open-ended questions that let students create and refine their own intuitions about the subject. My students can answer these questions using many different kinds of reasoning, and their answers often reflect this. To both validate the various ways my students have thought about these problems and to encourage them to explore new ways of thinking about mathematics, I incorporate various points of view in my lectures. For example, when I define the derivative, I spend equal amounts of time focusing on the geometry of tangent lines, the algebra of limits, and the computation of difference quotients, emphasizing how each reinforces the others. This gives students with different strengths equal footing in the classroom.

Initially, many students are not accustomed to being asked these open-ended questions and are not confident in their solutions. This is especially prevalent among students from underrepresented and minoritized groups, such as women and people of color, who are more likely to suffer from mathematical anxiety and undervalue their own ideas (Hembree 1990). In the classroom I ensure that all students have the opportunity to contribute their ideas through frequent small group activities, and I credit students by name for their contributions. By providing frequent positive feedback and emphasizing a growth mindset, my students build confidence and over time they become more comfortable using their own voice.

It is also important to me that my students ask (and answer!) their own mathematical questions. In one term, I gave my students a project to find their own family of plane curves and then write qualitatively about why they found their graphs interesting. I then let them make their own quantitative inquiries about their curves, such as finding various slopes, areas, and distances. Each student has their own sense of what makes a graph or a question interesting, and I knew the project succeeded when I saw my students asking and answering novel questions that we had not discussed in class. One particular example of this was a student who wrote about the number of connected components in her graph.

Along with these kinds of projects and the daily homework assignments, I use written exams to track the progress of my students. I design exams with three goals in mind. First, I use them as records of a student’s ability to perform the kinds of standard calculations that they will need in future STEM courses. Second, I provide ample feedback on student responses so that my exams are formative as well as summative. Finally, my exams serve as a portfolio for my students’ original mathematical thoughts via extended versions of the
open-ended questions from the homework. For example, on one exam I gave my students graphs of discontinuous functions and I asked them to rank the graphs from “most” to “least” continuous. Each student provided different reasons for why they thought a graph with a jump was more or less continuous than a graph with a hole, and they were graded on their ability to make a mathematical argument using vocabulary from class.

It is also important to me that my students encounter mathematical perspectives other than their own. Rather than act as the sole expert in the room, I encourage my students to collaborate and learn as much from each other as they do from me. One way I encourage this is via the “2-by-2 method”, which I learned from MurphyKate Montee. After I finish a section of my lecture, I pair the students up and have one student, the “summarizer”, explain the topic to the other, the “skeptic”, who asks questions. Then, the pairs pair up and discuss the remaining questions they have, and then these questions are presented to the whole class. By rotating students through the various roles, I ensure that each student has the opportunity to share their own thoughts and hear the thoughts of others, especially those different from themselves. This also allows students to ask questions in a low-stakes environment, and the questions that persist to the end of the process help me evaluate which topics I should spend more time discussing in the remainder of the lecture.

In addition to teaching calculus to college students, I also teach summer courses to high-school sophomores and juniors, many of whom are students of color. These students are more likely to be affected by stereotype threat (Steele & Aronson 1995), so it is especially important to me that I use evidence-based practices when I assess them, for example, by providing so-called “wise feedback” on homework and exams. One course was on cryptography. I broke each two-and-a-half hour session into two or three 20-minute lectures separated by 5-to-10 minute blocks where students discussed puzzles from the lectures, and each session ended with a group project. In one project, I asked students to encrypt short messages of their own choosing. Once students could put themselves into the project, they became invested in others’ messages, and in response I divided the class into teams competing to decode “friendly” messages and crack “enemy” messages. This activity was also enriching to me; I got to know my students better by letting them insert themselves into the project.

I have also mentored over a dozen advanced undergraduates in research and directed reading projects. These more-experienced students often have a more developed mathematical voice. With that in mind, I focus more on teaching them how to ask the right questions. I often hear two kinds of questions: procedural questions (“Can you explain this step?”) and exploration questions (“What about the general case?”), and I dedicate half of our meeting time to each kind. While being able to identify and articulate confusion is important, asking the second kind of question is the key to developing original mathematics. At the beginning of the project, I often have to lead them to these questions, but once they are able to ask these questions on their own, I feel like I have helped prepare them for future graduate study.

No matter the academic level of my students, my core goal is the same: By celebrating diverse voices in the classroom and providing thought-provoking projects and exams, I create an environment in which my students can cultivate personal, meaningful relationships with mathematics. My interactions with students and teachers of all backgrounds have helped me develop my own approach to teaching, and I continue to seek out these experiences so that I can continue to grow as a teacher.
Statement on Equity and Inclusion in Teaching

I recognize the structural and social barriers that women and minoritized groups face in the mathematics classroom. As a teacher of mathematics, my goal is to create an environment where all students can access and interact with mathematics on equitable ground. I believe the best way for me to achieve this is by using evidence-based inclusive pedagogical practices and soliciting generous feedback from students, peers, and professionals on my teaching. As a Teaching Fellow at the Chicago Center for Teaching, I have engaged with literature on inclusive pedagogy and I have in turn facilitated workshops on inclusive pedagogy for other graduate student instructors.

Creating an inclusive classroom begins on the first day of class. I set aside 20 minutes for the class to come up with a set of “discussion guidelines” that we will abide by in the classroom during the small and large group discussions that we would have over the quarter. I ask my students to individually answer the question “What does a good discussion sound like, look like, and feel like?” and complete the sentence “Small group discussions work best when...” After they finish, I form the students into groups of four, have them share their answers, and then ask them to synthesize those answers into guidelines that I write on the board and then include in the syllabus. Some sample guidelines from past courses include “Listen carefully and give attention to the speaker”, “Share responsibility for the group’s understanding”, and “Don’t pounce on mistakes”. This activity shows my students that creating an inclusive classroom is a priority of mine, that I value their voice, and that I am not the sole owner of the space. I have noticed since I began this activity that it increases student participation throughout the quarter.

It is important to me to cultivate my students’ confidence in their mathematical abilities, and I don’t want my students to see me as the sole expert in the room. I explicitly ask my students to use phrases like “The way I see it...” instead of “I am probably wrong, but...”, and when one student asks a thoughtful question, I open up the discussion to the class instead of providing an answer. This has the added benefit of giving me feedback on how my students are thinking. Furthermore, I actively ask my students to catch and correct my mistakes. Each time they correct a mistake that I make, I lose a “point”, and once I lose 10 points, those 10 points turn into bonus points on a quiz. This encourages all students, including the strategic learners, to be actively participating.

In a standard class period, I intersperse lectures with a variety of individual and group work strategies to give students an opportunity to reflect on their own thoughts (via minute papers and “think with your pencil” problems in class) and then get peer feedback on them (via think-pair-share and small group brainstorming). Giving students frequent low-stakes opportunities to have their ideas evaluated by both me and their peers is important for their learning and doesn’t increase student anxiety. It lets me emphasize that their ideas and understanding are more important than a grade and also communicate to them exactly what my expectations are. For my students, these experiences let them identify potential mistakes they are making quickly and without the anxiety of “their grade” hanging over their head.

This is one way in which I design assessments of student learning using inclusive practices.

In the Spring of 2019 I participated in a workshop at the Chicago Center for Teaching on inclusive assessments in STEM. During this workshop, I discussed various assessment practices with other instructors at the University of Chicago and we reflected on various studies about these practices can affect student learning and help lessen the achievement gap between minoritized and majoritized groups in STEM classrooms. We developed a list of guidelines that will be published on the Chicago Center for Teaching Website. Studies have shown that by increasing structure in classes by adding at least one graded assignment per week halves the gap in test scores between Black and White students and eliminates the gap between first-generation and continuing-generation students\(^2\). By maintaining this structure in my own courses I help provide students of all backgrounds a more equitable access to learning.

When I give feedback on these assessments, I keep in mind the benefits of so-called “wise feedback”, where I describe the nature of my feedback, reaffirm my high standards, and provide an assurance of student ability. Studies have shown that providing wise feedback can help to lessen some of the negative effects of stereotype threat\(^3\), and my students are more likely to ask me follow-up questions about their quizzes when I spend more time giving wise feedback.

When designing exams, I take care to not include questions that could implicitly favor one group of students over another. For example, research has indicated that giving unscaffolded questions on exams decreases student performance, but women are more affected than men\(^4\). With this in mind, I write exams that avoid long, unscaffolded questions, and I keep sentences short, use simple grammar, and avoid cultural references to not impact students for whom English is not their first language. I also keep in mind the length of the exam, so that I don’t accidentally assess my students’ speed as opposed to their content knowledge and analysis ability.

Inclusive and equitable teaching benefits all students, and to achieve this I intentionally and actively seek out pedagogical research on best practices in the classroom. Creating a classroom where all students have equal access to mathematics requires constant effort, and I am continually striving to achieve that goal.


