Embeddings of Computable Linear Orders

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Definition

A infinite order type $\mathcal{L}$ is *computable* if it has a *computable presentation*, i.e., if there is a computable binary relation $\prec$ on $\omega$ such that $\mathcal{L} \cong (\omega : \prec)$.

If $\mathcal{L}_1 = (L_1 : \prec_1)$ and $\mathcal{L}_2 = (L_2 : \prec_2)$ are computable presentations of computable linear orders, then an embedding $\pi : L_1 \rightarrow L_2$ is *computable* if $\pi$ is computable as a function.
Definition

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Theorem (Folklore)

Uniformly in an index for a $\Delta^0_3$ linear order $\mathcal{L}$ with distinguished least element, there is an index for a computable presentation of the linear order $\omega \cdot \mathcal{L}$.
### Theorem (Folklore)

If $L$ is an infinite order type, then at least one of $\omega$ or $\omega^*$ classically embeds.
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If $\mathcal{L}$ is an infinite order type, then at least one of $\omega$ or $\omega^*$ classically embeds.

Theorem (Denisov; Tennenbaum; Lerman)

The order types $\omega$, $\omega^*$, $\omega + \omega^*$, and $\omega + \zeta \cdot \eta + \omega^*$ form a bases for computable presentations of computable linear orders. In other words, if $\mathcal{L} = (L : \prec)$ is any computable presentation of a computable linear order, there is a computable subset of order type one of these.
Theorem (Denisov; Tennenbaum)

There is a computable presentation of the order type $\omega + \omega^*$ such that neither $\omega$ nor $\omega^*$ computably embeds.
**Theorem (Denisov; Tennenbaum)**

*There is a computable presentation of the order type $\omega + \omega^*$ such that neither $\omega$ nor $\omega^*$ computably embeds.*

**Proof.**

We construct a computable presentation of the order type $\omega + \omega^*$ meeting the following requirements $\mathcal{R}_e$.

$\mathcal{R}_e$: *If $W_e$ is infinite, then $W_e \not\subseteq \omega$ and $W_e \not\subseteq \omega^*$.*

We meet $\mathcal{R}_e$ by putting one element of $W_e$ into $\omega$ and one element into $\omega^*$. To facilitate this, we maintain a virtual fence separating these points with priority $e$. Note that if a higher priority fence prevents us from separating points in $W_e$, we can wait for additional points to be enumerated; if none appear, then we win as $|W_e| < \infty$. □
The Question

Remark

It is natural to ask what can be said about the effectiveness of embeddings of $L_1$ into $L_2$, allowing the presentations of $L_1$ and $L_2$ to vary to minimize the complexity of the embedding.
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It is natural to ask what can be said about the effectiveness of embeddings of $\mathcal{L}_1$ into $\mathcal{L}_2$, allowing the presentations of $\mathcal{L}_1$ and $\mathcal{L}_2$ to vary to minimize the complexity of the embedding.

Question

Are there computable linear orders $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_1$ classically embeds into $\mathcal{L}_2$ but for no computable presentations of $\mathcal{L}_1$ and $\mathcal{L}_2$ does $\mathcal{L}_1$ computably embed into $\mathcal{L}_2$?
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It is natural to ask what can be said about the effectiveness of embeddings of \( L_1 \) into \( L_2 \), allowing the presentations of \( L_1 \) and \( L_2 \) to vary to minimize the complexity of the embedding.

Question

Are there computable linear orders \( L_1 \) and \( L_2 \) such that \( L_1 \) classically embeds into \( L_2 \) but for no computable presentations of \( L_1 \) and \( L_2 \) does \( L_1 \) computably embed into \( L_2 \)?

Remark

Of particular (and natural) interest are the special cases when \( L_1 = \eta \) and \( L_1 = \omega^* \).
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5. Non-Well-Ordered Linear Orders (Revisited)

6. Other Classes of Algebraic Structures

7. Summary
The Goal and the Strategy

Remark

The goal is to produce a computable non-scattered linear order $\mathcal{L}$ such that $\eta$ does not computably embed into any computable presentation of $\mathcal{L}$.

