Non-Standard Models of Arithmetic

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Abstract

Almost everyone, mathematician or not, is comfortable with the standard model $(\mathbb{N} : +, \cdot)$ of arithmetic. Less familiar, even among logicians, are the non-standard models of arithmetic. In this talk we prove their existence, explore their structure, discuss their uniqueness, and examine various model and computability theoretic properties they possess. As much as is reasonably possible, the history of their discovery and study will be integrated into the talk.

Slides are available online at

`http://www.math.wisc.edu/~kach/mathematics/nsmoa`

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Definitions and Notation

Throughout we will refer to $\mathcal{N}_0 = (\mathbb{N} : +, \cdot)$ as *the standard model of arithmetic*. Any model of $\mathrm{Th}(\mathcal{N}_0)$ not isomorphic to $\mathcal{N}_0$ will be termed *a non-standard model of arithmetic*, or more briefly a *non-standard model*.

An element $x \in \mathcal{M}$ will be called *finite* if $x \in \mathcal{N}_0$; otherwise x will be called *infinite*.

Working in a model $\mathcal{M}$, we will say that x is less than or equal to y, denoted $x \leq y$, if there is a $z \in \mathcal{M}$ such that $x + z = y$. We define $y \dot{-} x$ to be z if such a z exists and 0 if no such z exists.

We will use $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ to denote both the usual set and its order type. If α and β are order types, we will use $\alpha\beta$ to denote the order type obtained by replacing each element of β by a copy of α.

Existence of Non-Standard Models

Theorem: (Skolem, 1934 / 1955) There is a countable non-standard model of arithmetic.

Proof 1: (Idea) Letting $F = \{f_i : i \in \omega\}$ be the set of definable functions in $\mathcal{N}_0$, define a one-to-one increasing function g that allows an ordering to be put on equivalence classes of F. Define addition and multiplication to be pointwise and verify that $(F/\equiv : +/\equiv, \cdot/\equiv) \in \mathrm{Mod}(\mathrm{Th}(\mathcal{N}_0))$.

Proof 2: (Idea) After augmenting the language with a constant c, use the Compactness Theorem to show the consistency of an infinite number via the set of sentences $\Phi = \{c > \underline{n} : n \in \omega\}$.

Order Types of Non-Standard Models

Theorem: (Henkin, 1950) The order type of any non-standard model of arithmetic is of the form $\mathbb{N} + \mathbb{Z}\theta$ for some dense linear order θ without endpoints.

Proof: (Sketch) For denseness, between any two elements $a \ll b$, either $q = (a+b)/2$ or $q = (a+b+1)/2$ exists. In either case, it must be that $a \ll q \ll b$. Extend q to a $\mathbb{Z}$-chain by adding and subtracting finite integers. For unboundedness, for any infinite a, the element $2a$ must satisfy $a \ll 2a$.

Corollary: Any countable non-standard model of arithmetic has order type $\mathbb{N} + \mathbb{Z}\mathbb{Q}$.

Proof: Up to isomorphism, the only countable dense linear order without endpoints is $\mathbb{Q}$.

Continuum Many Countable Non-Standard Models

Theorem: There are exactly $2^{\aleph_0}$ non-isomorphic countable non-standard models of arithmetic.

Proof: (Sketch) Augment the language with an extra constant c. Let P be any set of (finite) prime numbers and let Φ_P be the set of sentences

$$\{p \mid c : p \in P\} \cup \{p \nmid c : p \notin P\}.$$

Use the Compactness Theorem to show consistency of Φ_P and Löwenheim - Skolem to get a countable model. Finally argue that if there were fewer than continuum many countable models, then not all types would be realised.

The Overspill Principle

Theorem: Let $\mathcal{M}$ be a non-standard model of arithmetic and let b be any element of M.

(Weak Overspill) Let $\varphi(x, y)$ have free variables x and y. Then

$$\forall x \in \omega[\mathcal{M} \models \varphi(x, b)]$$

if and only if

$$\exists a \text{ infinite } [\mathcal{M} \models (\forall x < a)\varphi(x, b)].$$

Proof: (Sketch) Show that the set of natural numbers $\omega \subset M$ is not definable (using parameters) in $\mathcal{M}$. For the non-trivial direction, if there was no such infinite a, use that ω is not definable to contradict that $\mathcal{M}$ was assumed to be a non-standard model.