Teaching Experience

Courses are sorted by role and listed in reverse chronological order. Unless otherwise indicated, all courses listed were taught in the Department of Mathematics at the University of Chicago and lasted 10 weeks.

Teaching Awards

Graves Prize
In the Spring of 2018, I was awarded with the Lawrence and Josephine Graves Prize for excellence in undergraduate teaching. This prize is awarded by the Department of Mathematics at the University of Chicago to instructors in their second year of teaching. The prize is given based on student evaluations and faculty observations.

As Instructor of Record

As a stand-alone instructor, I created all of the course material including lectures, homework assignments, quizzes, exams, and projects. I also managed a team of course assistants for each course, coordinated tutorial sessions, held office hours, and assigned final grades.

I was presented with a teaching award in the Spring of 2018 for my performance in this role; see the Awards section above.

Both the “Elementary Functions and Calculus” sequence and the “Calculus” sequence are year-long calculus sequences for first-year students at the University of Chicago. The former is aimed at students in the humanities and social sciences and is taught out of the textbook Calculus, 9th edition by Varberg, Purcell, and Rigdon. The latter is aimed at students in the sciences and is taught out of Calculus: One and Several Variables, 10th edition by Salas, Hille, and Etgen.

Winter 2020 (Upcoming)
Math 15300: Calculus III
Expected Enrollment: 27

Autumn 2019 (Ongoing)
Math 15200: Calculus II
Total Enrollment: 27

Winter 2019
Math 13200: Elementary Functions and Calculus II
Total Enrollment: 27

Autumn 2018
Math 13100: Elementary Functions and Calculus I
Total Enrollment: 27

Spring 2018
Math 13300: Elementary Functions and Calculus III
Total Enrollment: 24
Winter 2018
Math 13200: Elementary Functions and Calculus II
Total Enrollment: 33

Autumn 2017
Math 13100: Elementary Functions and Calculus I
Total Enrollment: 34

Spring 2017
Math 13300: Elementary Functions and Calculus III
Total Enrollment: 17

Winter 2017
Math 13200: Elementary Functions and Calculus II
Total Enrollment: 28

Winter 2016
Math 13100: Elementary Functions and Calculus I
Total Enrollment: 26

In Inquiry-Based Learning Classrooms

Inquiry-Based Learning (also known as “IBL” or “the Moore method”) is a method of flipped-classroom teaching where the students are provided with “scripts” that contain theorems for the students to prove on their own and present in class. See the Sample Teaching Materials section for a sample script. The Math 160s sequence of courses is an introduction to analysis designed for advanced first-year undergraduates intending to pursue a math major.

Both as a co-instructor and as a teaching assistant, my partner and I facilitate in-class discussion, help guide the conversation, and provide feedback on student presentations and proofs. I also held weekly discussion session and office hours and graded student work. As a co-instructor, I was additionally responsible for conducting student oral examinations and assigning final grades.

Spring 2019
Math 16310: IBL Honors Calculus III
Role: Co-instructor (with John Boller)
Total Enrollment: 20

Winter 2018
Math 16210: IBL Honors Calculus II
Role: Teaching Assistant
Instructor: John Boller
Total Enrollment: 20

As a Teaching Assistant

In this role, my responsibilities included attending course meetings, leading discussion sessions, holding office hours, and grading student assignments and exams. My work for the
Honors Algebra sequence was done as part of the College Fellow program (see my Pedagogical Training and Development).

**Autumn 2017**
Math 17500: Elementary Number Theory
Instructor: George Boxer
Total Enrollment: 7

**Spring 2016**
Math 25900: Honors Algebra III
Instructor: Matthew Emerton
Total Enrollment: 30

**Winter 2016**
Math 25800: Honors Algebra II
Instructor: Frank Calegari
Total Enrollment: 37

**Autumn 2015**
Math 25700: Honors Algebra I
Instructor: Frank Calegari
Total Enrollment: 38

**With High-School Students**

The University of Chicago Collegiate Scholars Program is a three-year program for high school students in the Chicago Public Schools system. The program is aimed at students from traditionally underrepresented groups and underprivileged neighborhoods and prepares them for success at 4-year colleges and universities around the country.

Each year, I was tasked to create a curriculum for a course that would meet for 30 hours over the course of 6 weeks. My goal was to introduce students to a subject in mathematics that they would not otherwise learn in schools, with a focus on projects and interactivity.

**Summer 2019**
Let’s Play a Game!
Total Enrollment: 20
This course is an introduction to game theory. Students played games such as tic-tac-toe, Hex, and Nim, and discussed the math behind various strategies. Students used case analysis and symmetry to determine winning strategies for various games, and learned the mathematics behind Nim Addition. Students were exposed to the prisoner’s dilemma and related social experiments. Students designed new mathematical games as a final project.

**Summer 2018**
Introduction to Cryptography
Total Enrollment: 18
When I designed this course, my goals were for students to understand the risks of data transmission and learn about various ways our data can be protected. After building up a foundation in modular arithmetic, students learned about how modular arithmetic is used to
transmit secure data via processes such as Diffie-Hellman key exchange and RSA encryption. Projects included creating divisibility graphs, encrypting and sending messages to each other, and exploring how various forms of noise can disrupt data transmission.

**Summer 2017**
A Problem-Solving Approach to Graph Theory
Total Enrollment: 20
In this course, each week was dedicated to a different topic in graph theory, including Eulerian paths, chromatic number, and spanning trees. After a short lecture on the topic, students worked in groups to find examples of these topics in various graphs provided to them. Each topic concluded in a small group project where students came up with a real-world problem that they could solve using that topic.

**Additional Mentoring Experience**
Mentoring undergraduate students outside of the classroom is an important part of my identity as a teacher. When I mentor students doing directed reading projects or more open-ended projects, I focus on teaching them how to ask effective questions.

**University of Chicago Summer REU Program**
During the 8-week summer REU program, I work more closely with students from both the University of Chicago and other institutions. I meet them two or three times a week as I supervise their projects that culminate in an expository paper. The students I have worked with, along with their projects, are listed below. Projects listed in reverse chronological order. Younger students participated in a shortened 5-week “apprentice program”; those students are designated below. Each project culminated in an expository paper. All student papers can be found at [http://math.uchicago.edu/~may/REU[YEAR]/](http://math.uchicago.edu/~may/REU[YEAR]/) where [YEAR] is the 4-digit year in which the project took place. Students who have since entered graduate school are listed along with their current graduate institution.

**Summer 2019**
Student: Spencer Dembner
Topic: Torsion on Elliptic Curves and Mazur’s Theorem

Student: Oscar Michel (Apprentice)
Topic: An Introduction to the Zariski Topology

Student: Xingyu Wang
Topic: $p$-adic numbers, the Hasse-Minkowski Theorem, and its Applications

**Summer 2018**
Student: Yuchen Chen (Apprentice)
Topic: $p$-adics, Hensel’s Lemma, and Strassman’s Theorem.

Student: Cecelia Higgins (UCLA)
Topic: Ultrafilters in Logic and Set Theory

Student: Amin Idelahj (U Wisconsin at Madison)
Topic: Elliptic Curves and Dreams of Youth
Student: Mark Schachner (Apprentice)

Topic: Algebraic and Analytic Properties of Arithmetic Functions
Student: Jae Hyung Sim (Boston University)

Topic: The $p$-adic Numbers, their Extensions, and a Proof of the Kronecker-Weber Theorem
Student: Aleksander Skenderi (Apprentice)

Topic: Quadratic Forms, Reciprocity Laws, and Primes of the form $x^2 + ny^2$

Summer 2017
Student: Esme Bajo (UC Berkeley)

Topic: Amalgamated Free Products, HNN Extensions, and Decision Problems
Student: Matthew Scalamandre (Notre Dame)

Topic: Harmonic Analysis on LCA Groups

Summer 2016
Student: Karen Butt (U Michigan)

Topic: Elliptic Curves and the Mordell-Weil Theorem
Student: Sameer Kailasa (U Michigan)

Topic: On the Tate-Shafarevich Group of a Number Field
Student: Arieh Zimmerman

Topic: The Geometry of Elliptic Curves over Finite Fields

Directed Reading Program
In the Directed Reading Program at the University of Chicago, I meet with students weekly as they work through a certain book or topic. Projects are listed in reverse chronological order.

Spring 2019
Student: Jae Hyung Sim

Topic: Complex Multiplication

Winter 2019
Students: Jae Hyung Sim and David Lin

Topic: Global Class Field Theory

Autumn 2018
Student: David Lin

Topic: The Local Kronecker-Weber Theorem

Spring 2018
Student: Will Porteous

Topic: Harmonic Analysis on LCA Groups

Winter 2018
Student: David Lin

Topic: Quaternion Algebras and Local-Global Principles
Autumn 2017
Student: David Lin
Topic: Elliptic Curves and Number Theory

Spring 2017
Student: Karen Butt
Topic: Algebraic Number Theory

Winter 2017
Student: Adele Padgett
Topic: Galois Theory

Autumn 2016
Student: Karen Butt
Topic: Topics on Elliptic Curves
Sample Syllabus

Below is the syllabus that I gave to my students in Math 15200. It details some of the course policies, including the grade breakdown for the course. On the first day of class, I had a discussion with my students on why I thought an inclusive classroom environment was important and as a class we came up with a list of community standards for productive and inclusive group work and discussions, which I later appended to this syllabus and distributed through the course website. See my Sample Inclusive Classroom Standards below.

