Characterizing the Computable Structures: Boolean Algebras and Linear Orders

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Acknowledgements

Remark

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Outline

Definition

A countable structure (with finite signature) is computable *if its universe can be identified with ω in such as way as to make the functions and relations on it computable.*

Computable Model Theory

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Remark

Here we will be considering two specific classes of structures: Boolean algebras and linear orders. We view a Boolean algebra *as a structure in the signature $\mathcal{B} = (B : +, \cdot, -, 0, 1)$ and a* linear order *as a structure in the signature $\mathcal{L} = (L : \prec)$.*

The Feiner Hierarchy

Definition

For $n \in \omega$, define $\emptyset^{(\leq n)}$ to be the set

$$\emptyset^{(\leq n)} = \{\langle k, m \rangle : m \in \emptyset^{(k)}, k \leq n\}.$$

Definition ([1])

Let $S \subseteq \omega$ be a set computable in $\emptyset^{(\omega)}$. Then S is (a, b) in the Feiner hierarchy *if there exists an index e such that*

1. *The function $\varphi_e^{\emptyset^{(\omega)}}$ is total and is the characteristic function of S, i.e., $\varphi_e^{\emptyset^{(\omega)}}(n) = \chi_S(n)$ for all n.*
2. *The computations $\varphi_e^{\emptyset^{(\leq bn+a)}}(n)$ and $\varphi_e^{\emptyset^{(\omega)}}(n)$ are identical; in particular, the latter queries $\emptyset^{(\omega)}$ on no number $\langle k, m \rangle$ with $k > bn + a$.*

Outline

Shuffle Sums

Definition

Let $S = \{\mathcal{L}_x\}$ be a countable set of linear orders. Then the shuffle sum of S, denoted $\sigma(S)$, is the linear order obtained by partitioning the rationals $\mathbb{Q}$ into $|S|$ many dense sets $\{Q_x\}$ and replacing each point $q \in Q_x$ with the linear order $\mathcal{L}_x$.

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Definition

If $\mathcal{L} = (L : \prec)$ is a linear order and $\mathcal{L}_a = (L_a : \prec_a)$ is a linear order for each $a \in L$, the lexicographic sum of $\mathcal{L}$ and $\{\mathcal{L}_a\}_{a \in L}$ *is the linear order with universe $\{(a, b) : a \in L, b \in L_a\}$ under the lexicographic order induced by $\prec$ and $\prec_a$.*

Definition

The shuffle sum *of a set $S = \{\mathcal{L}_x\}_{x \in \omega}$ is the linear order $\sum_{a \in \mathbb{Q}} \mathcal{L}_a$, where $\mathcal{L}_a$ is the linear order $\mathcal{L}_x$ if $a \in Q_x$.*

Definition

A set $S \subseteq \omega + 1$ *is a limit infimum set, written* LIMINF *set, if there is a total computable function* $g(x,s) : \omega \times \omega \to \omega$ *such that the function*

$$f(x) = \liminf_s g(x,s)$$

enumerates S*. Here we use the convention that* $\liminf_s g(x,s) = \omega$ *if* $\lim_s g(x,s) = \infty$.

Definition ([2])

A set $S \subseteq \omega + 1$ *is a limitwise monotonic set relative to* $(\mathbf{0}')$*, written* LimMon $(\mathbf{0}')$ *set, if there is a total* $(\mathbf{0}')$*-computable function* $\tilde{g}(x,t) : \omega \times \omega \to \omega$ *satisfying*

$$\tilde{g}(x,t) \leq \tilde{g}(x,t+1) \qquad \text{for all } x \text{ and } t$$

such that the function

$$\tilde{f}(x) = \lim_t \tilde{g}(x,t).$$

enumerates S*. Again we use the convention that* $\lim_t \tilde{g}(x,t) = \omega$ *if* $\lim_t \tilde{g}(x,t) = \infty$*.*

Shuffle Sums: The Main Result

Theorem (K)

For sets $S \subseteq \omega + 1$, the following are equivalent:

1. *The shuffle sum $\sigma(S)$ is computable.*
2. *The set S is a* LimInf *set.*
3. *The set S is a* LimMon $(\mathbf{0}')$ *set.*

Proof of (1) implies (2)

Proof (Sketch).

From a computable presentation of $\sigma(S)$, define a computable function $g(x, s)$ as the "sum" of the number of points to the left of x in the block of x, one, and the number of points to the right of x in the block of x.

As the block of x is not computable, we guess that it extends from the point last enumerated to the left of x to the point last enumerated to the right of x (exclusive).

Verify that this approximation works, separating the case when the block size of x is finite from when the block size is infinite. □

Proof of (2) implies (1)

Proof (Sketch).

From a function g witnessing that S is a LIMINF set, build infinitely many copies of the linear order $g(x, s)$ at all rationals in the set Q_x.

When the value of $g(x, s)$ increases, add additional points. When the value of $g(x, s)$ decreases, dissassociate the extra points from the rational in $q \in Q_x$.

