The Complexity of Computable Categoricity

Asher M. Kach
University of Chicago

(Joint Work with Rodney Downey, Steffen Lempp, Andrew Lewis, Antonio Montalbán, and Daniel Turetsky)

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Outline

1. Computable Categoricity and Relative Δ^0_α-Categoricity
2. The Good and The Bad
3. Uniformity in Computable Categoricity
4. Theorems and Sketches of Proofs

Computable and Relative Computable Categoricity...

Definition

A computable structure $\mathcal{S}$ is *computably categorical* if between any two computable presentations $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}$ there is a computable isomorphism $\pi : \mathcal{A} \cong \mathcal{B}$.

Definition

A computable structure $\mathcal{S}$ is *relatively computably categorical* if between any two arbitrary presentations $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}$ there is a $(\mathcal{A} \oplus \mathcal{B})$-computable isomorphism $\pi : \mathcal{A} \cong \mathcal{B}$.

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Relativizing The Definitions...

Definition

A computable structure $\mathcal{S}$ is Δ^0_α*-categorical* if between any two computable presentations $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}$ there is a Δ^0_α-computable isomorphism $\pi : \mathcal{A} \cong \mathcal{B}$.

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A computable structure $\mathcal{S}$ is *relatively* Δ^0_α*-categorical* if between any two arbitrary presentations $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}$ there is a $(\Delta^0_\alpha(\mathcal{A}) \oplus \Delta^0_\alpha(\mathcal{B}))$-computable isomorphism $\pi : \mathcal{A} \cong \mathcal{B}$.

Almost The Same...

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Theorem (Folklore)

Let $\mathcal{S}$ be a `natural` *computable structure. Then $\mathcal{S}$ is computably categorical if and only if $\mathcal{S}$ is relatively computably categorical.*

Proof.

Let $\mathcal{S}$ be any computable vector space, computable equivalence structure (Calvert, Cenzer, Harizanov, and Morozov 2006), computable linear order (Dzgoev and Goncharov 1980), computable Boolean algebra (Remmel 1981), computable torsion-free abelian group (Goncharov, Lempp, and Solomon 2003), computable tree of finite height and type (Lempp, McCoy, Miller, and Solomon 2005), and so on. □

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Let $\mathcal{S}$ be a computable 2-decidable structure. Then $\mathcal{S}$ is computably categorical if and only if it is relatively computably categorical.

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Theorem (Downey, Kach, Lempp, and Turetsky)

Let $\mathcal{S}$ be a computable 1-decidable structure. Then $\mathcal{S}$ is relatively Δ^0_2-categorical if it is computably categorical.

But Not Quite...

Theorem (Goncharov 1977)

There is a computable structure $\mathcal{S}$ that is computably categorical but not relatively computably categorical.

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Proof.

In order to defeat the family Φ_0:

- Build a vertex u with loop of size $3n$ and a vertex v with loops of size $3n$ and $3n+1$.
- Wait for $\varphi \in \Phi_0$ with $\mathcal{A} \models \varphi(v)$.
- Add loop of size $3n+2$ to v.
- Wait for $\mathcal{M}_0$ to show the loop of size $3n+2$ on v.
- Add loop of size $3n+1$ to u.

Note that Φ_0 cannot isolate the orbits of tuples (singletons) as now $\mathcal{A} \models \varphi(u) \wedge \varphi(v)$ but u and v are not automorphic. □

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Theorem (Kudinov 1996)

There is a 1*-decidable computable structure* $\mathcal{S}$ *that is computably categorical but not relatively computably categorical.*

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There is a 1-decidable computable structure $\mathcal{S}$ that is computably categorical but not relatively computably categorical.

Theorem (Chisholm, Fokina, Goncharov, Harizanov, Knight, and Quinn)

For every computable ordinal α, there is a computable structure $\mathcal{S}$ that is Δ^0_α-categorical but not relatively Δ^0_α-categorical.