The Overspill Principle (Continued)

(Strong Overspill) Let $\varphi(x, y, z)$ have free variables x, y, and z, and suppose $\varphi(x, y, b)$ defines a function $F : M \to M$. Then

$$\forall x \in \omega[F(x) \text{ is infinite}]$$

if and only if

$$\exists a \text{ infinite } \forall x < a[F(x) \text{ is infinite}].$$

Proof: For the non-trivial direction, apply Weak Overspill to the formula $\varphi(x)$ given by $Fx > x$.

Corollary: For any infinite integer a, there is an infinite integer c with $2^c < a$.

Proof: As 2^x is finite for all finite integers x, and hence smaller than a, there must be an infinite integer c with $2^c < a$ by Weak Overspill.

Order Type of the Reals Not Realised

Theorem: (Klaus Potthoff) There is no non-standard model of arithmetic with order type $\mathbb{N} + \mathbb{Z}\mathbb{R}$.

Proof: (Sketch) Assume there was a non-standard model of order type $\mathbb{N} + \mathbb{Z}\mathbb{R}$, and let a be any infinite element of it. Identify real numbers r with the corresponding $\mathbb{Z}$-chain, $\mathbb{Z}_r$. Let r_n for $n \in \omega$ be the $\mathbb{Z}_r$ in which the element na resides. As the r_n are increasing and bounded by the copy of $\mathbb{Z}$ in which the element a^2 resides, the sequence $\{r_n\}$ converges to some real number r. Let b be any element of $\mathbb{Z}_r$, choosing b smaller than the multiple of a in $\mathbb{Z}_r$ if one exists. Define $S = \{x : a|x \text{ and } x < b\}$. Then $\omega = \{n : na \in S\}$, a contradiction to ω not being definable in any non-standard model of arithmetic.

Extension Types

Let $\mathcal{M} \subseteq \mathcal{N}$ be models of arithmetic, not necessarily non-standard.

Definition: An element $a \in N - M$ is said to be $\mathcal{M}$-*infinite* if $a > b$ for all $b \in M$; otherwise it is said to be $\mathcal{M}$-*finite*.

Definition: If every element of $N - M$ is $\mathcal{M}$-infinite, then $\mathcal{N}$ is said to be an *end extension* of $\mathcal{M}$. We write $\mathcal{M} \subseteq_e \mathcal{N}$ in this case.

If every element of $N - M$ is $\mathcal{M}$-finite, then $\mathcal{N}$ is said to be a *cofinal extension* of $\mathcal{M}$. We write $\mathcal{M} \subseteq_c \mathcal{N}$ in this case.

If $N - M$ contains both $\mathcal{M}$-finite and $\mathcal{M}$-infinite integers, then $\mathcal{N}$ is said to be a *mixed extension* of $\mathcal{M}$. We write $\mathcal{M} \subseteq_m \mathcal{N}$ in this case.

Extension Existence

Theorem: Every non-standard model $\mathcal{M}$ of PA has a proper elementary mixed extension (trivial), a proper elementary end extension (MacDowell and Specker, 1961), and a proper elementary cofinal extension (Rabin, 1962).

Theorem: (Gaifman, 1971) Let $\mathcal{M} \subseteq \mathcal{N}$ be models of PA. Then there is a unique model $\hat{M}$ of PA such that $\mathcal{M} \subseteq_c \hat{\mathcal{M}} \subseteq_e \mathcal{N}$. Moreover, the cofinal extension $\mathcal{M} \subseteq_c \hat{\mathcal{M}}$ is elementary.

Corollary: Cofinal extensions are necessarily elementary.

Proof: Let $\mathcal{M} \subseteq \mathcal{N}$ be any cofinal extension. Then $\hat{\mathcal{M}} = \mathcal{N}$ by uniqueness and so the extension is elementary.

Recovering a Structure
From End Segments

Theorem: (Smoryński, 1977) Let $\mathcal{M}$ be a model of arithmetic with $M = I \cup E$, an initial segment and end segment. Then $\mathcal{M}$ can be completely recovered from the structure $\mathcal{E} = (E : +, \cdot)$.

Proof: (Idea) Similar to the construction of a field of quotients from an integral domain. In $\mathcal{E}$, define $x < y$, $S(x) = y$ (i.e. $x + 1 = y$), and $x|y$. With

$$M' = \{(a, b) \in E^2 : E \models b|a\},$$

argue that

$$\mathcal{M} \cong (M'/\equiv : +/\equiv, \cdot/\equiv),$$

where $\equiv$ is the equivalence relation given by $(a, b) \equiv (c, d)$ if and only if $ad = bc$.