Remark

The strategy to produce a computable non-scattered linear order that is intrinsically computably scattered will be to encode trees $T$ into linear orders $L_T$ in such a way that any embedding of $\eta$ into $L_T$ gives information about an infinite path through $T$ in a fairly effective manner. By choosing $T$ simple enough so that $L_T$ is computable but complex enough so that its paths are complicated, we obtain an appropriate linear order. The map $T \mapsto L_T$ depends on the goal.
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By choosing $T$ simple enough so that $\mathcal{L}_T$ is computable but complex enough so that its paths are complicated, we obtain an appropriate linear order.

The map $T \mapsto \mathcal{L}_T$ depends on the goal.
The Encoding for $T \subseteq 2^{<\omega}$

**Definition**

If $T \subseteq 2^{<\omega}$ is any tree, define linear orders $L_{\langle \sigma, \tau \rangle}$ via corecursion by

$$L_{\langle \sigma, \tau \rangle} = \langle \sigma, \tau \rangle + \hat{L}_{\langle \sigma \rhd 0, \tau \rhd 0 \rangle} + \hat{L}_{\langle \sigma \rhd 0, \tau \rhd 1 \rangle} + \hat{L}_{\langle \sigma \rhd 1, \tau \rhd 0 \rangle} + \hat{L}_{\langle \sigma \rhd 1, \tau \rhd 1 \rangle} + \langle \sigma, \tau \rangle$$

where

$$\hat{L}_{\langle \sigma \rhd i, \tau \rhd j \rangle} = \begin{cases} \zeta + L_{\langle \sigma \rhd i, \tau \rhd j \rangle} + \zeta & \text{if } \sigma \rhd i \in T, \\ \zeta & \text{otherwise.} \end{cases}$$

Define $L_T$ to be the linear order $L_{\langle \epsilon, \epsilon \rangle}$, where $\epsilon$ denotes the empty string.
### Theorem

There is a computable, non-scattered, rank two linear order $\mathcal{L}$ that is intrinsically computably scattered.

### Proof.

Let $T \subseteq 2^{<\omega}$ be any infinite $\Delta^0_3$ tree with no $\Delta^0_3$ paths. Then $\mathcal{L}_T$ is computable, non-scattered, rank two, and intrinsically computably scattered.
Claim
If \( T \subseteq 2^{<\omega} \) is \( \Delta^0_3 \), then \( \mathcal{L}_T \) is computable.

Remark
Recall that \( \mathcal{L}_T \) was effectively defined in terms of the linear orders

\[
\hat{\mathcal{L}}_{\langle \sigma \ast i, \tau \ast j \rangle} = \begin{cases} 
\zeta + \mathcal{L}_{\langle \sigma \ast i, \tau \ast j \rangle} + \zeta & \text{if } \sigma \ast i \in T, \\
\zeta & \text{otherwise.}
\end{cases}
\]

Proof.
Let \( \exists k \forall m \exists n R(\sigma, k, m, n) \) be a \( \Sigma^0_3 \) predicate for \( T \). Build \( \hat{\mathcal{L}}_{\langle \sigma, \tau \rangle} \) by building a sum \( \cdots \hat{\mathcal{L}}_3 + \hat{\mathcal{L}}_1 + \hat{\mathcal{L}}_0 + \hat{\mathcal{L}}_2 + \hat{\mathcal{L}}_4 + \cdots \) and attempting to build \( \mathcal{L}_{\langle \sigma, \tau \rangle} \) at each \( \hat{\mathcal{L}}_k \), adding additional points to \( \hat{\mathcal{L}}_k \) only when a new witness \( n \) for the next \( m \) appears. \( \square \)
Claim

If $T \subseteq 2^{<\omega}$ is infinite (i.e., has an infinite path), then $\mathcal{L}_T$ is non-scattered.

Proof.

Let $X \subseteq T$ be an infinite path. Define an embedding $\pi : 2^{<\omega} \rightarrow \mathcal{L}_T$ by recursion.

Define $\pi(\epsilon)$ to be any point in either one of the $\mathcal{L}_{\langle \epsilon, \epsilon \rangle}$ copies of $\zeta$ between $\mathcal{L}_{\langle x(0), 0 \rangle}$ and $\mathcal{L}_{\langle x(0), 1 \rangle}$.