Outline

Relative Δ^0_α-Categoricity Is Well-Behaved

Theorem (Ash 1987)

For a computable structure $\mathcal{S}$, the following are equivalent:

- *The structure $\mathcal{S}$ is relatively Δ^0_α-categorical.*
- *The orbits are effectively isolated by Σ^c_α-formulas: There is a c.e. family Φ of Σ^c_α-formulas (over some fixed parameter $\overline{c}$) such that each $\overline{a} \in S$ satisfies some $\varphi \in \Phi$, and if $\overline{a}, \overline{b} \in S$ both satisfy the same $\varphi \in \Phi$ then they are automorphic.*
- *The Σ^c_α-types are effectively isolated by Σ^c_α-formulas: There is a c.e. family Φ of Σ^c_α-formulas (over some fixed parameter $\overline{c}$) such that each $\overline{a} \in S$ satisfies some $\varphi \in \Phi$, and if $\overline{a}, \overline{b} \in S$ both satisfy the same $\varphi \in \Phi$ then there Σ^c_α-types coincide.*

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Theorem (Khoussainov and Shore 1998)

If $\mathcal{A}$ is relatively Δ^0_α-categorical, then so is $(\mathcal{A}; \overline{a})$ for any $\overline{a} \in A$.

An Easy Proof...

Corollary (Downey, Kach, Lempp, and Turetsky)

The index set complexity of the relatively computably categorical structures is Σ^0_3*-complete.*

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Proof.

A $\mathbb{Q}$-vector space is relatively computably categorical if and only if it is finite dimensional. Fix a Σ^0_3-set S. Build a uniformly computable sequence of structures $\{\mathcal{V}_i\}_{i \in \mathbb{N}}$ such that $\mathcal{V}_i$ is finite dimensional if and only if $i \in S$.

Given a Σ^0_3-predicate $\exists s \exists^\infty t \, R(s,t)$, view each s as controlling the sth (limit) basis element. Every time $R(s,t)$ holds, trash the current sth column into the 0th column. □

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Theorem (Folklore)

A computable structure $\mathcal{S}$ is computably categorical if and only if it is computably categorical.

Proof.

- Cholak, Goncharov, Khoussainov, and Shore 1999.
- Csima, Khoussainov, and Liu 2008.
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Theorem (Downey, Kach, Lempp, Lewis, Montalbán, Turetsky)

The index set complexity of the computably categorical structures is Π^1_1-complete.

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Identifying a Computable Isomorphism...

Remark

A priori, if $\mathcal{M}_i$ and $\mathcal{M}_j$ are computable presentations of a computably categorical structure $\mathcal{S}$, the oracle $\mathbf{0}''$ suffices to find a computable isomorphism $\Phi_e : \mathcal{M}_i \cong \mathcal{M}_j$.

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The oracle $\mathbf{0}'$ is both necessary and suffices for the prototypical computably categorical structure that is not relatively computably categorical. Here, even with any (bounded) amount of finite nonuniform information, the oracle $\mathbf{0}'$ is required.

(Weakly) Uniform Computable Categoricity...

Definition (Ventsov 1992)

A computable structure $\mathcal{S}$ is *weakly uniformly computably categorical* if there is a partial computable operator Ψ such that $\Psi(i, j) : \mathcal{M}_i \cong \mathcal{M}_j$ whenever $\mathcal{M}_i$ and $\mathcal{M}_j$ are computable presentations of $\mathcal{S}$.

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Definition (Kudinov 1996)

A computable structure $\mathcal{S}$ is *uniformly computably categorical* if there is a partial computable operator Ψ such that $\Psi(\mathcal{M}_i, \mathcal{M}_j) : \mathcal{M}_i \cong \mathcal{M}_j$ whenever $\mathcal{M}_i$ and $\mathcal{M}_j$ are computable presentations of $\mathcal{S}$, i.e., the function $x \to \Psi(\mathcal{M}_i, \mathcal{M}_j; x)$ is an isomorphism from $\mathcal{M}_i$ to $\mathcal{M}_j$.

