Shuffle Sums

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Outline

1. Acknowledgements... (UW-Madison; Hirschfeldt and Lempp)
2. What are Shuffle Sums?
   - Background and Notation
   - Shuffle Sums
3. Characterizing the Computable Shuffle Sums of Ordinals
   - LIMINF Sets
   - The Characterization: Forwards
   - The Characterization: Backwards
   - LIMINF Sets Revisited
4. When Larger Ordinals are Allowed

Background and Notation

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Thus **2**, for example, will denote a linearly ordered pair of points.

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Definition

Let $S = \{\mathcal{L}_x\}_{x \in \omega}$ be a countably infinite set of linear orders. Then the shuffle sum of S, denoted $\sigma(S)$, is the linear order obtained by partitioning the rationals $\mathbb{Q}$ into countably many dense sets $\{Q_x\}_{x \in \omega}$ and replacing each point $q \in Q_x$ with the linear order $\mathcal{L}_x$.

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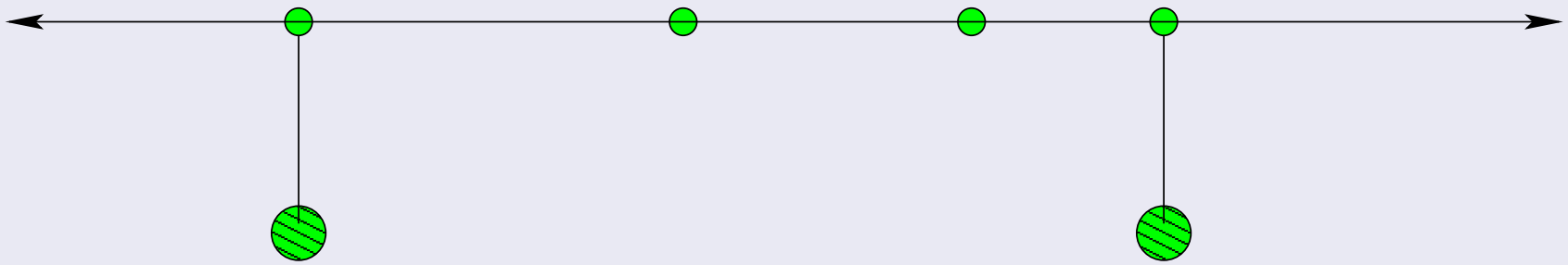

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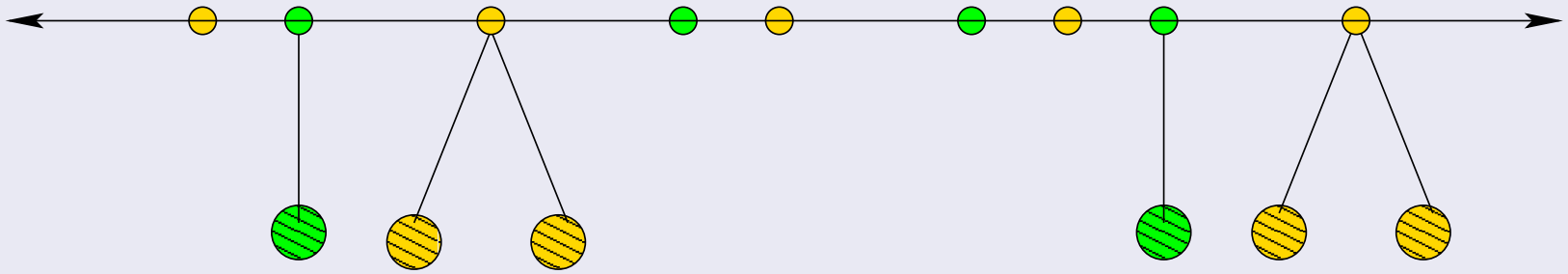

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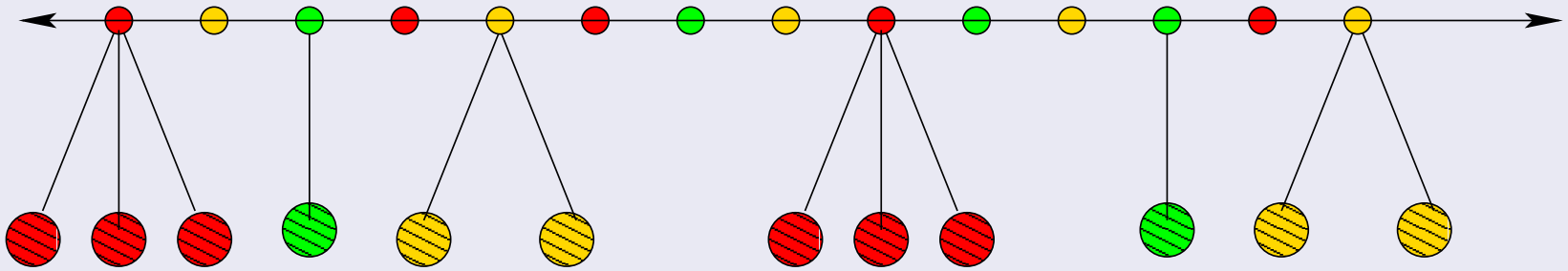