Math 15200, Section 47
Autumn 2019

Instructor: Karl Schaefer (he/him)
Office: Math-Stat 011
Email: karl@math.uchicago.edu
Office Hours: Mondays 4-5pm, Thursdays 5-6pm, and by appointment
VCA: Palash Goiporia (he/him)
Office Hours: Tuesdays 4-5pm in Regenstein A-levels
Email: palashg@uchicago.edu
Course Time and Place: MWF, 11:30am-12:20pm, Stuart Hall 104
Problem Session: Mondays 7-8pm in Eckhart 207
Textbook: Calculus: One and Several Variables (10th Edition) by Salas, Hille, and Etgen
Content Overview: Broadly, the course will cover Chapters 5-8 of the textbook. The rough outline is as follows:
- Review of Derivatives and Applications (Chapters 2-4): 1 week
- Integration (Chapter 5): 3 weeks
- Applications of the Integral (Sections 6.1-6.3): 1 week
- Transcendental Functions (Sections 7.1-7.7): 3 weeks
- Techniques of Integration (Sections 8.2-8.4): 1.5 weeks

Homework: Homework will be assigned after class every day on the course Canvas page. Homework assigned on Mondays will be due on that same Friday. Homework assigned Wednesdays and Fridays will be due the following Wednesday. All work is due at the beginning of class.
Your homework should be legible and written in complete sentences. You should progress sequentially, justify each step, use notation correctly, and clearly mark the final answer. What you hand in should be a final version, not a first draft. Late homework will not be accepted. If you need an extension, please contact the instructor as soon as possible. Homework will be graded for accuracy.
Quizzes: A short quiz will be administered during class on Fridays with the exception of exam days. Quizzes will usually contain one or two problems that are very similar to those on the current week’s homework assignment.
**Exams:** This course will have two 50-minute in-class midterm exams and one 2-hour final exam. All exams will be administered in the normal course room. Exam corrections will be offered. The exam dates are:

- Midterm 1: Friday, October 25 (Week 4)
- Midterm 2: Friday, November 22 (Week 8)
- Final Exam: Monday, December 9, 10:30am-12:30pm

It is the policy of the Department of Mathematics that the following rules apply to final exams in all undergraduate mathematics courses:

1. The final exam must occur at the time and place designated on the College Final Exam Schedule. In particular, no final examinations may be given during the tenth week of the quarter, except in the case of graduating seniors.

2. Instructors are not permitted to excuse students from the scheduled time of the final exam except in the cases of an Incomplete, or a graduating senior.

**Grading Policy:** Your final grade will be based on your grades on the homework, quizzes, and exams with the following weights:

- 10%: Homework
- 10%: Quizzes
- 5%: Participation
- 20%: Midterm 1
- 20%: Midterm 2
- 35%: Final Exam

The VCA will grade the homework assignments. The instructor will grade the quizzes and exams. The instructor is happy to answer questions about items on homework, quizzes, or exams. If your question involves a possible grading mistake, you must request regrading within one week after the date the work was returned to you.

**Participation:** Participating in in-class discussions and learning activities is beneficial not only to yourself but to your classmates. To earn participation points, you need to actively participate in discussions with your classmates and observe our classroom guidelines. I expect that everyone will earn full participation points, and I will notify you by email if you are not on track to meet that.

**Cheating:** Working together on the exercises is an excellent way to enhance your learning and understanding. Nevertheless, all written work you hand in must be strictly your own. The College takes academic integrity very seriously. Any work that you submit that is copied from another student or source will be assigned a grade of 0 with no exceptions. You will also be referred to the Dean of the College for disciplinary action.

**Accommodations:** If you require any accommodations for this course, as soon as possible please provide the instructor with a copy of your Accommodation Determination Letter (provided to you by the Student Disability Services office) so that you may discuss with him how your accommodations may be implemented in this course. The University of Chicago is committed to ensuring the full participation of all students in its programs. If you have a documented disability (or think you may have a disability) and, as
a result, need a reasonable accommodation to participate in class, complete course require-
ments, or benefit from the University’s programs or services, you are encouraged to contact
Student Disability Services as soon as possible. To receive reasonable accommodation, you
must be appropriately registered with Student Disability Services. Please contact the office
at 773-702-6000 or disabilities@uchicago.edu, or visit the website at disabilities.uchicago.edu.
Student Disability Services is located at 5501 S. Ellis Avenue.
Sample Inclusive Classroom Standards

On the first day of class, I want to make it clear to my students that I am committed to maintaining an inclusive classroom environment and that they had ownership over that space as well. I set apart 20 minutes for the class to come up with “discussion guidelines” that we would abide by during the small and large group discussions that we would have over the quarter. I asked my students to individually answer the question “What does a good discussion sound like, look like, and feel like?” and complete the sentence “Small group discussions work best when...” Then I formed the students into groups of four, had them share their answers, and synthesize those answers into guidelines that I wrote on the board and then included in the syllabus. The document and standards that my students in Math 15200 from Autumn 2019 set is included below.

Our Classroom Standards
Math 15200, Section 47
Autumn 2019

The University of Chicago is committed to diversity and rigorous inquiry that arises from multiple perspectives. I concur with this commitment and also believe that we have the highest quality interactions and can creatively solve more problems when we recognize and share our diversity. I thus expect to maintain a productive learning environment based on open communication, mutual respect, and non-discrimination. I view the diversity that students bring to this class as a resource, strength and benefit. It is my intent to present materials and activities that are respectful of diversity: gender, sexuality, disability, generational status, socioeconomic status, ethnicity, race, religious background, and immigration status. Any suggestions for promoting a positive and open environment will be appreciated and given serious consideration. Please let me know your preferred name and gender pronouns. If there are circumstances that make our learning environment and activities difficult, please let me know. I promise to maintain the confidentiality of these discussions.

Below are the guidelines for classroom participation and discussion that we generated on the first day of class.

• Listen carefully, give attention to the speaker, and don’t interrupt.
• Make purposeful contributions.
• Accept all people and ideas, and welcome them to the conversation.
• Step up, step back.
• Be respectful.
• Share responsibility for the group’s understanding.
• Be aware of your tone when you speak.
• Criticize ideas, not people.
• Understand that mistakes are inevitable, and respond courteously. Don’t pounce on mistakes.
Sample Teaching Materials

I am including four samples of teaching materials: a collection of visualizations I used in calculus classes from 2016-2019, a project that I assigned in Autumn 2016, a sample homework assignment with solutions from a Math 132 course, and a script from the IBL class I taught in Spring 2019. Each sample is preceded by a short discussion of the sample and its relation to and role in my teaching.

Desmos Graphs

When I teach Calculus, I like to include lots of visualizations during my lectures. Sometimes I can draw these on the board, but for more complicated pictures and diagrams I like to use the Desmos online graphing calculator, available at https://www.desmos.com/. This helps me to reach students who have trouble with just equations alone, and often it can solidify the bridge between concept and method. Below is a small sample of Desmos graphs that I have used to good effect in my Calculus courses. They are ordered chronologically by when they would appear in class. The particular feature of Desmos that I like the most is the ability to include sliders in graphs. By moving the slider, you can see how the graph changes in real time. I have included some still images of these graphs below.

Tangent Lines

When we first start defining the derivative as the slope of the tangent line, I use the following graph: https://www.desmos.com/calculator/llhwcqas. You can make the function $f(x)$ whatever you want (feel free to change it!) and it graphs the function and the tangent line at the point $x = a$. By moving the slider for $a$ around, you can see the tangent line move along the curve. As a bonus, you can really “see” the inflection point on the graph, which becomes a great introduction to the second derivative.

![Figure 1: The tangent line to the graph of a function at two points.](image)

Riemann Sums

When defining the Riemann integral as a limit of Riemann sums, I find it very instructive to actually see how increasing the partition size gives a better approximation of the area...
under the curve. In the graph at [https://www.desmos.com/calculator/tkrl4anarn](https://www.desmos.com/calculator/tkrl4anarn), you can input any function \( f(x) \) and there are sliders for \( a, b, \) and \( k \). The graph shows the Riemann sum for the graph of \( f(x) \) over the interval \([a, b]\) with \( k \) equally-sized rectangles and sampling at the right endpoints. You can see how the rectangles become a very good approximation for the area under the curve as \( k \) grows very large.

