Jump inversions of algebraic structures and the Σ-definability

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Abstract

It is proved that for every countable structure $\mathcal{A}$ and a successive computable ordinal α there is a countable structure $\mathcal{A}^{-\alpha}$ which is $\leq_\Sigma$-least among all countable structures $\mathcal{C}$ such that $\mathcal{A}$ is Σ-definable in the α-th jump $\mathcal{C}^{(\alpha)}$. We also show that this result does not hold for the limit $\alpha = \omega$. Moreover, we prove that there is no countable structure $\mathcal{A}$ with the degree spectrum $\{\mathbf{d} : \mathbf{a} \leq \mathbf{d}^{(\omega)}\}$ for

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$\mathbf{a} > \mathbf{0}^{(\omega)}$.

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1. Introduction

The notion of the jump of a structure has received much attention in computable structure theory over the last decade. It is defined by adding to a structure $\mathcal{A}$ a complete Σ_1 relation. It was defined independently by various researchers [1, 2, 3, 4, 5, 6, 7], as there are various ways of understanding what a complete Σ_1 relation: Montalbán used relatively intrinsic c.e. subsets of $\mathbb{N}\times\mathcal{A}^{<\omega}$, Soskov used the forcing relation for Π_1 formulas over the Moschovakis extension of $\mathcal{A}$, and in Russia is common to use Σ-definable relations on the hereditarily finite extension of $\mathcal{A}$ — we use the latter one here. (We refer the reader to [8, Definition 5.1] for the history of the different definitions explained in more detail.) It is an important concept because, as it turned out, many constructions in the area can be better understood using the notion of jump.

Once the jump was defined, various jump inversion theorems were proved. What is sometimes called the first jump inversion theorem states that for every structure $\mathcal{A}$ that computes $\mathbf{0}'$ there is a structure $\mathcal{B}$ whose jump is equivalent to $\mathcal{A}$. Goncharov, Harizanov, Knight, McCoy, R. Miller and Solomon [9, Lemma 5.5] build a jump inversion to translate results about computable categoricity to results about Δ^0_α-categoricity even before the notion of jump was introduced. Alexandra Soskova [3, 4] built another jump inversion using Marker extensions. The question then was whether these two constructions are in any sense equivalent. Both jump inversions are equivalent up to Muchnik reducibility (i.e., they have the same degree spectrum), but this is two coarse of a measure to say that the structures are "equivalent." A better measure is Σ-definability, which in the west is called effective interpretability with parameters. *Effective interpretability* is like interpretability of structures in model theory, except that one is allowed to use a subset of $\mathcal{A}^{<\omega}$ to define the new structure, and the domain and relations of the interpretation have to be relatively intrinsically computable [10, Definition 5.1]. In this paper we will use the equivalent notion of Σ-definability introduced by Ershov. It turned out that the jump inversion from [9] and [3, 4] are not equivalent up to Σ-definability. Is there a canonical jump inversion of a structure? In the Turing degrees we know there isn't one. In this paper we show that, for structures, there is: Among the jump inversions of a structure there is one that is the least up to Σ-definability.

One can also iterate the jump and define the α jump of a structure for each computable ordinal α. We show that the least Σ- jump inversion theorem also works for infinite successor ordinal α but not for limit ones.

2. Background

A family $HF(M)$ of *hereditarily finite sets over* M is defined by induction as follows:

- $H_0(M) = \{\emptyset\}$;
- $H_{n+1}(M) = H_n(M) \cup \mathcal{P}_\omega(H_n(M) \cup M)$;
- $HF(M) = \bigcup_{n<\omega} H_n(M)$

(where $P_\omega(X)$ denotes the set of all finite subsets of X). If $\mathcal{M}$ is a structure in a relational signature σ, then on $HF(|\mathcal{M}|) \cup |\mathcal{M}|$ we can define a structure $\mathbb{HF}(\mathcal{M})$ in a signature $\sigma \cup \{U^1, \in^2, \emptyset\}$ (called a *hereditarily finite superstructure over* $\mathcal{M}$), so that $U^{\mathbb{HF}(\mathcal{M})} = |\mathcal{M}|$, $\in^{\mathbb{HF}(\mathcal{M})} \subseteq (HF(|\mathcal{M}|) \cup |\mathcal{M}|) \times HF(|\mathcal{M}|)$ is the membership relation on $\mathbb{HF}(\mathcal{M})$, the constant symbol $\emptyset$ is interpreted as the empty "set", and symbols in the signature σ are interpreted in the same way as on $\mathcal{M}$.

Note that in this paper we will consider only countable algebraic structures with finite relational languages.