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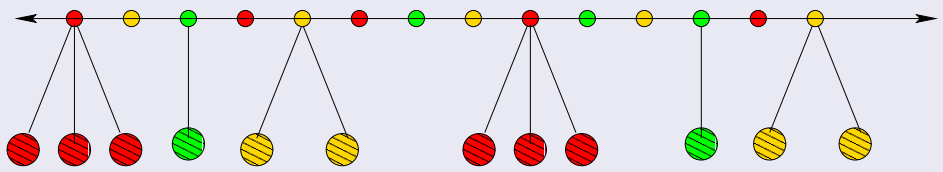

Example

If $\mathcal{L}_x = \mathbf{1}$ for all x, then $\sigma(S) \cong \mathbb{Q}$.

Shuffle Sum Uniqueness

Proposition

The definition of the shuffle sum $\sigma(S)$ is well-defined, i.e. the (isomorphism type of the) linear order does not depend on the partition $\mathbb{Q} = \bigsqcup_{x \in \omega} Q_x$.

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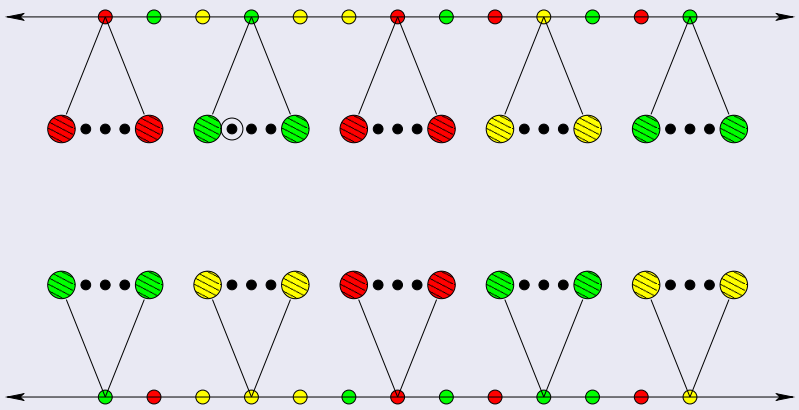
Construct an isomorphism in stages using the standard back-and-forth argument (slightly modified). For example:



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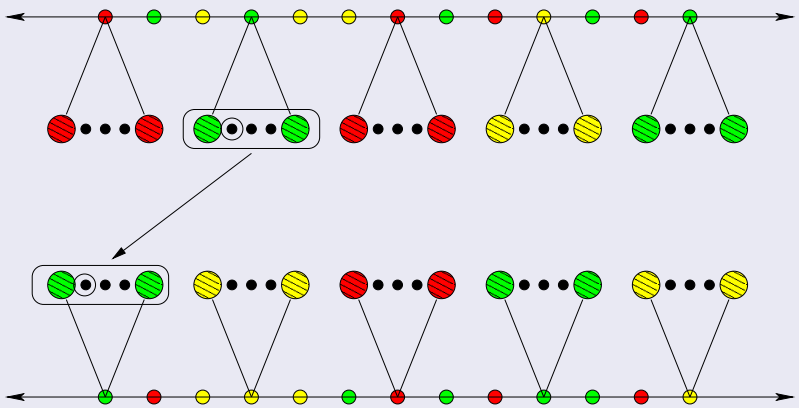

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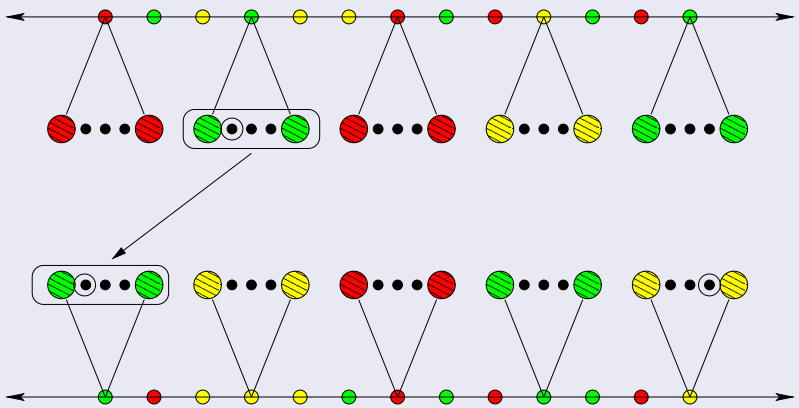