![Figure 2: Two Riemann sums for \( y = \sin(x) \) with 4 and 15 equally-spaced intervals.](image)

**First Fundamental Theorem of Calculus**
In this example, I am trying to demonstrate the textbook’s definition of the “accumulation function”: Given a function \( f(x) \), its accumulation function is defined by \( A(x) = \int_a^x f(t) \, dt \).

In the graph at [https://www.desmos.com/calculator/p3bf3okbms](https://www.desmos.com/calculator/p3bf3okbms), the parameters \( a \) and \( b \) change the region being accumulated over. The black line is the graph of the accumulation function over the interval \([a, b]\). As a fun “pre-1st-Fundamental-Theorem” exercise for my students, I like to have them choose \( f(x) = \sin(x) \) and \( a = 0 \). As you increase \( b \), the accumulation function increases until you hit \( b = \pi \). Then it starts decreasing. Why? At \( b = 2\pi \), the accumulation function is 0; why?

![Figure 3: Two values of the accumulation function of \( f(x) = \sin(x) \).](image)

**Taylor Series**
Near the end of the single-variable calculus unit, we cover infinite series. I have found that the idea of representing functions as “polynomials of infinite degree” is a tricky idea
for students at first. To help show how this makes sense, I give them this graph: [https://www.desmos.com/calculator/qs85kaypqq](https://www.desmos.com/calculator/qs85kaypqq). It has the graph of the function $f(x) = \sin(x)$ in green, and on the right is a slider for $k$, which represents the order of the Taylor expansion of $\sin(x)$ at 0. As you increase $k$, the number of terms in the Taylor series increases, and you can see it start to hug the graph of $\sin(x)$.

![Image of Taylor approximations](https://www.desmos.com/calculator/qs85kaypqq)

Figure 4: The 3rd and 11th order Taylor approximations to $f(x) = \sin(x)$. 
Curves Project

This was one of my favorite projects from any calculus class that I taught, and I spoke a little bit about it in my Statement of Teaching Philosophy. In class, we spent a lot of time on implicit differentiation, and I brought in a computer and played with the graphs of several families of implicit curves on the Desmos online graphing calculator; see above or at https://www.desmos.com/calculator/i3pvs1dah9 for an example.

I got feedback from my students that they really enjoyed playing with graphs of implicit curves. In response, I added a project to the course that would allow students to approach some difficult mathematics about a topic of interest on their own terms and give them an opportunity to express themselves through the mathematics.

The full text of the project begins below. In the first part of the project, I gave students a family of implicit curves to consider and I asked them to answer various questions that started off straightforward and became very challenging. Students were instructed to answer as much as they wanted and encouraged to write down their thoughts on each problem. I told them that I was more interested in reading what they had to say and what their attempts were on the more difficult problems than if they got them correct or not. In the second part of the project, students were asked to find their own family of implicit curves. They wrote about why their curves were interesting and did some quantitative calculations.

I was extremely pleased with the results. All but 3 of the 26 students in the course submitted something, and the vast majority of them attempted even the most challenging questions. I wasn’t disappointed when they didn’t have complete answers. Instead, I was excited to get responses along the lines of “This is the picture I have in mind, and here is what I tried to do, but I couldn’t seem to get the limit to work out because...” which is exactly the kind of thinking I was trying to encourage from the start of the quarter.

For the second part of the project, students really let their creativity shine. I received lots of very interesting graphs with spirals, interesting intersections, and multiple components. Some students even asked new questions that I hadn’t asked in class. For example, one student found a family of curves with two components and calculated the distance between the components as a function of the parameter. Overall, it was a very successful project.
MATH 13100/48, AUTUMN 2016
CURVES PROJECT

Details: This project is due at the beginning of the last day of class. You may work alone or with a partner. You can earn up to a total of 10 points of extra credit to your final grade in the course. (This is worth approximately 65 points on one of the midterms or 30 points on the final!) There are two independent parts to this project, described in more detail below.

1. Part I (7 points)

Assignment: For this part, I will give you a family of curves that depends on a parameter $t$ (so for each value of $t$ you get a different equation which gives a different curve, as in the example above) and ask questions about them. Some of the questions will be harder than others, and some will be extremely challenging. You don’t have to answer all of the questions, but you should think about all of them.

What to turn in: You should show all your work, and write out answers in complete sentences. Write out what method you’re using to solve a problem. Even if you can’t solve a problem completely, write out what you tried and where you got stuck. If you graphed something in Desmos (like tangent/normal lines, circles, etc.) then print out the graphs and label the different lines/curves.

Questions: Consider the family of curves

$$y^2 = x^3 + (t - 4)x^2 + 4x$$

where $t$ is between $-10$ and $10$. (You should graph this in Desmos to see what these look like.)

For the curve you get when $t = -1$, answer the following questions:

- What is $dy/dx$?
- When does this curve have a horizontal tangent line?
- When does this curve have a vertical tangent line?
- What is the tangent line at $(5, -\sqrt{20})$?
- What is the normal line at $(5, -\sqrt{20})$?
- Looking at the graph, you see that there is no point on the curve with $x$-coordinate equal to 2. Could you have known this just from the equation, without looking at the graph? Why or why not?

For the curve you get when $t = 0$, answer the following questions:

- What is $dy/dx$?
- When does this curve have a horizontal tangent line?
- When does this curve have a vertical tangent line?
- There are two circles of radius 1 that are tangent to this curve at $(1, 1)$. What are they?
- Draw the line between the two centers of the circles from the previous part. This line has a special name in relation to this curve. What is it?
• For any radius \( r \) there are two circles of radius \( r \) that are tangent to this curve at \((1, 1)\). What are their centers? Your answer should depend on \( r \). Graph the equations (in terms of \( r \)) for the circles in Desmos, and create a slider for \( r \).
• There are two circles of radius 1 that intersect this curve perpendicularly at \((1, 1)\). What are they?
• Draw the line between the two centers of the circles from the previous part. What is the name for the line you just drew (in relation to this curve)?
• For any radius \( r \) there are two circles of radius \( r \) that intersect this curve perpendicularly at \((1, 1)\). What are their centers in terms of \( r \)?
• The curve appears to have a “self-intersection” at the point \((2, 0)\). What happens when you plug this point into your formula for \( dy/dx \)?
• Even though the whole curve doesn’t have a well-defined tangent line at \((2, 0)\) since it crosses itself, it looks like it should have a tangent line in each direction at \((2, 0)\). What are the slopes of these two “tangent” lines?

For the curve you get when \( t = 1 \), answer the following questions:
• What is \( dy/dx \)?
• What are the tangent lines at \((2, 2)\) and \((2, -2)\)?
• There is a unique circle that is tangent to this curve at both \((2, 2)\) and \((2, -2)\). What is this circle?
• For any \( s \) in the interval \((0, \infty)\), there are exactly two points on this curve with \( x \)-coordinate equal to \( s \): \((s, \sqrt{s^3 - 3s^2 + 4s})\) and \((s, -\sqrt{s^3 - 3s^2 + 4s})\). There is a unique circle that is tangent to the curve at both of these points. What are its center \( z_1(s) \) and radius \( r_1(s) \) as functions of \( s \)? (So \( z_1(2) \) and \( r_1(2) \) should agree with the center and radius from the previous part. Graphing your answer in Desmos with an \( s \)-slider is a good way to check if you are right.) Call this circle the “tangent circle at \( s \”).
• What are \( \lim_{s \to 0} z_1(s) \) and \( \lim_{s \to 0} r_1(s) \)?
• The circle with center \( \lim_{s \to 0} z_1(s) \) and radius \( \lim_{s \to 0} r_1(s) \) will be tangent to this curve at \((0, 0)\). Two other circles tangent to this curve at \((0, 0)\) are given by \((x - 1/2)^2 + y^2 = (1/2)^2\) and \((x - 4)^2 + y^2 = 4^2\). The first of these circles is clearly “inside” this curve, and the second is clearly “outside” this curve. Is the circle you found in the previous part “inside”, “outside”, or neither?
• Could you have checked this another way (via some kind of derivative condition)? Hint: Sometimes \( dx/dy \) is more helpful than \( dy/dx \).
• There is a unique circle that intersects this curve perpendicularly at both \((2, 2)\) and \((2, -2)\). What is this circle?
• For any \( s \) as above, what are the center \( z_2(s) \) and radius \( r_2(s) \) of the unique circle that intersects this curve perpendicularly at \((s, \sqrt{s^3 - 3s^2 + 4s})\) and \((s, -\sqrt{s^3 - 3s^2 + 4s})\)? (This is the same as the question above, with “tangent” replaced by “perpendicular”.) Call this circle the “normal circle at \( s \”).
• For which \( x \)-values do the tangent circle at \( x \) and normal circle at \( x \) have the same radius?
• These circles having the same radius corresponds to some other phenomenon happening on this curve in terms of derivatives. What is it?