Define $\pi(\rho \upharpoonright i)$ to be any element in either one of the $\mathcal{L}_{\langle x(0)\ldots x(|\rho|), \rho \upharpoonright i \rangle}$ copies of $\zeta$ between $\mathcal{L}_{\langle x(0)\ldots x(|\rho|+1), \rho \upharpoonright i \upharpoonright 0 \rangle}$ and $\mathcal{L}_{\langle x(0)\ldots x(|\rho|+1), \rho \upharpoonright i \upharpoonright 1 \rangle}$.
Claim

If $T \subseteq 2^{<\omega}$ is an infinite $\Delta^0_3$ tree with no $\Delta^0_3$ path, then $\mathcal{L}_T$ is intrinsically computably scattered.

Proof.

If there were a computable embedding $\pi : \eta \to \mathcal{L}_T$, then we could recover a $\Delta^0_3$ path in $T$. Specifically, determining whether a set of elements form a maximal block is $\Pi^0_2$. Starting with $\rho_0 = \epsilon = \tau_0$, we set $\rho_{s+1} = \rho_s \upharpoonright i$ and $\tau_{s+1} = \tau_s \upharpoonright j$, where $i, j \in \{0, 1\}$ are such that there exists two maximal blocks of size $\langle \rho \upharpoonright i, \tau \upharpoonright j \rangle$ with the range of $\pi$ containing at least two points in this interval.
A Theorem

Theorem

There is a computable, non-scattered, rank two linear order $L$ that is intrinsically computably scattered.

Proof.

Let $T \subseteq 2^{<\omega}$ be any infinite $\Delta^0_3$ tree with no $\Delta^0_3$ paths. Then $L_T$ is computable, non-scattered, rank two, and intrinsically computably scattered.
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Remark

The goal is to produce a computable non-well-ordered linear order $\mathcal{L}$ such that $\omega^*$ does not computably embed into any computable presentation of $\mathcal{L}$. 
The Goal and the Strategy

Remark

The goal is to produce a computable non-well-ordered linear order $\mathcal{L}$ such that $\omega^*$ does not computably embed into any computable presentation of $\mathcal{L}$.

Remark

The strategy to produce a computable non-well-ordered linear order that is intrinsically computably well-ordered will be to encode functions $F$ into linear orders $\mathcal{L}_F$ in such a way that

- Any descending chain in $\mathcal{L}_F$ is (almost) cofinal [downwards] in $\mathcal{L}_F$.
- The linear order $\mathcal{L}_F$ is not computable.
- The linear order $\omega^+ \mathcal{L}_F$ is computable.

Again, the map $F \mapsto \mathcal{L}_F$ depends on the goal.
The Encoding for $F : \omega \rightarrow \omega$

**Definition**

If $F : \omega \rightarrow \omega$ is a function with infinite support, define the linear order $\mathcal{L}_F$ by

$$\mathcal{L}_F = \cdots + \omega^n \cdot F(n) + \cdots + \omega^2 \cdot F(2) + \omega \cdot F(1) + F(0).$$
Theorem

There is a computable, non-well-ordered, scattered, rank $\omega + 1$ linear order $L$ that is intrinsically computably well-ordered.

Proof.

Let $F$ be any $\Delta^0_{(2n+1)}$-limit infimum function such that $L_F$ is not computable. Then the linear order $\omega^\omega + L_F$ is computable, non-well-ordered, scattered, rank $\omega + 1$, and intrinsically computably well-ordered.
Limit Infimum Functions

Definition

A function $F : \omega \to \omega$ is a limit infimum function if there is a total computable function $f : \omega \times \omega \to \omega$ such that

$$F(n) = \lim_{s} \inf f(n, s)$$

for all $n$. 

∆₀(2n + 1)-limit infimum function

A function $F : \omega \to \omega$ is a ∆₀(2n + 1)-limit infimum function if there is a functional $\phi_{e} : \omega \times \omega \to \omega$ such that

$$F(n) = \lim_{s} \inf \phi_{e}(2n, f(n, s))$$

for all $n$. 
Limit Infimum Functions

Definition

A function $F : \omega \to \omega$ is a \textit{limit infimum function} if there is a total computable function $f : \omega \times \omega \to \omega$ such that

$$F(n) = \liminf_s f(n, s)$$

for all $n$.

Definition

A function $F : \omega \to \omega$ is a $\Delta^0_{(2n+1)}$-\textit{limit infimum function} if there is a functional $\varphi_e : \omega \times \omega \to \omega$ such that

$$F(n) = \liminf_s \varphi_e^{0(2n)}(n, s)$$

for all $n$. 
There is an $F$

Claim

There is a $\Delta^0_{(2n+1)}$-limit infimum function $F : \omega \to \omega$ such that $\mathcal{L}_F$ is not computable.