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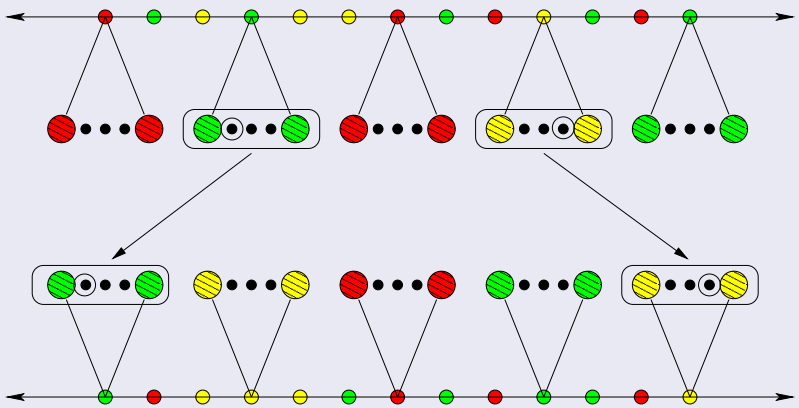

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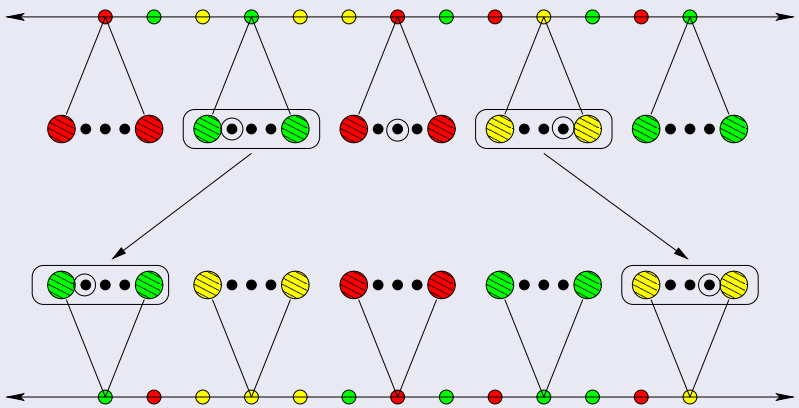

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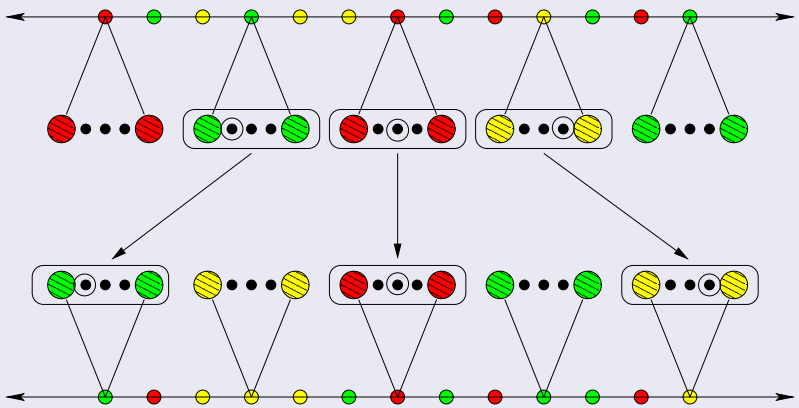

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LIMINF Sets

Definition (K)

A set $S \subseteq \omega + 1$ is a LIMINF *set if there is a total computable function $g(x,t)$ such that S is the range of the function $f(x)$ given by*

$$f(x) = \liminf_t g(x,t).$$

Here we use the convention that $\liminf_t g(x,t) = \omega$ if $\lim_t g(x,t) = \infty$.

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Morally, though, we aren't using the fact that $g(x,t)$ need only serve as an approximation to $f(x)$. We'll see later that more complicated sets are also LIMINF sets.

A Characterization

Theorem (K)

A set $S \subseteq \omega + 1$ is a LIMINF *set if and only if $\sigma(S)$ is computable.*

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Let $g(x,t)$ witness that S is a LIMINF set. At the rational $q_i \in Q_x$, build the linear order $\mathbf{g}(x,t+1)$ at stage t.