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Definition (Ventsov 1992; Kudinov 1996)

A computable structure $\mathcal{S}$ is *(weakly) uniformly computably categorical with parameters* if $(\mathcal{A}; \overline{a})$ is (weakly) uniformly computably categorical for some $\overline{a} \in A$.

Uniformity Matters...

Theorem (Ash, Knight, and Slaman 1993)

A computable structure $\mathcal{S}$ is relatively computably categorical if and only if it is uniformly computably categorical.

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Remark

Thus, the a priori $\mathbf{0}''$-computable question of finding an isomorphism $\pi : \mathcal{M}_i \cong \mathcal{M}_j$ separates these various notions.

Summarizing: $RCC \iff UCC \implies WUCC \implies CC$.

Indeed, if $\mathcal{S}$ is rigid, then $RCC \iff UCC \iff WUCC \implies CC$.

Outline

The Theorems...

Theorem (Downey, Kach, Lempp, Lewis, Montalbán, Turetsky)

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For every computable ordinal α, there is a computable structure $\mathcal{S}$ that is computably categorical but not relatively Δ^0_α-categorical.

Feferman and Spector's $\mathcal{O}^*$...

Definition (Feferman and Spector 1962)

There is a partial order $\mathcal{O}^* = (\mathcal{O}^*; \preccurlyeq)$ with the $\preccurlyeq$-relation c.e. and:

- For all $\alpha \in \mathcal{O}^*$, the set $\{\beta \preccurlyeq \alpha\}$ is linearly ordered and has no infinite hyperarithmetic descending sequence.
- The set $\mathcal{O}^*$ has a $\preccurlyeq$-least element. The set of successor and limit elements and the predecessor function are computable.
- The set of $\alpha \in \mathcal{O}^*$ for which $\{\beta \in \mathcal{O}^* : \beta \preccurlyeq \alpha\}$ is well-ordered is isomorphic to $\mathcal{O}$.
- There is a computable sequence $\{\alpha_n \in \mathcal{O}^* : n \in \mathbb{N}\}$ such that the set $\{n \in \mathbb{N} : \alpha_n \in \mathcal{O}\}$ is Π^1_1-complete.

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- There is a computable sequence $\{\alpha_n \in \mathcal{O}^* : n \in \mathbb{N}\}$ such that the set $\{n \in \mathbb{N} : \alpha_n \in \mathcal{O}\}$ is Π^1_1-complete.

Remark

It does little harm to imagine $\mathcal{O}^*$ as being a computable presentation of $\omega_1^{CK} \cdot (1 + \eta)$ with no hyperarithmetic descending sequences and with computable successor and limit elements and predecessor function.

Proof of Index Set Complexity...

Theorem (Downey, Kach, Lempp, Lewis, Montalbán, Turetsky)

The index set complexity of the computably categorical structures is Π^1_1*-complete.*

Proof (Sketch).

Fix a Π^1_1-set S. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a computable sequence of elements of $\mathcal{O}^*$ such that $\alpha_n \in \mathcal{O}$ if and only if $n \in S$.

Let $\mathcal{C}_n := \mathcal{A}_{\alpha_n}$ (with $\mathcal{A}_{\alpha_n}$ discussed soon).

Then $\mathcal{C}_n$ is computably categorical if $\alpha_n \in \mathcal{O}$ by construction. If $\alpha_n \notin \mathcal{O}$, then an overspill argument (discussed later) yields that $\mathcal{C}_n$ is not computably categorical. □

Proof of Computable Categoricity...

Theorem (Downey, Kach, Lempp, Lewis, Montalbán, Turetsky)

For every computable ordinal α, there is a computable structure $\mathcal{S}$ that is computably categorical but not relatively Δ^0_α-categorical.