Proof.

A diagonalization argument that builds a $\{0, 1\}$-valued function $F$. Roughly speaking, the strategy $S_{i,n}$ (for $n \geq i$) uses $F(2n)$ and $F(2n + 1)$ to assure that $\mathcal{L}_F \neq \mathcal{L}_i$.

For example, to assure $\mathcal{L}_F \neq \mathcal{L}_0$, the strategy $S_{0,0}$ begins setting $f(0, s) = 0$ and $f(1, s) = 1$. If a point appears to the right of $a_0$, the strategy $S_{0,0}$ switches to setting $f(0, s) = 1$ and $f(1, s) = 0$. Note that if $a_0$ is part of the $F(0)$ or $\omega \cdot F(1)$ blocks of $\mathcal{L}_0$, then $\mathcal{L}_F \neq \mathcal{L}_0$. 

□
Claim

If $F : \omega \to \omega$ is a $\Delta^0_{(2n+1)}$-limit infimum function, then $\omega^\omega + \mathcal{L}_F$ is computable.

Proof.

If $F(n) > 0$ for all $n$, build a computable copy of $\mathcal{L}_F$ by viewing it as the sum

$$\cdots + \left[\omega^n \cdot (\omega + F(n) - 1)\right] + \cdots + \left[\omega^2 \cdot (\omega + F(2) - 1)\right] + \left[\omega \cdot (\omega + F(1) - 1)\right] + \left[\omega + F(0)\right]$$

and building each summand separately.

For general $F$, use the Recursion Theorem to assure the garbage either settles down or collects in the copy of $\omega^\omega$. \hfill \Box
Claim

If $F : \omega \rightarrow \omega$ is any $\Delta^0_{(2n+1)}$-limit infimum function such that $L_F$ is not computable, then $\omega^\omega + L_F$ is intrinsically computably well-ordered.

Proof.

If there were a computable embedding $\pi : \omega^* \rightarrow \omega^\omega + L_F$, then the linear order with universe

$$\{ x \in \omega^\omega + L_F : \pi(z) \prec x \text{ for some } z \in \omega^* \}$$

and order inherited from $\omega^\omega + L_F$ would be computable. But this is $L_F$, a contradiction.
A Theorem

**Theorem**

There is a computable, non-well-ordered, scattered, rank $\omega + 1$ linear order $\mathcal{L}$ that is intrinsically computably well-ordered.

**Proof.**

Let $F$ be any $\Delta^0_{(2n+1)}$-limit infimum function such that $\mathcal{L}_F$ is not computable. Then the linear order $\omega^\omega + \mathcal{L}_F$ is computable, non-well-ordered, scattered, rank $\omega + 1$, and intrinsically computably well-ordered.
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The Encoding for $T \subseteq \omega^{<\omega}$

**Definition**

If $T \subseteq \omega^{<\omega}$ is any tree, define linear orders $L_{\langle \sigma, \tau \rangle}$ via corecursion by

$$L_{\langle \sigma, \tau \rangle} = \omega + \langle \sigma, \tau \rangle + \zeta + \left( \sum_{i \in \omega} L_{\langle \sigma \upharpoonright i, \tau \upharpoonright 0 \rangle} \right)^* + \left( \sum_{i \in \omega} L_{\langle \sigma \upharpoonright i, \tau \upharpoonright 1 \rangle} \right) + \zeta + \langle \sigma, \tau \rangle + \omega^*.$$

Let $L_T$ be the linear order $L_{\langle \epsilon, \epsilon \rangle}$, where $\epsilon$ denotes the empty string.
There is a computable, non-scattered linear order $\mathcal{L}$ that is intrinsically hyperarithmetically scattered.

Proof. Let $T \subseteq \omega^{<\omega}$ be a computable tree with infinite paths but no hyperarithmetic paths. Then $\mathcal{L}_T$ is computable, non-scattered, and intrinsically hyperarithmetically scattered.
**Theorem**

*If $\mathcal{L}$ is any computable rank one, non-scattered linear order, then there is a computable embedding of $\eta$ into some computable presentation of $\mathcal{L}$.***

**Proof.**

Any such linear order is (almost) of the form

$$\mathcal{L}_F = \sum_{q \in \mathbb{Q}} F(q)$$

for some function $F : \mathbb{Q} \to \omega \cup \{\omega^*, \zeta, \omega\}$. If it isn’t, argue that it might as well be by considering either $\mathcal{L}^* + \mathcal{L}$, $\mathcal{L} + \mathcal{L}^*$, or $\sum_{z \in \zeta} \mathcal{L}$.

