Shift Complex Sequences

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Outline

Terminology and Notation...

Definition

A *string* is a finite stream of binary digits, i.e., an element of $2^{<\omega}$.

An *infinite sequence* is an infinite stream of binary digits indexed by ω, i.e., an element of 2^{ω}.

A *bi-infinite sequence* is an infinite stream of binary digits indexed by ζ (the order type of the integers), i.e., an element of 2^{ζ}.

Definition

We identify *sets* (subsets of $\mathbb{N}$) with infinite sequences in the natural way.

Terminology and Notation...

Definition

Let $f : D \to \mathbb{N}$ and $g : D \to \mathbb{N}$ be two (total) functions. We write $f \leq^+ g$ if there is a constant $d \in \mathbb{N}$ such that $f(x) \leq g(x) + d$ for all $x \in D$. We write $f <^+ g$ if $f \leq^+ g$ and $g \not\leq^+ f$.

Remark

Note that the $\leq^+$ relation defines a pre-partial order on the space of functions with domain D.

Effective Dimension...

Definition

The effective Hausdorff dimension and the effective packing dimension of a real A are

$$dim(A) := \liminf \frac{K(A \upharpoonright n)}{n} \quad \text{and} \quad Dim(A) := \limsup \frac{K(A \upharpoonright n)}{n}$$

respectively, where $K(\sigma)$ denotes the prefix-free complexity of σ.

Shift Complex Sequences...

Definition

Fix a real $\delta \in [0, 1]$. A set A is δ-*shift complex* if $K(\sigma) \geq^+ \delta |\sigma|$ for $\sigma \subset A$, i.e., if there is an integer $b \in \mathbb{N}$ such that

$$K(\sigma) \geq \delta |\sigma| - b$$

for all (not necessarily initial segment) substrings $\sigma \subset A$.

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Definition

A set A is *shift complex* if there is a real $\delta > 0$ such that A is δ-shift complex.

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Definition

A set A is *shift complex* if there is a real $\delta > 0$ such that A is δ-shift complex.

A set A is *exactly δ-shift complex* if A is δ-shift complex but not δ'-shift complex for any $\delta' > \delta$.

A set A is *almost δ-shift complex* if A is δ'-shift complex for all $\delta' < \delta$ but not δ-shift complex.

Preliminary Results...

Proposition

No 1-random real is shift complex.

Proof.

If A is 1-random, then for every integer n, the string 0^n appears as a substring of A. But $K(0^n) =^+ K(n) \leq^+ 2\log(n) <^+ \delta n$ for all $\delta > 0$. □

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Remark

For similar reasons, no real of packing dimension 1 is shift-complex. Thus, there is no 1-shift complex.

Convention

Whenever δ is fixed, it is assumed to satisfy $0 < \delta < 1$.

Outline

Existence Without Too Many...

Theorem (Durand, Levin, and Shen (2008))

For every δ, there is a δ-shift complex sequence A.

Proof.

Choose m sufficiently large. Take the next m bits of A to satisfy

$$K(A \upharpoonright m(n+1)) - K(A \upharpoonright mn) \geq \delta m.$$

Verify this is both possible and sufficient. □

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Verify this is both possible and sufficient. □

Remark

The measure of the shift complex sequences is 0.

Proof.

The set of reals with packing dimension 1 has measure one. The shift complex reals, sitting inside the complement, then has measure 0. □

No Extra Complexity...

Theorem (Hirschfeldt and Kach)

For every δ, there is an exactly δ-shift complex sequence A with Dim$(A) = \delta$.

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No Extra Complexity...

Theorem (Hirschfeldt and Kach)

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Proof.

Modify the construction of Durand, Levin, and Shen:

- If the packing dimension seems to be too high, append the string 0^m.
- If the packing dimension seems to be too low, append a string so that $K(A \upharpoonright m(n+1)) - K(A \upharpoonright mn) \geq \delta m$.

Verify this is sufficient. □

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Computing Shift Complex Reals...

Theorem (Rumyanstev (2011))

The set of reals that compute a shift complex sequence has measure 0*.*

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Definition

A shift complex sequence A is *abundant* if there is an integer $n > 1$ and a real $\delta > 1/n$ such that A is δ-shift complex and A contains at least $2^{m(n-1)/n}$-many different substrings of length m for all $m \in \mathbb{N}$.

Lemma

Every shift complex sequence computes an abundant shift complex sequence.

Calibrating the Level of Randomness...

Remark

Because any property that holds of almost all oracles holds of sufficiently random oracles, this says a sufficiently random sequence does not compute a shift complex sequence.

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Because any property that holds of almost all oracles holds of sufficiently random oracles, this says a sufficiently random sequence does not compute a shift complex sequence.

Theorem (Khan)

No difference random real computes a shift complex real. Thus, a 1*-random real computes a shift complex sequence if and only if it is complete.*

Outline

Dimension Extraction...

Theorem (Bienvenu, Doty, and Stephan (2009))

Fix A with $Dim(A) > 0$. Then for each $\varepsilon > 0$, there is a B with $B \leq_T A$ and $dim(B) \geq \frac{dim(A)}{Dim(A)} - \varepsilon$. In particular, if $0 < Dim(A) < 1$, then there is a B with $B \leq_T A$ and $dim(B) > dim(A)$.

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Theorem (Hirschfeldt and Kach)

Fix a δ-shift complex set A. Then for some $\varepsilon > 0$, there is a $(\delta + \varepsilon)$-shift complex sequence B with $B \leq_T A$.

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Theorem (Hirschfeldt and Kach)

Fix a δ*-shift complex set A. Then for some* $\varepsilon > 0$, *there is a* $(\delta + \varepsilon)$*-shift complex sequence B with* $B \leq_T A$.

Question

If A is almost δ-shift complex, does A necessarily compute a δ-shift complex B?

Computing 1-Randoms...

Theorem (Khan)

For every δ, there is a δ-shift complex real that computes no 1-random.

Question

Fix δ. Does every δ-shift complex real compute a real of packing dimension one?

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Remark

Since arbitrary sets can be encoded into shift complex sequences, for every δ and every set B, there is a δ-shift complex real A with $A \geq_T B$.

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Choose ε sufficiently small. □

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Proof.

Choose ε sufficiently small. $\square$

Question

Does every δ-shift complex real compute a bi-infinite δ-shift complex real?

Bi-Infinite Shift Complex Sequences...

Proposition (Khan)

Every $(1-\varepsilon)$-shift complex real computes a bi-infinite $(1-2\varepsilon)$-shift complex real.

Proof.

Verify that if $A = B \oplus C$ is $(1-\varepsilon)$-shift complex, then $\overleftarrow{B}\overrightarrow{C}$ is $(1-2\varepsilon)$-shift complex. □

References

Bruno Durand, Leonid A. Levin, and Alexander Shen.
Complex tilings.
J. Symbolic Logic, 73(2):593–613, 2008.

Mushfeq Khan.
Shift-complex sequences.

A. Yu. Rumyantsev and M. A. Ushakov.
Forbidden substrings, Kolmogorov complexity and almost periodic sequences.
In *STACS 2006*, volume 3884 of *Lecture Notes in Comput. Sci.*, pages 396–407, Berlin, 2006. Springer.