Remark

In fact, for each $\alpha \in \mathcal{O}^*$, we build a computable structure $\mathcal{A}_\alpha$. For $\alpha \in \mathcal{O}$, it will be the case that $\mathcal{A}_\alpha$ is computably categorical but not relatively Δ^0_α-categorical. For $\alpha \in \mathcal{O}^*$, it will be the case that $\mathcal{A}_\alpha$ is not computably categorical.

In order to prevent relative Δ^0_α-categoricity, rather than attempt to prevent the existence of a computably enumerable Scott family of Σ^c_α-formulas, it is easier to prevent the existence of any Scott family of Σ^{in}_α-formulas.

The Trees...

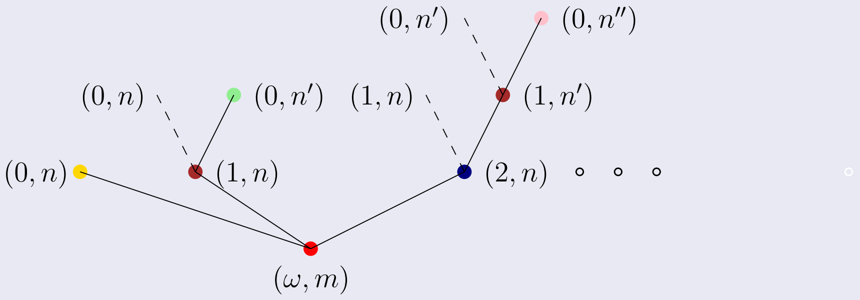
Remark

We thus design trees that have no such classical family. Unfortunately these trees, by their very nature, are not computably categorical. We therefore build trees that are similar enough to the designed trees to prevent relative Δ^0_α-categoricity while different enough to allow for computable categoricity.

Reiterating, for each $\alpha \in O^*$, we build a presentation $\mathcal{A}_\alpha$ of a computable structure. If $\alpha \in \mathcal{O}$, then $\mathcal{A}_\alpha$ is computably categorical and not relatively Δ^0_α-categorical. If $\alpha \in \mathcal{O}^* \setminus \mathcal{O}$, then $\mathcal{A}_\alpha$ is not computably categorical.

Remark

For $\alpha = \omega$, we illustrate the basic trees $\mathrm{T}_{(\omega,m)}$ and $\mathrm{T}_{(\omega,m,(k,n))}$.



The Basic Tree $\mathrm{T}_{(\omega,m)}$

Note the presence and absence of *height labels* and *marker labels*.

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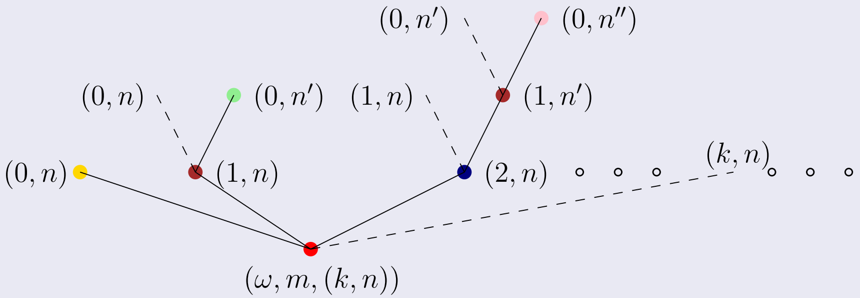

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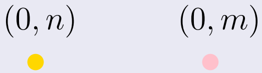
Increasing Type Similarity...

Remark

The reason we use these trees is that, as α increases, more and more quantifiers are necessary to differentiate the types of the trees $T_{\alpha,n}$ and $T_{\alpha,m}$.