Handle the case when $F$ is unbounded on every interval separate from when $F$ is bounded on some interval.
When $F$ is Bounded on an Interval

**Proof.**

Demonstrate $\eta$ computably embeds into *every* computable presentation of $\mathcal{L}$. Add a point in $\mathcal{L}$ into the range of $\pi$ whenever it is separated on the left and on the right by $N$ points in $\mathcal{L}$ not yet in the range of $\pi$.

**Question**

If $\eta$ computably embeds into every computable presentation of a linear order $\mathcal{L}$, must $\mathcal{L}$ be strongly $\eta$-like on some interval?
When $F$ is Unbounded on Every Interval

Proof.

For functions $F : \mathbb{Q} \to \omega \cup \{\omega^*, \zeta, \omega\}$ unbounded on every interval, the following are equivalent:

- The linear order $\mathcal{L}_F$ is computable.
- There are limit infimum functions $L : \mathbb{Q} \to \omega$ and $R : \mathbb{Q} \to \omega$ such that $F(q) = L(q)^* + 1 + R(q)$ for all $q$.
- There are $0'$-limitwise monotonic functions $L : \mathbb{Q} \to \omega$ and $R : \mathbb{Q} \to \omega$ such that $F(q) = L(q)^* + 1 + R(q)$ for all $q$. 

\[\square\]
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A Theorem

Theorem

For every computable ordinal $\alpha$, there is a computable, non-well-ordered linear order $L$ that is intrinsically $\emptyset^{(\alpha)}$-computationally well-ordered.

Proof.

Let $F : \omega \to \omega$ be a $\Delta^0_{(2n+1)}(\emptyset^{(\alpha)})$-limit infimum function such that $L_F$ is not $\emptyset^{(\alpha)}$ computable. Then $\omega^\alpha \cdot (\omega^\omega + L_F)$ suffices. □
Theorem

For every computable ordinal $\alpha$, there is a computable, non-well-ordered linear order $L$ that is intrinsically $\emptyset^{(\alpha)}$-computably well-ordered.

Proof.

Let $F : \omega \to \omega$ be a $\Delta^0_{2n+1}(\emptyset^{(\alpha)})$-limit infimum function such that $L_F$ is not $\emptyset^{(\alpha)}$ computable. Then $\omega^\alpha \cdot (\omega^\omega + L_F)$ suffices. □

Remark

By a result of Harrison, this is best possible.
## Theorem

If $L$ is a computable, rank $\omega$, scattered, non-well-ordered linear order, then there is a computable embedding of $\omega^*$ into some computable presentation of $L$.

## Proof.

Demonstrate the ability to build a computable presentation into which $\omega^*$ computably embeds if a non-greatest point in $c^n(L)$ has no immediate successor in $c^n(L)$. Note that $\omega^*$ and $\zeta$ cannot be the order type of a maximal block in any $c^n(L)$ if $L$ is intrinsically computably well-ordered.

## Lemma

If $L$ is a $\Delta^0_3$ linear order with distinguished least element having a $\Delta^0_3$ embedding of $\omega^*$ and $R$ is any $\Sigma^0_3$ predicate, then $R \cdot L$ has a computable presentation into which $\omega^*$ computably embeds.
A Conjecture

Conjecture

There is a computable, non-well-ordered, non-scattered, rank $\omega + 1$ linear order $L$ that is intrinsically computably well-ordered.

Proof.

Define a linear order similar to $L_T$ for $T \subseteq 2^{<\omega}$ except use linear orders $L_F = \cdots + \omega^n \cdot F(n) + \cdots + \omega \cdot F(1) + F(0) + \omega^\omega$ as markers rather than finite linear orders $\langle \sigma, \tau \rangle$. As $L_{F_1} \cong L_{F_2}$ if and only if $F_1 \equiv^* F_2$, code $\sigma$ into $L_F$ by having the support of $F$ be a subset of the multiples of $\sigma$.  
\qed
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Let $X$ be the class of directed (acyclic) graphs, the class of undirected graphs, the class of commutative rings, the class of two-step nilpotent groups, the class of integral domains, or the class of commutative semigroups.