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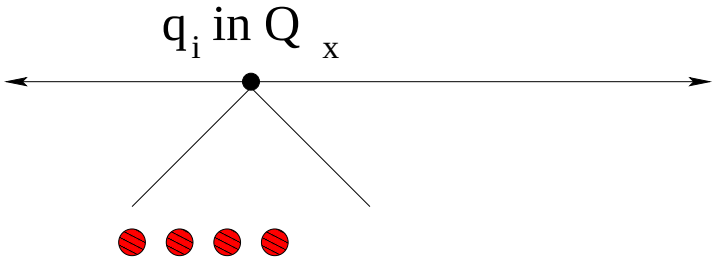

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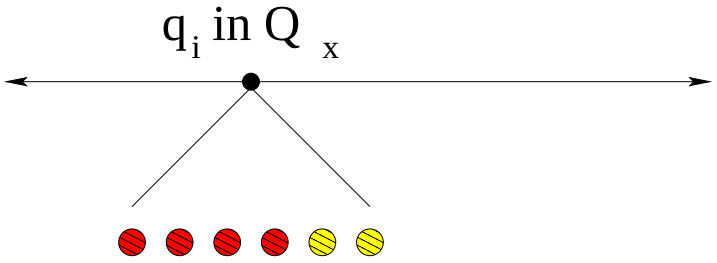

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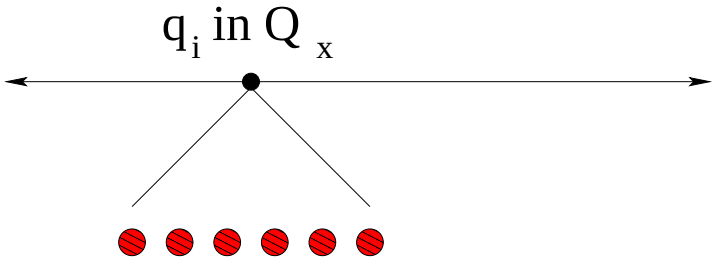

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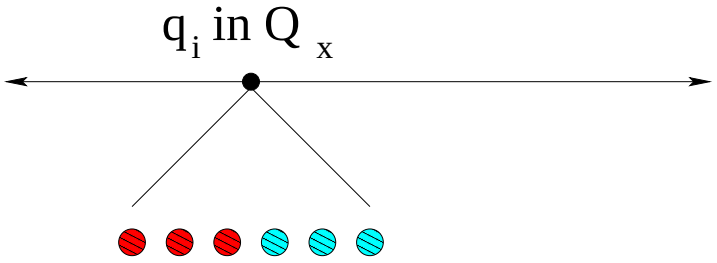

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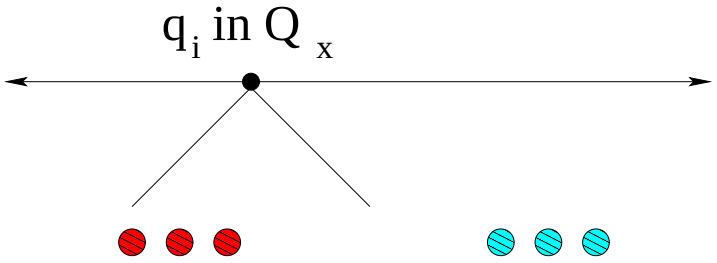

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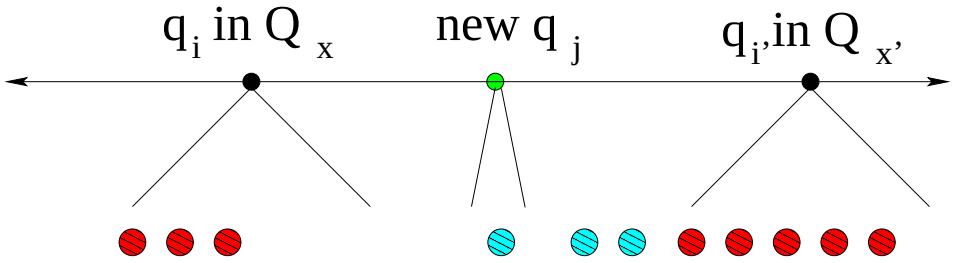

And Backwards...

Proposition (K)

If $\sigma(S)$ has a computable presentation with $S \subseteq \omega + 1$, then S is a LIMINF *set.*

Proof.