Prioritize the disassociated points so that they eventually become permanently associated to some other rational $q' \in Q_{x'}$. □

Proof of (2) if and only if (3)

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From a computable function $g(r, s)$ witnessing that S is a LIMINF set, define a $(\mathbf{0}')$-computable function $\tilde{g}(r, t)$ witnessing that S is a LIMMON $(\mathbf{0}')$ set.

Proof of (2) if and only if (3)

Proof (Sketch).

From a computable function $g(r,s)$ witnessing that S is a LimInf set, define a $(\mathbf{0}')$-computable function $\tilde{g}(r,t)$ witnessing that S is a LimMon $(\mathbf{0}')$ set.

Conversely, from a $(\mathbf{0}')$-computable function $\tilde{g}(r,t)$ witnessing that S is a LimMon $(\mathbf{0}')$ set, define a computable function $g(r,s)$ witnessing that S is a LimInf set. □

Outline

Measures

Definition ([5])

A measure σ is a map from the countable atomless Boolean algebra $\mathcal{F}$ to the countable ordinals satisfying $\sigma(x + y) = \max\{\sigma(x), \sigma(y)\}$.

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Remark

By associating the countable atomless Boolean algebra with finite unions of cones of $2^{<\omega}$, a measure can be viewed as a map $\sigma : 2^{<\omega} \rightarrow \omega_1$ satisfying $\sigma(\tau) = \max\{\sigma(\tau^\frown 0), \sigma(\tau^\frown 1)\}$. Under this interpretation, a measure can be thought of as a labelled binary tree.

Measure Derivatives

Definition ([5])

If $\sigma : \mathcal{F} \to \omega_1$ *is a measure, define maps* $\Delta^\alpha\sigma$ *with domain* $\mathcal{F}$ *for* $\alpha < \omega_1$ *recursively by setting* $\Delta^0\sigma = \sigma$,

$$\Delta^{\alpha+1}\sigma(x) = \{(\Delta^\alpha\sigma(x_1), \ldots, \Delta^\alpha\sigma(x_n)) : x = x_1 \oplus \cdots \oplus x_n\},$$

and $\Delta^\gamma\sigma(x)$ *as the inverse limit of* $\Delta^\beta\sigma(x)$ *for* $\beta < \gamma$.

The set $\Delta^\alpha\sigma(1_\mathcal{B})$ *is the* α^{th} derivative of $\mathcal{B}_\sigma$.

Classical Depth Zero

Theorem (K)

For each set $S \subseteq \omega_1$ satisfying $|S| = 1$, there is exactly one depth zero Boolean algebra with range S, namely $B_{u(S)} = B_{v(S)}$.

For each set $S \subseteq \omega_1$ with greatest element satisfying $|S| > 1$, there are exactly two depth zero Boolean algebras with range S, namely $B_{u(S)}$ and $B_{v(S)}$.

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Proof (Sketch).

Show the existence of at least two, then of at most two. For the former, define the Boolean algebras $\mathcal{B}_{u(\alpha+1)}$ and $\mathcal{B}_{v(\alpha+1)}$ by induction on α. For the latter, either use pseudo-indecomposability and primitiveness or appeal directly to the depth zero definition. □

Depth One, Rank ω

Proposition (K)

There are continuum many depth one, rank ω Boolean algebras with range $\omega + 1$.

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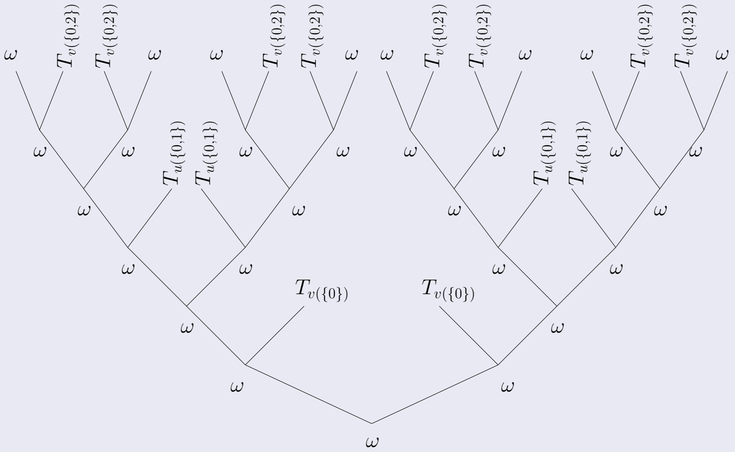
Proof (Sketch).

Code subsets of the positive integers into a Boolean algebra $\mathcal{B}^S$. Have $\sigma_{u(\{0,n\})}$ be a subalgebra of $\mathcal{B}^S$ if and only if $n \in S$; have $\sigma_{v(\{0,n\})}$ be a subalgebra of $\mathcal{B}^S$ if and only if $n \notin S$. □

Depth One, Rank ω Example

Example



The measure σ^S *when* $S = \{1, 3, 5, \dots\}$

Depth ω, Rank One

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There are continuum many depth ω, rank one measures.

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Proof (Sketch).

Define a map π from the space of uniform Boolean algebras to the space of uniform rank one Boolean algebras. The algebra $\pi(\mathcal{B})$ is the algebra generated by the (any) characteristic function of a subset of the rationals whose clopen algebra is $\mathcal{B}$.