For the next questions, look at the whole family of curves when \( t \) is between \(-10 \) and \( 10 \):
• For which \( t \) does this graph have two disconnected parts? For which \( t \) is it connected?
• Compare this curve (for varying \( t \)) to the graph of \( y = x^3 + (t - 4)x^2 + 4x \). (Note that this is “\( y = \)” instead of “\( y^2 = \)”.) What comparisons can you make?
• Find an expression for \( dy/dx \) that works for any \( t \). Your answer should depend on \( t \) so that when you plug in a certain value of \( t \), you get the expression for \( dy/dx \) for the graph of the curve corresponding to that value of \( t \).
• When \( t = 0 \), the curve has a point with horizontal tangent line. When \( t = 1 \), the curve does not have a point with horizontal tangent line. In general, for what values of \( t \) does the corresponding curve have a horizontal tangent line?

2. Part II (3 points)

Assignment: For this part, your goal is simple: Play with Desmos, find interesting curves or families of curves (that depend on one or more parameter, like the \( t \) above), and tell me about what you found!

Specifications: Your curve should have at least one point with rational coordinates. (That is, the \( x \) and \( y \) coordinates of the point are rational numbers.) Find \( dy/dx \) for the whole curve, and give the tangent and normal lines at the point with rational coordinates.

Talk about any interesting features that your curve has. Does it have any self-intersections? How many disconnected parts does it have (or is it connected)? Are there any lines that are tangent to the curve at exactly two points? Is there anything else striking about it?

Let your curve vary in a family by adding changing one of your coefficients in your equation to a \( t \) and slide it around. How does the curve change as you change \( t \)? Does the number of components change? Do you get any self-intersections? Any sharp points? (For an example, look at the curve \( y^2 = x^3 \).) Can you say anything about how \( dy/dx \) changes as you change the curve? Do you get any other interesting behavior?

Or maybe you want to let your equation vary in two ways at once, with an \( s \) and a \( t \) that can slide independently. Can you say anything interesting there?

What to turn in: Print out from Desmos a copy of your curve and anything extra you’ve drawn (such as tangent lines, normal lines, circles, etc.). Write a couple of paragraphs about your curve and the family of curves you are looking at: Say why you think it’s neat, answer some of the questions above, and write about anything else you find interesting. On another page, be sure to show all the work for any calculations you needed to do. It should be neat, not just a page of scratch work.

How to get started: There are several strategies you could use to find an interesting curve/family of curves. One way is to just start by writing down some random equation (like \( 3y^4 + 2xy^2 - 6x^3 - xy + 2x^2 - y = 3 \)) and then playing around with the coefficients or adding/deleting terms to see what you changes. Then you can add a \( t \) and maybe an \( s \) and move the sliders around to see how things change. (For example, look at our equation from above, \( y^2 = x^3 + (t - 4)x^2 + 4x \) and add a +\( s \) onto the end, and move the \( s \) and \( t \) sliders independently.) Another way that will get interesting behavior is to start with two separate expressions (like \( x + 4 - y^2 \) and \( x^2 - y - 4 \), for example), taking their product, and setting it equal to some number, like 1, 2, -1, 0, etc., and then changing some coefficient (or the number you set it equal to) to a \( t \) and moving the slider around.

If you’re having trouble finding a point on your curve with rational coordinates, try changing the constant term (the one without an \( x \) or \( y \)) to a different value.
Calculus Homework

This assignment would be given after we defined definite integrals as areas in the plane. Because there isn’t much mechanical computation for students to be doing at this point (we don’t yet have either of the Fundamental Theorems of Calculus and we aren’t taking any antiderivatives yet), this assignment is slightly more weighted toward conceptual questions. The benefit of this is that it helps me achieve one of my learning objectives, which is for students to have a strong conceptual foundation in calculus before they begin practicing computation.

The assignment starts below. The main text is what the students would see, and the text in the boxes is my notes, including a grading rubric and sample responses/solutions for each problem, as well as a discussion of the purpose of each group of questions.
Each problem below is worth 3 points.

The problems are graded on the following scale.

3 points: Meets or exceeds expectations. Answers all parts of the question clearly and correctly. Small typos are acceptable if they don’t change the problem significantly.

2 points: Nearly meets expectations. Answers most parts of the question, or lacks clarity. Might have a small mathematical error or two but the essence of the solution is apparent.

1 point: Doesn’t meet expectations. Solution is not relevant to the question or shows a serious misunderstanding or misapplication of the mathematics from class.

0 points: Doesn’t attempt the problem or merely rewrites the statement of the problem.

For problems 1-4, sketch the graph of the given function over the interval \([-1, 1]\), and use geometric arguments to calculate the integral \(\int_{-1}^{1} f(x) \, dx\).

1. \(f(x) = 1 - x\)
2. \(f(x) = \sqrt{1 - x^2}\)
3. \(f(x) = \begin{cases} 2 & -1 \leq x \leq 0 \\ x & 0 < x \leq 1 \end{cases}\)
4. \(f(x) = \begin{cases} x & -1 \leq x \leq 0 \\ 2 & 0 < x \leq 1 \end{cases}\)

For these problems, students are expected to calculate the definite integral as an area under a curve using simple area formulas, such as the area of a triangle, a rectangle, and a semicircle. The first three functions given are positive on the interval \([-1, 1]\), so students can focus on area in the familiar way. The final function is a slight variant on the third function that has a negative component. Students should calculate the area of the positive piece and the negative piece and then take their difference.

A 3-point solution would be complete, but an arithmetic error would be acceptable as long as the student set up the problem correctly. A 2-point solution might have an incorrect graph but calculate the area of their graph correctly, or make a mistake like adding instead of subtracting the areas in problem 4. In a 1-point solution, a student might sketch the correct function but then attempt to solve the problem by taking a derivative.
5. Suppose that \( f(x) \) is an odd function and let \( a \) be any positive number. Explain why (using words and pictures) the integral from \(-a\) to \(a\) of \( f(x) \) is always 0. That is, explain why
\[
\int_{-a}^{a} f(x) \, dx = 0.
\]

6. Suppose that \( f(x) \) is an even function, and let \( a \) be any positive number. Explain why (using words and pictures) it is always true that
\[
\int_{-a}^{a} f(x) \, dx = 2 \cdot \int_{0}^{a} f(x) \, dx.
\]

7. Evaluate \( \int_{-2019}^{2019} \sin(x) \, dx \).

For the first pair of problems, the students are expected to use symmetry to solve the problems. Previously, students used these kinds of geometric arguments to explain why the derivative of an odd function is even and vice versa. The goal is to continue to reinforce geometric reasoning as a valid way to solve calculus problems.

A 3-point solution would include a picture of a sufficiently “generic” odd/even function (i.e. anything except \( f(x) = x \) or \( f(x) = x^2 \)) and the vertical lines \( x = -a \) and \( x = a \) clearly marked. The solution explains how the area to the left of \( x = 0 \) is an exact copy of the area to the right of \( x = 0 \), but with opposite sign in the situation of problem 5. A 2-point solution might have a messy picture or unclear explanation (“The area is negated because it is.”) A 1-point solution only cites irrelevant facts (such as talking about derivatives) or is missing a picture entirely.

The final problem is meant to be pattern recognition. Students should recall that \( \sin(x) \) is an odd function and then apply the reasoning of problem 5.

8. If \( a \) and \( b \) are any two positive numbers with \( a < b \), show using geometric arguments (like in problems 1-4) that
\[
\int_{a}^{b} x \, dx = \frac{1}{2}(b^2 - a^2).
\]

9. In the previous question, what if one (or both) of \( a \) and \( b \) is negative? Is the formula still true? Explain. (Hint: The answer is “yes”. This is what I meant in class when I gave the third “reason” why considering signed area instead of unsigned area is a good thing. The formula still works no matter what \( a \) and \( b \) are!)

10. As a follow-up to the previous question, try to find a formula using \( a \) and \( b \) for the unsigned area between the graph of \( f(x) = x \) and the \( x \)-axis between \( x = a \) and \( x = b \). For example, if \( a = -1 \) and \( b = 1 \), then the formula in problem 8 gives you 0 (because the signed area is 0), but your formula for this problem should give you 2 (because the unsigned area is 2)! Your formula can be written piecewise if you would like. You can still get full credit if you don’t find a concrete formula, as long as you provide an explanation of what happens for various values of \( a \) and \( b \).
Problem 8 is meant to be an abstraction of problems 1-4 where the bounds are variable. Students are meant to complete it in a similar way. I expect that a correct solution would include a sketch of the graph of $f(x) = x$ with $x = a$ and $x = b$ labeled. Then students can calculate the area in several ways. One way is to use a formula for the area of a trapezoid. Alternatively, students can view the area as a large right triangle with base $b$ with a right triangle with base $a$ cut out of the corner. A third possibility is for students to cut the area into a rectangle and a triangle sitting on top. Since the students are given what the formula is supposed to be, their solution is likely either a 3-point solution or a 1-point solution (if they didn’t make any progress toward a solution.)