Then there are computable structures $S_1, S_2 \in X$ such that $S_1$ classically embeds into $S_2$ but for no computable presentations of $S_1$ and $S_2$ is there a computable embedding.
Theorem

Let $X$ be the class of directed (acyclic) graphs, the class of undirected graphs, the class of commutative rings, the class of two-step nilpotent groups, the class of integral domains, or the class of commutative semigroups.

Then there are computable structures $S_1, S_2 \in X$ such that $S_1$ classically embeds into $S_2$ but for no computable presentations of $S_1$ and $S_2$ is there a computable embedding.

Proof.

Show the result for $X$ the class of directed acyclic graphs. The other classes then follow from previous work (Hirschfeldt, Khoussainov, Shore, Slinko).
Proof.

Let $T$ be an infinite computable tree with no computable paths. Let $S_2$ be the graph of $T$ after replacing edges by either a directed diamond or a directed hexagon depending on whether the edge represents a string ending in a 0 or a 1. Let $S_1$ be the graph of exactly one (directed) infinite path.

Proof.

Let $\omega^\omega + \mathcal{L}_F$ be a computable non-well-ordered intrinsically computably well-ordered linear order. Let $S_2$ be the graph whose vertices are the elements of $\omega^\omega + \mathcal{L}_F$, with a directed edge connecting vertex $i$ to vertex $j$ if and only if $j < i$ in the linear order. Again, let $S_1$ be the graph of exactly one (directed) infinite path.
Definition

A tree is a partial order \((T : \prec)\) with a least element such that for all \(x \in T\), the set \(\{y \in T : y \preceq x\}\) is a finite linearly ordered set.

Theorem (Binns, Kjos-Hanssen, Lerman, Schmerl, Solomon)

There are computable trees \(T_1\) and \(T_2\) such that \(T_1\) classically embeds into \(T_2\) but for no computable presentations of \(T_1\) and \(T_2\) is there a computable embedding.

Proof.

Let \(T_1 \cong 2^{<\omega}\) and let \(T_2\) be an appropriate perfect binary branching tree. Build \(T_2\) computable so that any function \(f : \omega \to \omega\) that dominates the properly \(\emptyset''\)-computable branching function \(b : \omega \to \omega\) satisfies \(b \leq_T f \oplus \emptyset'\).
Boolean Algebras

Theorem

There are no computable Boolean algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{B}_1$ classically embeds into $\mathcal{B}_2$ but for no computable presentations of $\mathcal{B}_1$ and $\mathcal{B}_2$ is there a computable embedding.

Proof.

If $\mathcal{B}_2$ is superatomic, then $\mathcal{B}_1$ is superatomic; and the result is immediate.

If $\mathcal{B}_2$ is non-superatomic, it suffices to show that the countable atomless Boolean algebra computably embeds into some computable presentation of $\mathcal{B}_2$. Note that it suffices to consider uniform $\mathcal{B}_2$. With $\alpha$ the minimal ordinal in the range of $\sigma_{\mathcal{B}_2}$, note $\mathcal{B}_2 = \mathcal{B}_2 \oplus \mathcal{B}_{\sigma_u(\{\alpha\})}$. There is a nice presentation of the latter into which the countable atomless Boolean algebra computably embeds.
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A Summary of Embedding Results

Theorem

There is a computable, non-scattered, rank two linear order $\mathcal{L}$ that is intrinsically computably scattered.

There is a computable, non-well-ordered, scattered, rank $\omega + 1$ linear order $\mathcal{L}$ that is intrinsically computably well-ordered.

There is a computable, non-well-ordered, non-scattered, rank $\omega + 1$ linear order $\mathcal{L}$ that is intrinsically computably well-ordered?

There is a computable, non-scattered, linear order $\mathcal{L}$ that is intrinsically hyperarithmetically scattered.

For many nice classes of algebraic structures $X$ (but not $X$ the class of Boolean algebras), there are computable $S_1$ and $S_2$ in $X$ such that $S_1$ classically embeds into $S_2$ but for no computable presentations of $S_1$ and $S_2$ is there a computable embedding.
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