Argue that $\pi(\mathcal{B}_{u(S)})$ and $\pi(\mathcal{B}_{v(S)})$ are (at most) depth ω for sets $S \subseteq \omega + 1$. □

Depth ω, Rank One Example

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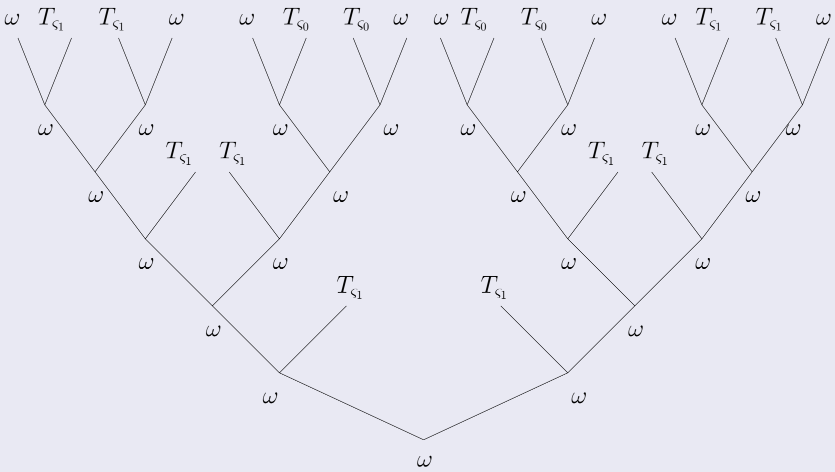

The measure $\pi(\sigma_{u(2)})$

Effective Boolean Algebras: The Main Results

Proposition (K)

If σ is a depth zero measure of rank at most $\lambda < \omega_1^{\mathrm{CK}}$, then σ is computable (i.e., there is a computable measure σ' such that $\mathcal{B}_\sigma = \mathcal{B}_{\sigma'}$) if and only if $\Delta\sigma(1_{\mathcal{B}})$ is computably enumerable.

Moreover, from an index for either σ or $\Delta\sigma(1_{\mathcal{B}})$, an index for the other can be given uniformly.

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Moreover, from an index for either σ or $\Delta\sigma(1_\mathcal{B})$, an index for the other can be given uniformly.

Theorem (K)

Let $S \subseteq \omega + 1$ be a set with greatest element. Then the following are equivalent:

1. *The Boolean algebra $\mathcal{B}_{u(S)}$ is computable.*
2. *The Boolean algebra $\mathcal{B}_{v(S)}$ is computable.*
3. *The set S is $(2, 2)$ in the Feiner hierarchy.*

Proof of (1), (2) implies (3)

Proof (Sketch).

Uniformly in n, define Σ^0_{2n+3} sentences φ_n satisfying

$$\mathcal{B}_{u(S)}, \mathcal{B}_{v(S)} \models \varphi_n \qquad \text{if and only if} \qquad n \in S.$$

When defining these formulas, make use of the fact that there are formulas (uniform in α) of complexity $\Pi^0_{2\alpha+1}$ identifying whether an element is an α-atom. □

Proof of (3) implies (1), (2)

Proof (Sketch).

Assume without loss of generality that S is infinite.

Construct $\mathcal{B}_{u(S)}$ ($\mathcal{B}_{v(S)}$, respectively) from an index e witnessing that S is $(2, 2)$ in the Feiner hierarchy. Do so by building a linear order

$$\mathcal{L} = \sum_{\tau \in 2^{<\omega}} \mathcal{L}_\tau$$

and taking its interval algebra.

The linear order $\mathcal{L}_\tau$ depends on S and the value of $\sigma_{u(\omega+1)}(\tau)$ ($\sigma_{v(\omega+1)}(\tau)$, respectively). It is built by iterating the following technical lemma. □

Technical Lemma (Statement)

Lemma (K)

Uniformly in

- *a* Δ^0_3 *index for the atomic diagram* $D(\mathcal{A})$ *of a linear order* $\mathcal{A} = (A : \prec) = (\{a_0, a_1, \dots\} : \prec)$ *with distinguished first element*
- *and an index for a* Σ^0_3 *predicate* $\exists n \forall u \exists v R(n, u, v)$*,*

there is an index for a Δ^0_1 *linear order* $\mathcal{B}$ *such that* $\mathcal{B} \cong \sum_{a \in A} \mathcal{L}_a$*, where* $\mathcal{L}_{a_n} = 1 + \eta + \omega$ *if* $\forall u \exists v R(n, u, v)$ *and* $\mathcal{L}_{a_n} = \omega$ *otherwise.*

Technical Lemma (Proof)

Proof (Sketch).

An infinite injury argument using work of Thurber as an outline. Approximate the atomic diagram of $\mathcal{A}$ using the Limit Lemma twice, imposing (without loss of uniformity) constraints on the approximating functions.

Introduce chronological priorities and build each block as the sum of a singleton segment, a dense segment, and a discrete segment. As the approximations change, attach and detach blocks appropriately. □

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