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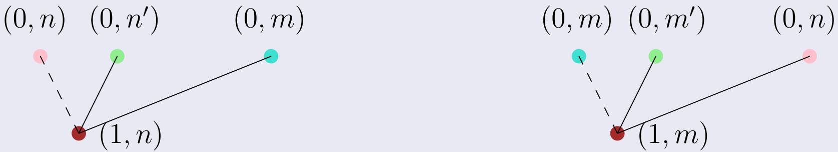

The Expanded Trees...

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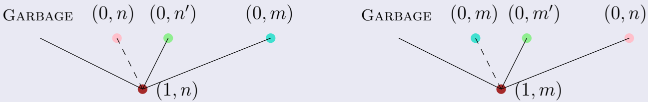
Unfortunately, the basic tree is not computably categorial. The idea is that garbage can be added to the tree while maintaining the failure of relative Δ^0_α-categoricity provided the garbage is deposited uniformly.



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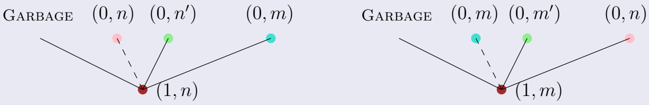

The challenge, therefore, is to construct an appropriate expansion of the basic tree in which garbage is deposited uniformly.

Remark

The construction maintains a global *bag of (temporary) labels*. At every (finite) stage, these *temporary labels* distinguish elements of the tree from each other. There purpose is to provide a matching of the elements in the structure under construction with the elements of the structure $\mathcal{M}_i$.

The Construction...

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The construction maintains a global *bag of (temporary) labels*. At every (finite) stage, these *temporary labels* distinguish elements of the tree from each other. There purpose is to provide a matching of the elements in the structure under construction with the elements of the structure $\mathcal{M}_i$.

In the limit, every element of the basic tree will share the same temporary labels (the infinitely many in the bag). The garbage will share only finitely many of these temporary labels. In addition, every garbage element will have a unique (modulo copies) temporary label not found anywhere else.

The Overspill...

Remark

Let $\sigma = \langle(\alpha, 0), (\beta, n)\rangle$ and $\sigma' = \langle(\alpha, 0), (\beta, n')\rangle$. Then $\mathcal{A}_\alpha[\sigma'/\sigma]$, the structure obtained by replacing the tree above σ with the tree above σ', is isomorphic to $\mathcal{A}_\alpha$ if and only if either $n = n'$ or $\beta \in \mathcal{O}^* \setminus \mathcal{O}$.

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Let B be the set of all successor $\beta \in \mathcal{O}^*$ such that for all σ and σ' with $\sigma = \langle(\alpha, 0), (\beta, 1)\rangle$ and $\sigma' = \langle(\alpha, 0), (\beta, 2)\rangle$, there is no computable isomorphism between $\mathcal{A}_\sigma$ and $\mathcal{A}_{\sigma'}$.

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- The set B is arithmetically definable.
- The set B contains all the well-ordered β.
- Since $\mathcal{O}^*$ has no hyperarithmetic descending sequences, there must be a non-well-ordered $\beta_0 \in B$.

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Let B be the set of all successor $\beta \in \mathcal{O}^*$ such that for all σ and σ' with $\sigma = \langle(\alpha, 0), (\beta, 1)\rangle$ and $\sigma' = \langle(\alpha, 0), (\beta, 2)\rangle$, there is no computable isomorphism between $\mathcal{A}_\sigma$ and $\mathcal{A}_{\sigma'}$.

- The set B is arithmetically definable.
- The set B contains all the well-ordered β.
- Since $\mathcal{O}^*$ has no hyperarithmetic descending sequences, there must be a non-well-ordered $\beta_0 \in B$.

Let $\sigma := \langle(\alpha, 0), (\beta_0, 1)\rangle$ and $\sigma' := \langle(\alpha, 0), (\beta_0, 2)\rangle$. Then $\mathcal{A}_\alpha$ and $\mathcal{A}_\alpha[\sigma'/\sigma]$ are isomorphic but not computably so.

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