The goals of problems 9 and 10 are to help students process the idea of “signed area”, where area below the $x$-axis is counted as negative. This is a tricky concept at first. By providing the answer (“Yes”) for problem 9, I can help guide the students toward productive thinking. They still have to grapple with the question, but I can be sure that they are moving in the right direction. I also want to make sure I can link this more conceptual question with the more concrete discussion we had during class.

Problem 10 is meant to be tricky. Students can give a formula using the sign function, the absolute value function, or as a piecewise function, for example, $\frac{1}{2} (\text{sign}(b)b^2 - \text{sign}(a)a^2)$. If students aren’t able to come up with that formula, they can get full credit by providing an explanation for what changes between problems 9 and 10 when $a$ or $b$ is negative. This would be something along the lines of “if $b$ is positive but $a$ becomes negative, then we add the two triangle areas together to get...”. A 2-point solution would only include a couple of examples with no discussion of the general idea.

11. Suppose that $f(x)$ and $g(x)$ are functions on the interval $[0, 1]$ and that $f(x) \leq g(x)$ for all $x$ in that interval. Explain using words and pictures why $\int_0^1 f(x) \, dx \leq \int_0^1 g(x) \, dx$.

Several solutions are possible. Students must identify this problem as a question about areas. A full solution should include a picture of the “generic” case. For now, it is okay if students assume both functions are positive. A 2-point solution might have a slightly-incorrect picture or an unclear explanation (such as simply stating “$g$ has more area than $f$”).
IBL Script

As discussed in the Teaching Experience section, in the IBL classroom, students are given scripts with theorems that they prove on their own. They then present their proofs during class, and the instructor(s) facilitate the discussion between the students about the material.

I am including a portion of Script 15: Sequences and Series of Functions below. This script is by far the most technical of all of the scripts, and it’s the culmination of a lot of work that the students have done. I am not choosing to include it in spite of its difficulty, but because of it; the students at first really struggled with these concepts, which gave me an opportunity to use the class time to really help the students wrap their minds around these ideas.

The plain text is what the students see. The text in the boxes are my annotations, including what comments I made in class and what I wanted to get the students to think about. On the end-of-term course evaluations, 10 students wrote that these examples and questions I asked in class significantly impacted their learning. See my Recent Course Evaluations for more student feedback.
SCRIPT 15: Sequences and Series of Functions

We will now turn to sequences and series of functions.

Definition 15.1. Let $A \subset \mathbb{R}$, and consider $X = \{f : A \to \mathbb{R}\}$, the collection of real-valued functions on $A$. A sequence of functions (on $A$) is an ordered list $(f_1, f_2, f_3, \ldots)$ which we will denote $(f_n)$, where each $f_n \in X$. We can take the sequence to start at any $n_0 \in \mathbb{Z}$ and not just at 1, just like we did for sequences of real numbers.

Definition 15.2. The sequence $(f_n)$ converges pointwise to a function $f : A \to \mathbb{R}$ if for all $x \in A$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that:

$$\text{if } n \geq N, \text{ then } |f_n(x) - f(x)| < \epsilon.$$ 

In other words, we have that for all $p \in A$, $\lim_{n \to \infty} f_n(x) = f(x)$.

Are pointwise limits unique? That was a theorem for sequences of numbers. How can we prove it here? (Answer: Just need to apply the theorem for sequences of numbers.)

Definition 15.3. The sequence $(f_n)$ converges uniformly to a function $f : A \to \mathbb{R}$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that:

$$\text{if } n \geq N, \text{ then } |f_n(x) - f(x)| < \epsilon \text{ for every } x \in A.$$ 

What is the difference between this definition and the previous one? (Ans: Order of the quantifiers!) Make sure we get the quantifiers right: $\forall \epsilon \exists N \forall x \forall n$ as opposed to $\forall x \exists N \forall \epsilon \forall n$. Where have we seen this quantifier issue come up before? (Ans: Continuity vs. uniform continuity of functions.) Draw lots of pictures; in particular, the graphs of $y = f(x)$, $y = f(x) + \epsilon$, and $y = f(x) - \epsilon$. The graph of $y = f_n(x)$ need to squeeze in between these two graphs in order to get uniform convergence!

Exercise 15.4. Suppose that a sequence $(f_n)$ converges pointwise to a function $f$. Prove that if $(f_n)$ converges uniformly, its uniform limit must be $f$.

A bit oddly worded. Let’s rephrase: If $(f_n)$ converges uniformly to $f$, then $(f_n)$ also converges pointwise to $f$. Combine with the fact that pointwise limits are unique. This is an important concept! It means that for the following exercises, in order to prove that $(f_n)$ doesn’t converge uniformly to anything, it suffices to prove that it doesn’t converge uniformly to $f$, where $f$ is the pointwise limit of $(f_n)$.
Exercise 15.5. For each of the following sequences of functions, determine what function the sequence \((f_n)\) converges to pointwise. Does the sequence converge uniformly to this function?

i) For \(n \in \mathbb{N}\), let \(f_n: [0, 1] \to \mathbb{R}\) be given by \(f_n(x) = x^n\).

ii) For \(n \in \mathbb{N}\), let \(f_n: \mathbb{R} \to \mathbb{R}\) be given by \(f_n(x) = \frac{x}{n}\).

iii) For \(n \in \mathbb{N}\), let \(f_n: [0, 1] \to \mathbb{R}\) be given by 
\[
f_n(x) = \begin{cases} 
  n^2x & 0 \leq x \leq \frac{1}{n} \\
  n(2-nx) & \frac{1}{n} < x \leq \frac{2}{n} \\
  0 & \frac{2}{n} < x \leq 1.
\end{cases}
\]

Students are expected to use the definitions above. After discussion, read Theorems 15.6-15.8. Could we have used any of them to simplify our work on 15.5? Yes! For example, can use 15.6 to show that the functions in (i) can’t converge uniformly, because we found the pointwise limit and it’s discontinuous. Similarly, we can use 15.7 to rule out uniform continuity on (iii).

Theorem 15.6. Let \((f_n)\) be a sequence of functions, and suppose that each \(f_n: A \to \mathbb{R}\) is continuous. If \((f_n)\) converges uniformly to \(f: A \to \mathbb{R}\), then \(f\) is continuous.

After discussion of the proof is complete: Is the statement still true if we replace “continuous” by “uniformly continuous”? What in the proof has to change? Answer: Yes it’s still true, and the proof should still be the same! (This depends on what proof the student presents on the board. Depending on the proof given, either reference it or give this as a homework exercise.)

Theorem 15.7. Suppose that \((f_n)\) is a sequence of integrable functions on \([a, b]\) and suppose that \((f_n)\) converges uniformly to \(f: [a, b] \to \mathbb{R}\). Then
\[
\int_a^b f = \lim_{n \to \infty} \int_a^b f_n.
\]

This theorem is more subtle than it looks! You don’t know that \(f\) is integrable at all, so the first thing you have to do is prove that it is. You can’t just assume that it is. You also aren’t given that the limit on the right exists! So this really is a multi-step theorem.
Theorem 15.8. Let \((f_n)\) be a sequence of functions on \([a,b]\) such that each \(f_n\) is differentiable and \(f'_n\) is integrable on \([a,b]\). Suppose further that \((f_n)\) converges pointwise to \(f\) and that \((f'_n)\) converges uniformly to a continuous function \(g\). Then \(f\) is differentiable and
\[ f'(x) = \lim_{n \to \infty} f'_n(x). \]

It’s kind of like the previous two but for derivatives, except this one needs a lot more hypotheses. Why? Is it still true if we remove hypotheses? Homework assignment:
Find an example of a sequence of differentiable functions \((f_n)\) such that \((f_n)\) converges pointwise to \(f\), and \((f'_n)\) converges pointwise to some \(g\), and yet \(f\) is not differentiable. This shows that the uniform convergence of the derivatives was crucial! Just drawing pictures is enough.
Example solution: \(f_n(x) = \sqrt{x^2 + \frac{1}{n}}, f(x) = |x|\).
Evaluation and Feedback

Evaluations from Recent Courses

Below are evaluations from students in the three most recent courses, in reverse chronological order. The evaluations are standardized and administered by the mathematics department, and they contain quantitative and qualitative aspects. Complete evaluations dating back to Autumn 2016 are available upon request.

Spring 2019: Math 16310
20 enrolled, 18 evaluations received

In this course, I was a co-instructor. I have included only the parts of the evaluation that were specific to my role in the classroom.

Quantitative Responses
The students rated me on the following questions using a Likert scale from 1: “strongly disagree” to 5: “strongly agree”. The questions are:

#1: The instructor facilitated discussions that supported my learning.
#2: The instructor gave me useful feedback on my work.
#3: The instructor stimulated my interest in the core ideas of the course.
#4: The instructor helped me succeed in the class.
#5: The instructor was available and helpful outside of class.
#6: The instructor gave appropriate feedback on assignments.
#7: The instructor returned assignments promptly.
#8: Overall, the instructor made a significant contribution to my learning.

