# The Smiley Face Theorem
## Lindström's First Theorem

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Slides available at
http : //www.math.wisc.edu/ kach/lindstrom

Based on Chapter XIII of *Mathematical Logic* by H.-D. Ebbinghaus, J. Flum, and W. Thomas.

**Lindström's First Theorem:**

Every regular logical system $\mathcal{L}$ with $\mathcal{L}_I \leq \mathcal{L}$ satisfying LoSko($\mathcal{L}$) and Comp($\mathcal{L}$) is equally strong as $\mathcal{L}_I$, i.e.

$$\Big(\mathcal{L}_I \leq \mathcal{L} \bigwedge \text{LoSko}(\mathcal{L}) \bigwedge \text{Comp}(\mathcal{L})\Big) \longrightarrow \mathcal{L}_I \sim \mathcal{L}.$$

**Corollary:**

Either Comp($\mathcal{L}$) or LoSko($\mathcal{L}$) fails to hold for any regular logical system strictly stronger than first-order logic.

**Defn:** A **logical system** $\mathcal{L}$ is a function $L$ and binary relation $\models_{\mathcal{L}}$ satisfying the properties:

1. If $S_0 \subseteq S_1$, then $L(S_0) \subseteq L(S_1)$.

2. If $\mathcal{A} \models_{\mathcal{L}} \varphi$, then, for some $S$, $\mathcal{A}$ is an $S$-structure and $\varphi \in L(S)$.

3. If $\mathcal{A} \models_{\mathcal{L}} \varphi$ and $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{B} \models_{\mathcal{L}} \varphi$.

4. If $S_0 \subseteq S_1$ and $\varphi \in L(S_0)$ and $\mathcal{A}$ is an $S_1$-structure, then

$$\mathcal{A} \models_{\mathcal{L}} \varphi \quad \text{iff} \quad \mathcal{A}|_{S_0} \models_{\mathcal{L}} \varphi.$$

We interpret $L(S)$ as the set of $S$-sentences of $\mathcal{L}$ and $\models_{\mathcal{L}}$ as the satisfaction relation.

**Defn:** We say a logical system $\mathcal{L}$ is **regular** if the properties $\mathbf{Boole}(\mathcal{L})$, $\mathbf{Rel}(\mathcal{L})$, and $\mathbf{Repl}(\mathcal{L})$ are satisfied, where Boole($\mathcal{L}$) is an abbreviation for

> "$\mathcal{L}$ contains propositional connectives",

where Rel($\mathcal{L}$) is an abbreviation for

> "$\mathcal{L}$ permits relativization",

and where Repl($\mathcal{L}$) is an abbreviation for

> "$\mathcal{L}$ permits replacement of function and constant symbols by relation symbols".

**Defn:** We say that the Compactness Theorem holds for $\mathcal{L}$, written $\mathbf{Comp}(\mathcal{L})$, provided

> If $\Phi \subseteq L(S)$ and every finite subset of $\Phi$ is satisfiable, then $\Phi$ is satisfiable.

**Defn:** We say that the Löwenheim-Skolem Theorem holds for $\mathcal{L}$, written $\mathbf{LoSko}(\mathcal{L})$, provided

> If $\varphi \in L(S)$ is satisfiable, then there is a model of $\varphi$ whose domain is at most countable.

**Defn:** If $\mathcal{L}$ is a logical system and $\varphi \in L(S)$, define

$$\mathbf{Mod}^{\mathbf{S}}_{\mathcal{L}}(\varphi) := \{\mathcal{A} \mid \mathcal{A} \text{ is an } S\text{-structure and } \mathcal{A} \models_{\mathcal{L}} \varphi\}.$$

**Defn:** Let $\mathcal{L}$ and $\mathcal{L}'$ be two logical systems.

1. Let $\varphi \in L(S)$ and $\psi \in L'(S)$ for some set $S$. Then we say $\varphi$ and $\psi$ are **logically equivalent** if $\text{Mod}^S_{\mathcal{L}}(\varphi) = \text{Mod}^S_{\mathcal{L}'}(\psi)$.

2. If for every $S$ and every $\varphi \in L(S)$ there is a $\psi \in L'(S)$ such that $\varphi$ and $\psi$ are logically equivalent, then we say $\mathcal{L}'$ is **at least as strong as** $\mathcal{L}$ and write $\mathcal{L} \leq \mathcal{L}'$.

3. We say $\mathcal{L}$ and $\mathcal{L}'$ are **equally strong** and write $\mathcal{L} \sim \mathcal{L}'$ if $\mathcal{L} \leq \mathcal{L}'$ and $\mathcal{L}' \leq \mathcal{L}$.

Let $\mathcal{L}$ be a regular logical system with $\mathcal{L}_I \leq \mathcal{L}$.

**Lemma:** Suppose Comp($\mathcal{L}$) and $\Phi \cup \{\varphi\} \subset L(S)$ with $\Phi \models_{\mathcal{L}} \varphi$. Then there is a finite subset $\Phi_0$ of $\Phi$ such that $\Phi_0 \models_{\mathcal{L}} \varphi$.

**Lemma:** Suppose Comp($\mathcal{L}$) and $\psi \in L(S)$. Then there is a finite subset $S_0$ of $S$ such that for all $S$-structures $\mathcal{A}$ and $\mathcal{B}$

If $\mathcal{A}\,|_{S_0} \cong \mathcal{B}\,|_{S_0}$, then ($\mathcal{A} \models_{\mathcal{L}} \psi$ iff $\mathcal{B} \models_{\mathcal{L}} \psi$).