Corollaries

Corollary: If $\mathcal{M}_1$ and $\mathcal{M}_2$ are models of arithmetic with isomorphic end segments, then $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic.

Corollary: The theory of non-standard parts of non-standard models of PA is Π^1_1 complete.

Corollary: The theory of end segments of non-standard models of PA is recursively axiomatizable.

A Number Theoretic Result

Theorem: (Rabin, 1962) For every non-standard model of arithmetic $\mathcal{M}$, there are parameters $a_{i_1 \dots i_n}$ for $0 \leq i_j \leq k$ in M such that the diophantine equation

$$\sum_{0 \leq i_j \leq k} a_{i_1 \dots i_n} t_1^{i_1} \dots t_n^{i_n} = 0$$

is not solvable in $\mathcal{M}$ but is solvable in some model of arithmetic $\mathcal{M}'$ with $\mathcal{M} \subset \mathcal{M}'$.

Remark: Note that a diophantine equation with parameters from the standard model is solvable in the standard model if and only if it is solvable in some non-standard model.

Standard Systems

Definition: Let $\mathcal{M}$ be a model of arithmetic. A set $X \subseteq \omega$ is *standard* in $\mathcal{M}$ if there is a formula $\varphi(x, \bar{y})$ and $\bar{b} \in \mathcal{M}$ such that $X = \{x \in \omega : \mathcal{M} \models \varphi(x, \bar{b})\}$. The *standard system* of $\mathcal{M}$ is the collection of standard sets in $\mathcal{M}$.

Proposition: If X is a standard set in $\mathcal{M}$, then X has arbitrarily small infinite codes.

Proof: Let X be defined by $\varphi(x, \bar{y})$ and $\bar{b}$. For any infinite integer a, let c be an infinite integer with $2^c < a$. Define

$$F(x) = \begin{cases} 2^x & \text{if } \varphi(x, \bar{b}) \\ 0 & \text{otherwise.} \end{cases}$$

Then $d = \sum_{i=0}^{c-2} F(x) + 2^{c-1} < 2^c < a$ is an infinite code for X.

Embeddability Results

Theorem: Let $\mathcal{M}$ and $\mathcal{N}$ be countable non-standard models of arithmetic.

- (Friedman) Then $\mathcal{M}$ is embeddable in $\mathcal{N}$ if and only if $\mathrm{SSym}(\mathcal{M}) \subseteq \mathrm{SSym}(\mathcal{N})$ and $\mathrm{Th}_{\exists}(\mathcal{M}) \subseteq \mathrm{Th}_{\exists}(\mathcal{N})$.
- (Friedman) Then $\mathcal{M}$ is isomorphic to an initial segment of $\mathcal{N}$ if and only if $\mathrm{SSym}(\mathcal{M}) = \mathrm{SSym}(\mathcal{N})$ and $\mathrm{Th}_{\Sigma_1}(\mathcal{M}) \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{N})$.
- (Wilkie, 1977) Then $\mathcal{M}$ is isomorphic to arbitrarily large initial segments of $\mathcal{N}$ if and only if $\mathrm{SSym}(\mathcal{M}) = \mathrm{SSym}(\mathcal{N})$ and $\mathrm{Th}_{\Pi_2}(\mathcal{M}) \subseteq \mathrm{Th}_{\Pi_2}(\mathcal{N})$.

Corollary: Any countable non-standard model of arithmetic is isomorphic to a proper initial segment of itself.

Random Theorems

Theorem: (Rabin, 1962) There is an ascending chain $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots$ of models of arithmetic such that the union $\mathcal{M} = \cup_i \mathcal{M}_i$ is not a model of arithmetic.

Corollary: There is no AE set $S \subset \mathrm{Th}(\mathcal{N}_0)$ such that $S \models \mathrm{Th}(\mathcal{N}_0)$.

Proof: The union of a ascending chain of models of an AE set S is a model of S.

Theorem: (Rabin, 1962) There is a non-standard model of arithmetic $\mathcal{M}$ with elementary submodels $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\mathcal{M}_1 \cap \mathcal{M}_2$ is not a model of arithmetic.

Theorem: (Knight, 1973, 1975) Let Σ be a complete type with respect to $\mathrm{Th}(\mathcal{N}_0)$ omitted in $\mathcal{N}_0$. Then for every cardinal κ, there is a model of $\mathrm{Th}(\mathcal{N}_0)$ with cardinality κ omitting the type Σ.

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