A class of Δ_0-*formulas* in the language of $\mathbb{HF}(\mathcal{M})$ is the least class which contains atomic formulas and is closed under logical connectives $\vee$, $\&$, $\rightarrow$, $\neg$, and also under *bounded quantification*, i.e., if Φ is a Δ_0-formula then so are $\forall x \in t\,\Phi$ and $\exists x \in t\,\Phi$, where t is a term containing no occurrence of a variable x. A class of Σ-*formulas* (or equivalently, Σ_1-formulas) is the least class which contains all Δ_0-formulas and is closed under logical connectives $\vee$, &, bounded quantification, and $\exists$. Π-formulas (or Π_1-formulas) can be obtained from Σ-formulas by replacing the unbounded quantifier $\exists$ with $\forall$. In a natural way, these definitions can be generalized to Σ_n- and Π_n-formulas.

Definition 1 (Ershov [11]). *Let*

$$\Psi_0, \Psi_1, \Phi_0, \ldots, \Phi_n, \Psi_1^*, \Phi_0^*, \ldots, \Phi_n^*$$

be a Σ-formulas in the language of $\mathbb{HF}(\mathcal{M})$,

- $A_0 = \{x : \mathbb{HF}(\mathcal{M}) \models \Psi_0(x)\}$;
- $\eta = A_0^2 \cap \{(x,y) : \mathbb{HF}(\mathcal{M}) \models \Psi_1(x,y)\}$;
- $P_i = A_0^{m_i} \cap \{(x_1,\ldots,x_{m_i}) : \mathbb{HF}(\mathcal{M}) \models \Phi_i(x_1,\ldots,x_{m_i})\}$, $i \leqslant n$;
- $A_0^2 \cap \{(x,y) : \mathbb{HF}(\mathcal{M}) \models \Psi_1^*(x,y)\} = A_0^2 \setminus \eta$;
- $A^{m_i} \setminus P_i = A_0^{m_i} \cap \{(x_1,\ldots,x_{m_i}) : \mathbb{HF}(\mathcal{M}) \models \Phi_i^*(x_1,\ldots,x_{m_i})\}$, $i \leqslant n$;
- *η is a congruence relation on the structure $\mathcal{A}_0 = (A_0; P_0, \ldots, P_n)$.*

We say that the system of formulas $\Psi_0, \Psi_1, \Phi_0, \ldots, \Phi_n, \Psi_1^*, \Phi_0^*, \ldots, \Phi_n^*$ Σ-defines a structure $\mathcal{A}$ in $\mathbb{HF}(\mathcal{M})$ if $\mathcal{A} \cong \mathcal{A}_0/\eta$. In this case we say that $\mathcal{A}$ is Σ-definable in $\mathbb{HF}(\mathcal{M})$ (written $\mathcal{A} \leqslant_\Sigma \mathcal{M}$).

It is proved in [12] that for countable structures the notion of Σ-definability of $\mathcal{A}$ in $\mathbb{HF}(\mathcal{B})$ is equivalent to existence of a computable functor from $\mathcal{B}$ to $\mathcal{A}$.

Definition 2 ([5, 6, 7]). *For any structure $\mathcal{M}$ the jump of $\mathcal{M}$ is the structure $\mathcal{M}' = \langle \mathbb{HF}(\mathcal{M}), T\rangle$, where T is a binary Σ-predicate on $\mathbb{HF}(\mathcal{M})$ universal for the class of all unary Σ-predicates on $\mathbb{HF}(\mathcal{M})$.*

The concept of the jump with respect to Σ-definability is independent of the choice of a universal Σ-predicate. As in the classical case, the Σ-jump operation satisfies the following:

1. $\mathcal{A} \leqslant_\Sigma \mathcal{A}'$;
2. $\mathcal{A} \leqslant_\Sigma \mathcal{B} \Rightarrow \mathcal{A}' \leqslant_\Sigma \mathcal{B}'$.

We define $\mathcal{A}^{(n)}$ by induction on $n \in \omega$ as follows: $\mathcal{A}^0 = \mathcal{A}$, $\mathcal{A}^{(n+1)} = (\mathcal{A}^{(n)})'$. It was shown in [6] that $\mathcal{A}$ is Σ_{m+1}-definable in $\mathbb{HF}(\mathcal{M})$ iff $\mathcal{A} \leqslant_\Sigma \mathcal{M}^{(m)}$.

Note also that the definition of the jump agrees with the Turing jumps of the presentations of the structures (i.e. the degree spectra).

Theorem 1 (A. Montalbán [5]; A.A. Soskova, I.N. Soskov [4])**.** *Let $\mathcal{A}$ be a countable structure. If*

$$\mathbf{Sp}(\mathcal{A}) = \{deg_T(X) : X \textit{ computes some isomorphic copy of } \mathcal{A}\}$$

and $\mathbf{Sp}(\mathcal{A})^{(n)} = \{\mathbf{x}^{(n)} : \mathbf{x} \in \mathbf{Sp}(\mathcal{A})\}$, *then* $\mathbf{Sp}(\mathcal{A}^{(n)}) = \mathbf{Sp}(\mathcal{A})^{(n)}$.