**Defn:** Two $S$-structures $\mathcal{A}$ and $\mathcal{B}$ are $m$-isomorphic, written $\mathcal{A} \cong_m \mathcal{B}$, if there is a sequence $I_0, \ldots, I_m$ of non-empty sets of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$ with the property

> For $0 \leq n \leq m-1$ and $p \in I_{n+1}$ and $a \in A$ (resp. $b \in B$), there is $q \in I_n$ such that $p \subset q$ and $a \in \text{dom}(q)$ (resp. $b \in \text{rg}(q)$).

**Lemma:** Let $S$ be a relational symbol set and $\psi$ be an $L(S)$-sentence not logically equivalent to any sentence in first-order logic. Then, for every finite $S_0 \subset S$ and every $m \in \mathbb{N}$, there are $S$-structures $\mathcal{A}$ and $\mathcal{B}$ such that

$$\mathcal{A}\,|_{S_0} \cong_m \mathcal{B}\,|_{S_0}, \quad \mathcal{A} \models_{\mathcal{L}} \psi, \quad \text{and } \mathcal{B} \models_{\mathcal{L}} \neg\psi.$$

1. $\forall p(Pp \longrightarrow \forall x \forall y(Gpxy \longrightarrow (Ux \land Vy)))$

2. $\forall p(Pp \longrightarrow \forall x \forall x' \forall y \forall y'$
   $((Gpxy \land Gpx'y') \longrightarrow (x \equiv x' \longleftrightarrow y \equiv y')))$

3. $\forall p(Pp \longrightarrow \forall \bar{x} \forall \bar{y}$
   $((Gpx_1 y_1 \land \cdots \land Gpx_n y_n) \longrightarrow (R\bar{x} \longleftrightarrow R\bar{y})))$

4. $\Phi_{\mathsf{pord}}$

5. $\forall x(Wx \longleftrightarrow (x \equiv c \lor \exists y(y < x \lor x < y)))$
   $\land \forall x(Wx \longrightarrow (x < c \lor x \equiv c))$

6. $\forall x(\exists y y < x \longrightarrow$
   $(fx < x \land \neg \exists z(fx < z \land z < x)))$

7. $\forall x(Wx \longrightarrow \exists p(Pp \wedge Ixp))$

8. $\forall x \forall p \forall u((fx < x \wedge Ixp \wedge Uu) \longrightarrow$
$\exists q \exists v(Ifxq \wedge Gquv \wedge \forall x' \forall y'(Gpx'y' \longrightarrow Gqx'y')))$

9. $\forall x \forall p \forall v((fx < x \wedge Ixp \wedge Vv) \longrightarrow$
$\exists q \exists u(Ifxq \wedge Gquv \wedge \forall x' \forall y'(Gpx'y' \longrightarrow Gqx'y')))$

10. $\exists x Ux \wedge \exists y Vy \wedge \psi^U \wedge (\neg \psi)^V$

**Main Lemma:** Let $\mathcal{L}$ be a regular logical system with $\mathcal{L}_I \leq \mathcal{L}$ and LoSko($\mathcal{L}$). Furthermore, let $S$ be a relational symbol set, and let $\psi$ be an $L(S)$-sentence which is not logically equivalent to any sentence in first-order logic. Then one of the following holds:

1. For all finite symbol sets $S_0$ with $S_0 \subset S$, there are $S$-structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \models_{\mathcal{L}} \psi$, $\mathcal{B} \models_{\mathcal{L}} \neg\psi$, and $\mathcal{A} \restriction_{S_0} \cong \mathcal{B} \restriction_{S_0}$.

2. For a unary relation symbol $W$ and a suitable symbol set $S^+$ with $S \cup \{W\} \subset S^+$ and finite $S^+ \backslash S$, there is a $L(S^+)$-sentence $\chi$ such that

   2a. In every model $\mathcal{C}$ of $\chi$, $W^C$ is finite and nonempty.

   2b. For every $m \geq 1$, there is a model $\mathcal{C}$ of $\chi$ in which $W^C$ has exactly $m$ elements.

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**Proof:**

Assume otherwise, namely that there is some $\psi \in L(S)$ not logically equivalent to any first-order sentence in $L_I(S)$. As $\mathcal{L}$ is regular, we may assume that $S$ contains only relational symbols. As Comp($\mathcal{L}$) holds, there is a finite set $S_0 \subseteq S$ such that

$$\text{If } \mathcal{A}\,|_{S_0} \cong \mathcal{B}\,|_{S_0}, \text{ then } (\mathcal{A} \models_{\mathcal{L}} \psi \text{ iff } \mathcal{B} \models_{\mathcal{L}} \psi).$$

The above contradicts the [1] of the Main Lemma, so [2] must hold. Then we have

$$\chi \cup \{\text{“}|W| \geq n\text{''} \,:\, n \in \mathbb{N}\}$$

is finitely satisfiable (c.f. [2a]) but isn't satisfiable (c.f. [2b]), a contradiction to Comp($\mathcal{L}$).

**Lindström's Second Theorem:**

Let $\mathcal{L}$ be an effective regular logical system such that $\mathcal{L}_I \leq_{\text{eff}} \mathcal{L}$. If LoSko($\mathcal{L}$) and the set of valid sentences is enumerable for $\mathcal{L}$, then $\mathcal{L}_I \sim_{\text{eff}} \mathcal{L}$.