Let $\mathcal{A}$ be a computable presentation of $\sigma(S)$ with universe $\{a_i\}_{i\in\omega}$. For each element a_i define functions $\ell_{a_i}(t)$ and $r_{a_i}(t)$ approximating the number of points to the left and right of a_i in its block. More specifically, set

$$\ell_{a_i}(t) = \begin{cases} \left|\{j : 0 \leq j \leq t \ \ \& \ \ a_j < a_i\}\right| & \text{if } a_i \leq a_t, \\ \left|\{j : 0 \leq j \leq t \ \ \& \ \ a_t < a_j < a_i\}\right| & \text{if } a_t < a_i. \end{cases}$$

Define $r_{a_i}(t)$ analogously.

□

Backwards (Continued)...

Proof (Continued).

Let $g(x,t)$ be the function given by

$$g(x,t) = \ell_{a_x}(t) + 1 + \min_{s \in [u,t]} r_{a_x}(s)$$

Then $g(x,t)$ is computable and $f(x) = \liminf_t g(x,t)$ enumerates S.

$\square$

Which Sets are LIMINF Sets (I)?

Proposition (K)

If S is Σ^0_3, then $S \cup \{\omega\}$ is a LIMINF *set.*

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Proof.

Let $\exists_s \exists_t^\infty R(n,s,t)$ with R computable witness that S is Σ^0_3. Define

$$g(x,t) = g(\langle n,s\rangle, t) = \begin{cases} n & \text{if } R(n,s,t), \\ t & \text{otherwise.} \end{cases}$$

Then $g(x,t)$ witnesses that $S \cup \{\omega\}$ is a LIMINF set.

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Which Sets are LIMINF Sets (II)?

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Proof.

Let $f(x) = \liminf_t g(x,t)$ witness that S is a LIMINF set. Then

$$\begin{aligned} n \in S \quad & \text{iff} \quad \exists x\, [f(x) = n] \\ & \text{iff} \quad \exists x \left[\liminf_t g(x,t) = n\right] \\ & \text{iff} \quad \exists x \left[\forall t \exists t_0 > t\, [g(x,t_0) = n] \text{ and } \exists s_0 \forall s > s_0\, [g(x,s) \geq n]\right] \end{aligned}$$

The last expression is clearly Σ^0_3.

□

Which Sets are LIMINF Sets (III)?

Proposition (K)

Not every Σ^0_3 set is a LIMINF *set.*

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Proof.

The result follows as a corollary from the following proposition of Hirschfeldt's.

Proposition (Hirschfeldt)

If $S \subseteq \omega - \{0, 1\}$ is a Σ^0_3 set that is not the range of a $\mathbf{0}'$-limitwise monotonic function, then $\sigma(S)$ is not computable.

That such sets exist is a result of Khoussainov, Nies, and Shore.

□

When Larger Ordinals are Allowed

A natural question to ask is which sets of ordinals $S \subseteq ON$ have computable shuffle sums. From (one of) the definitions of ω_1^{CK}, we can restrict our attention to $S \subseteq \omega_1^{\mathrm{CK}} + 1$.

The speaker is currently investigating this very question. In order to obtain a characterization, it seems that the notion of LIMINF sets needs to be relativized.

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Definition

A set S of ordinals is a LIMINF(A) *set if there is a total function $g(x,t)$ computable in A such that S is the range of the function $f(x)$ given by*

$$f(x) = \liminf_t g(x,t).$$

What's Known

Proposition (K)

If $\sigma(S)$ is computable with $S \subseteq \lambda < \omega_1^{\mathrm{CK}}$, then the sets

$$\tilde{S}_\gamma = \left\{k \in \omega + 1 : \alpha = \beta + \omega^\gamma \cdot k \in S \text{ for some } \beta \geq w^{\gamma+1} \text{ or } \beta = 0\right\}$$

are uniformly $\textsc{Liminf}\left(\emptyset^{(2\gamma)}\right)$.

Proof. (?)

Follow the proof that demonstrates that if $\sigma(S)$ is computable for $S \subseteq \omega + 1$, then S is a $\textsc{Liminf}$ set. Relativize appropriately [?].

□

What's Not Known

Conjecture

If $S \subseteq \lambda < \omega_1^{\mathrm{CK}}$ *is such that the sets*

$$\hat{S}_\gamma = \left\{ \alpha \in S : \begin{array}{l} \alpha = \omega^\gamma \cdot \beta \text{ for some } \beta \\ \alpha \neq \omega^{\gamma+1} \cdot \beta \text{ for any } \beta \end{array} \right\}$$

are uniformly $\textsc{Liminf}\left(\emptyset^{(2\gamma)}\right)$*, then* $\sigma(S)$ *is computable.*

Conjecture

Every Δ^0_3 *set is a* $\textsc{Liminf}$ *set.*

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