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Qualitative Responses
A common theme in the qualitative responses was that the examples I gave and questions I asked them in class significantly helped their learning (10 students). The next most common themes were my availability outside of class (4 students) and my emphasis on style and clarity in their mathematical writing (3 students).

The only critical comment was from a student who suggested that I could have provided more “model” proofs to help the students understand what they should be aiming for. Starting half-way through the quarter, I began modeling good proof-writing techniques outside of class in order to help the students improve their mathematical communication skills, and several students expressed that they wished I had started earlier.

Selected student comments below:

- “He always made useful comments and tried to expand our understanding of things which I found very helpful.”
- “He was an excellent teacher, always held helpful office hours, and really helped me understand the material.”
• “He was very good at helping with proof writing and his comments on student proofs were always helpful.”
• “He always told us about cool stuff!”
• “Karl was good at asking difficult questions in class and giving constructive criticism on homework.”
• “[Karl asked] questions about each step of my proofs to ensure that I knew deeply why I was doing things.”

**Winter 2019: Math 13200**

27 enrolled, 19 evaluations received

I was the instructor of record in this course.

**Quantitative Responses**

The students rated me on the following questions using a Likert scale from 1: “strongly disagree” to 5: “strongly agree”. The questions are:

Questions about the course:
#1: I understood the purpose of the course and what I was expected to gain from it.
#2: I understood the standards for success on assignments.
#3: Class time enhanced my ability to succeed on graded assignments.
#4: The course challenged me intellectually.
#5: My work was evaluated fairly.
#6: I felt respected in this class.
#7: Overall, this was an excellent class.

Questions about me as an instructor:
#1: The instructor organized the course clearly.
#2: The instructor stimulated my interest in the core ideas of the course.
#3: The instructor challenged me to learn.
#4: The instructor helped me to learn the course content.
#5: The instructor was accessible outside of class.
#6: The instructor created a welcoming and inclusive learning environment.
#7: Overall, the instructor made a significant contribution in my learning.

The first row of numbers is for the questions about the course, and the second row of numbers is for the questions about me as an instructor. The final column is the average of all ratings for the course.

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**Qualitative Responses**

Many students wrote that I explained concepts very well and thoroughly in class (8 students). Four students emphasized my use of examples in class, and 6 students highlighted either my organization of the blackboard or of the class as a whole. Four students cited my enthusiasm in the classroom as playing a big role in their experience.
The most common complaint was that either that the exams were harder than the homework assignments or that the homework assignments should contain problems more similar to the ones on exams (4 students). I tried to design the exams so that approximately 15% of the grade required students to synthesize topics in a novel way. In future courses, I will give students more of an opportunity to practice this skill on ungraded in-class quizzes which will help prepare them for the exams.

Selected students comments below:

- “Karl was a great teacher. He made everything very clear and easy to follow.”
- “This is the best math class I’ve ever taken and I’m devastated that Karl isn’t teaching [Math 13300] next quarter. I understand and appreciate math because of this class.”
- “Karl is such a talented teacher. Not only is he energetic and interesting, but he cares about his students and is excellent at explaining things.”
- “Karl is very helpful and wants us to succeed. He is clear when teaching and answers any questions. He is supportive and an awesome instructor!”
- “Karl is marvelous. He is an awesome instructor who knows all the topics very well and can express them clearly to all students.”
- “He would tell us fun quick stories about how certain math formulas came about.”
- “He did plenty of examples over each topic and gave us background as to why the formulas work the way they do.”

Autumn 2018: Math 13100
27 enrolled, 23 evaluations received
I was the instructor of record in this course.

Quantitative Responses
The students rated me on the same questions and using the same scale as in the above Math 13200 course. The first row of numbers is for the questions about the course, and the second row of numbers is for the questions about me as an instructor. The final column is the average of all ratings for the course.

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Qualitative Responses
Many of the student comments emphasized my in-class explanations (7 students) and examples (6 students) as being very clear and helpful to their learning. Additionally, students commented that I was engaging and passionate about the material (6 students), that I very supportive (4 students), and that I set clear expectations and structured the course well (4 students).

The most common complaint among the students was that the course was too easy or that they were not challenged very much by the material (10 students). This course had a mix of students – about half had seen no calculus before, and half had taken AP calculus in high
school, and when I taught this course I tried to focus more on the first half. In response to these comments, when I taught Math 13200 to the same group of students the following quarter, I selected in-class examples from a wider range of difficulties and I focused more on higher-order thinking skills on Bloom’s Taxonomy in a way that I didn’t think many high-school calculus classes did, and students reported feeling more challenged but still supported.

Selected student comments below:

- “Karl was a great instructor and showed that he cared about the material and students.
- “In class, he set up a nice vibe that was encouraging and he answered questions very well.”
- “Karl was great – clear, nice, reasonable.”
- “Karl is great at explaining concepts and why we do certain things in math.”
- “The professor is engaging and very helpful when students have questions. Grading policies are also very helpful and the structure of the class (homework, lecture, tutorials) overall improved my understanding of calculus.”
- “[I] loved how he used so many examples and explained in detail each topic.”
- “Karl is a great, helpful instructor. He is dedicated and wants us to do well.”
- “The instructor did an excellent job structuring the class and all the assignments were fair, helpful, and relevant.”
Individual Teaching Consultation Report

Individual Teaching Consultations are a service provided by the Chicago Center for Teaching at the University of Chicago. Over the course of two weeks in the Spring of 2019 when I was co-instructing IBL Calculus III, I met with a CCT-certified teaching consultant to discuss my course. The consultant came to my classroom to observe and film me, and we met the following week to discuss his feedback and brainstorm ideas for improving my lecturing. Below is the copy of the final report of this process furnished to me by my teaching consultant. I have highlighted comments that I believe demonstrate my pedagogical practices.

In the final report, my teaching consultant highlighted the positive nature of the participation among students in the class but noted that some students participated more than others. During our final meeting, we discussed strategies for how to increase participation among the more quiet students. After this session, I implemented some of these strategies, including increasing the structure for student sign-ups for proof presentations, which did increase the overall participation in the course.
INDIVIDUAL TEACHING CONSULTATION
FINAL REPORT

Date: April 25th, 2019
Instructor: Karl Schaefer
Course: Math 16310 Honors Calculus III (IBL)
Consultant: Rostom Mbarek
Attendance: 20 students

Summary:

The observed course is taught in the mathematics department at the University of Chicago. The observation took place at the end of week 4 of the spring quarter. The course is an Inquiry Based Learning (IBL) course, designed for freshman students who intend to pursue mathematics as a major. IBL is a form of active learning that relies on active participation and collaboration between students in mathematical activities during the class period. Karl Schaefer is one of two instructors of the course and has almost no control over the course design. The mathematics department chooses the topics and the books to be covered. The instructors, on the other hand, prepare the lectures and grade work.

The observation and meetings between the instructor and observer intend to provide constructive feedback on the instructor’s interaction with the students to expand on teaching practices that promote student learning. Such a process also aims to lessen the impact of practices that may hinder student learning. In the case of Karl, his care for the students and strong desire to make them succeed make him a great instructor. His lecture is well structured and organized. One can see that Karl has the attention of his students when lecturing. He makes great connections between the current lecture and previous ones while commenting on problems. He also employs clear visual aids to help his students visualize the mathematics behind the theorems and lemmas he writes on the board. Additionally, he has a clear and organized handwriting which made his lecture a success overall.

To continue leading his lectures successfully, I have three main suggestions for Karl. First, he could consider having more quizzes at the beginning and end of his lecture to get more individual assessment. Second, he could make student participation more structured to engage all students. Third, he could try to alter the seating if possible to engage the students and avoid side conversations.
Part I: Pre-observation Meeting  
April 19th, 2019

Karl has a set of objectives for his IBL class that are set by the mathematics department and has his own teaching objectives. Because of the IBL nature of the course, the instructors have a hands-off strategy in which they let the students run the class as much as possible. In the observed class, Karl was the only instructor and he lectured more than he usually does to introduce a new concept to the students. At the pre-observation meeting, Karl stated that he believes the course is going well even if sometimes lectures run more slowly. He explained that it is difficult to predict the pace at which the students will lecture because they all have different styles. Finally, he also stated that when students write their proofs on the board, the class period could lose structure and students start talking to one another. On a more general note, Karl stated that he wants his students to learn how to do math proofs and criticize them to understand them better. He also wants them to make connections between different concepts from previous lessons. In order to achieve such a goal, he provided many examples for his students that follow directly from theorems he introduced.

Part II: Observations and Recommendations

More individual assessment:

Shifting the focus to a more detailed analysis of Karl’s lecture, one can appreciate and evaluate the instructor’s teaching methods. As previously stated, Karl lectured for most of the class period instead of only commenting on the students’ lectures. The students had roughly 40% of the class period to solve problems on the board. Karl started by making a few announcements and then started introducing a new concept. As the lecture went on, the instructor provided many examples to better explain the theorems, definitions, and lemmas he is introducing. A few students also volunteered to present on proofs they worked on before class. Overall, more than half the students participated during the class period. However, not all students were able to participate. A few were ostensibly not paying attention to the lecture and proofs.

IBL is a great active learning concept that gives the students a lot of freedom. There are, however, issues with these practices since Karl cannot check everybody’s answer and the set of students who participate is usually the same. A more structured form of daily assessment such as quizzes could solve these issues. Daily quizzes are a great way to set the pace and objectives of each lecture, to help students retain information, and to enable the teacher to assess the students’ performance. In the case of Karl’s class, I suggest implementing two 5-minute quizzes at the beginning and end of class. In addition to the already stated benefits of a beginning-of-class quiz, reactivating prior knowledge in the first 5 minutes of class is as important as retaining it at the end. This will review earlier
sessions and set the pace for the rest of the lecture. Additionally, an extra quiz at the end of class would enable students to draw on material they just learned and get instant practice. Instead of learning only through the lecture, the students would learn by practicing the concepts they would have just learned. Everybody gets to participate and think about the problems. The students can work together or separately depending on what the instructor deems more appropriate for the material to be learned. In this case, the quizzes would not only be a gauge of the students’ performance, but also become a teaching tool. Additionally, they can assess how successful the lecture was. Quizzes and other forms of assessment during class can make explicit connections between different topics and build a better conceptual understanding of the subject, which is crucial for learning. Karl’s teaching practices are already engaging for the students. If he were to complement them with such quizzes, I believe that the students’ performance can only improve.

Setting the tone:

Karl is an engaging instructor. He is attentive to his students’ questions and takes input from them. The students actively participated in making the lecture a success. However, the same set of students who were participating with Karl volunteered to write proofs and discussed concepts with their peers. More importantly, when a student raised her hand to participate for the first time, another student, who had talked the most during the class period, starts responding to the question without any hesitation.

Students can be oblivious to their peers and can be excited to express their opinions, answer questions, or show their interest in the class material. Overall, that is excellent for the more involved students, but could be unfavorable to students who are shy or reticent to speak in public. This IBL class is an opportunity for such students to overcome this reticence and present their work in public. That is why I suggest a more structured way of signing up for proof presenting. All students would present their proofs a certain number of times per quarter. All students will participate equally and will have a better experience overall.

Maintaining Student Engagement:

Our discussions led me to believe that Karl cares about his students. He keeps an ear on what is happening in the whole classroom and tries to get everyone to participate. There were a few students who may have been disruptive to their classmates. Others were not really paying attention to the discussion and presented material.

It is always difficult to deal with such issues at the college level, but it is necessary to deal with this behavior in a subtle manner. I suggest changing the seating arrangement when possible to enable the teacher to walk by the students to try to engage all students.
Part III: Post-observation Meeting
May 7th, 2019

During the post-observation meeting, Karl and I brainstormed a few ideas based on the strategies listed above to make students perform better in his class. The instructor was carefully listening to the comments and even suggested new ideas of his own to improve the atmosphere in his classroom. He indicated that he would convince the students who do not participate to give more lectures and present their proofs. He also implied that, in order to keep the lectures more engaging, he would get more involved in the proof presentation process while still keeping the IBL elements of the course. The most important aspect in Karl’s teaching methods is that he recognizes the value of small details during his lecture. These practices ultimately influence the engagement of students and their abilities to synthesize the course material. We ended the conversation on a positive note since he recognized the benefits of the observation and the various discussions we had.
Letter from a Student

On March 22, 2019, I proctored the final exam for my Math 13200 course. On the final page of the exam, a student wrote me the following unsolicited note. I have redacted the student’s name, and the note has been reproduced below with permission from the student.
Dear Karl,

Thank you for being a marvelous teacher and always knowing how to answer my very specific and complicated questions. I'm taking exam time to write you a thank you instead of figuring out $\theta = \arcsin x$ because I've run out of ideas and this note is more important than that math anyway. I've learned so much about math (of course) but also hard work and dedication and the value of theory from you. I sincerely appreciate your drive and gusto that you bring to class and your patience with us students. Going to office hours (even though I missed a few this quarter) is always a joy and what taught me all those valuable lessons (which I learned in class, too!). I wish you the best of luck in your endeavors, but I know you'll do marvelously anyway because you're a phenomenal teacher.

Best,

3/22/19
Pedagogical Training and Development
Activities are listed in reverse chronological order.

Qualifications

Chicago Center for Teaching Fellow
During the 2019-2020 academic year, I am working with the Chicago Center for Teaching as a CCT Fellow. In this role, I engage with current theories and debates in teaching and learning, and develop programs and resources to address the needs and concerns of graduate student teachers across campus. I also lead graduate student teaching seminars and workshops through the Fundamentals of Teaching Series.

College Teaching Certificate
I received this certificate in Autumn of 2019. To complete this program, I participated in various workshops on student-centered and inclusive pedagogy, got feedback on my teaching, took a quarter-long course on Course Design and College Teaching, and wrote several reflective essays on inclusive pedagogy. These activities are given in more detail below.

Selected list of Workshops and Conferences

Fundamentals of Teaching in the Mathematical Sciences
I co-facilitated the four-week workshop series Fundamentals of Teaching Mathematical Sciences, supported by the Chicago Center for Teaching. My co-facilitator and I designed the workshops and led the discussions and activities for approximately 20 other graduate student instructors. Topics included: How do students learn math? What does inclusivity in a math classroom look like? How and why do we assess math students? What is the purpose of collegiate math?

Teaching@Chicago
Teaching@Chicago is a day-long conference designed to orient new teaching assistants to the culture, structure, elements, and practices of teaching at the University of Chicago. I helped to coordinate the event and led small group discussions on inclusive and student-centered teaching.

CCT Teaching Fellow Training
This two-day event featured seminars on evidence-based pedagogical practices, inclusive teaching practices, and mentorship for other graduate students. Additionally, there were workshops during which I began development of the Fundamentals of Teaching in the Mathematical Sciences workshop series for other graduate student instructors that I co-facilitated.

CCTE 50000: Course Design and College Teaching
In this course, taken in the Spring of 2019, I read literature on teaching and learning and used that to reflect on my own teaching practice. Over the term, we discussed how students learn, how to create inclusive and welcoming learning environments, and how to design a course that promotes deep learning. For a final project, I used what I learned during the course to design my course “Let’s Play a Game!” (see my Teaching Experience).
Microteaching
In my second Microteaching session (see below) from Spring of 2019, each of 5 participants gave a 15-minute lecture to the group. After my lecture, during which I simulated a piece of my course “Let’s Play a Game!” (see above), the other participants gave me feedback on my style. The lecture was filmed, and afterward I watched the tape and reflected on the choices I made in the lecture, including the proportion of time I spent talking versus allowing for classroom discussion and how to facilitate a mathematical discussion.

Individual Teaching Consultation
In the Spring of 2019, I reached out to the Chicago Center for Teaching for an Individual Teaching Consultation. A trained consultant observed and filmed a lecture in my class, and after watching the recording I had a follow-up meeting with the consultant to discuss strategies to improve the learning environment that I was creating in the classroom. During the consultation, we focused on effective discussion management and how to most effectively encourage all students to participate in classroom discussion. The report from the consultation is reproduced above.

Inclusive Assessment in STEM Workshop
For this Spring 2019 workshop, I read various studies on how assessment techniques in STEM classrooms can impact student learning and help close the achievement gap between minoritized and majoritized groups. For example, in the 2014 article “Getting Under the Hood: How and for Whom Does Increasing Course Structure Work”, Eddy and Hogan conclude that increasing course structure halves the gap in exam performance for black students compared to white students, and eliminates the gap for first-generation students compared to continuing-generation students. Using this, I collaborated with graduate students from several STEM disciplines to develop guidelines for inclusive assessment practices that will be published on the CCT website.

Workshop on the First Day of Class
In this workshop, facilitated by the CCT in Spring 2017, I worked with other graduate students in the math department to discuss how to write a good syllabus and establish a welcoming and inclusive classroom environment on the first day of class.

Microteaching
I participated in my first Microteaching session in Winter of 2016 in preparation for giving a full lecture to undergraduate students. I reviewed the video of the session with a CCT staff member, and we brainstormed ways in which I could improve my mechanics at the front of a classroom.

College Fellow Program
Facilitated by the Mathematics Department, this year-long program is designed to help young graduate students prepare for managing our own classrooms. During the 2015-2016 academic year, I was paired with a senior faculty mentor for whom I was a Teaching Assistant (see my Teaching Experience). During this year, I gave periodic lectures to a class of 30 advanced undergraduates, observed by my mentor, who I met with to debrief and discuss ways to improve the learning environment I created.
Teaching Publications