The proof of the Theorem above is based on a construction of a copy of the structure whose atomic diagram is 1-generic.

It follows from literature (see e.g. [3, 7, 9]) that the jump operation can be inversed. Namely, if $\emptyset'$ is Δ on $\mathbb{HF}(\mathcal{A})$ then $\mathcal{B}' \equiv_\Sigma \mathcal{A}$ for some structure $\mathcal{B}$. Note that such structure $\mathcal{B}$ is not unique up to Σ-equivalence. It is proved in [13] that even if $\emptyset''$ is Δ on $\mathbb{HF}(\mathcal{A})$ and $\mathcal{A}$ is $\emptyset''$-computable then there is at least two different inversions $\mathcal{B}$, one is obtained from the family of all total computable functions, another is obtained from the family of all infinite c.e. sets. It is proved in [14] that the last inversion $\mathcal{B}$ for such $\mathcal{A}$ has the least property: $\mathcal{B} \leq_\Sigma \mathcal{C}$ for every countable structure $\mathcal{C}$ such that $\emptyset''$ is Δ on $\mathcal{C}'$. In this paper we generalize this result to arbitrary structure $\mathcal{A}$ and arbitrary successive jump iterations.

Namely, we can proceed iterating the jump on any (computable) ordinal. For example,

Definition 3. *For a structure $\mathcal{M}$ define the ω-jump as the structure $\mathcal{M}^{(\omega)} = \langle \mathbb{HF}(\mathcal{M}), T_\omega \rangle$, where T_ω is a predicate on $\omega \times HF(\mathcal{M}) \times HF(\mathcal{M})$ such that for* *every $n \in \omega$ the predicate $T_\omega(n,\cdot,\cdot)$ is Σ_n-universal in $\mathbb{HF}(\mathcal{M})$. This definition can be easily extended for the α-th jump $\mathcal{M}^{(\alpha)}$ for every computable ordinal α.*

Note that replacing the 1-genericity by the arithmetic genericity in the proof of Theorem 1 we get the following

Corollary 1. $\mathbf{Sp}(\mathcal{A}^{(\alpha)}) = \mathbf{Sp}(\mathcal{A})^{(\alpha)}$, *where* $\mathbf{Sp}(\mathcal{A})^{(\alpha)} = \{\mathbf{x}^{(\alpha)} : \mathbf{x} \in \mathbf{Sp}(\mathcal{A})\}$,

3. The Least Jump Inversion

Our first result shows that for every countable structure $\mathcal{A}$ there is a Σ-least structure $\mathcal{A}^{-1}$ such that $\mathcal{A} \leqslant_\Sigma (\mathcal{A}^{-1})'$. Coding Turing degrees $\mathbf{x}$ into the structures $\mathcal{M}$ one can deduce that $\mathcal{A}^{-1}$ has an $\mathbf{x}$-computable copy if and only if $\mathcal{A}$ has an $\mathbf{x}'$-computable copy. It follows from [13] that only the last property does not determine $\mathcal{A}^{-1}$ up to Σ-equivalence.

Theorem 2. *For every countable structure $\mathcal{A}$ there is a countable structure $\mathcal{A}^{-1}$ such that*

1. $\mathcal{A} \leqslant_\Sigma (\mathcal{A}^{-1})'$;
2. $\mathcal{A} \leqslant_\Sigma \mathcal{M}' \Rightarrow \mathcal{A}^{-1} \leqslant_\Sigma \mathcal{M}$ *for every countable structure* $\mathcal{M}$.

Proof. Let $\mathcal{A} = (A, \sigma)$ be a structure in a finite relational signature σ. Without loss of generality we can assume that there is a congruence relation $\sim$ in σ such that each congruence class is infinite. If not we can consider a structure $\widetilde{\mathcal{A}}$ replacing each element of $\mathcal{A}$ by an infinite $\sim$- congruence class, where $\sim$ is a new congruence symbol. It is easy to see that $\mathcal{A} \equiv_\Sigma \widetilde{\mathcal{A}}$.

Define the structure $\mathcal{A}^{-1}$ of $\mathcal{A}$ with the relational signature

$$\{W^1, E^2\} \cup \{P_R^{n+1}, N_R^{n+1} : R^n \in \sigma\}$$

and the universe

$$C = A \cup B, A \cap B = \emptyset,$$

where

1. $\mathcal{A}^{-1} \models W(x) \iff x \in A$;

2. (B, E) is the equivalence structure with two-element equivalence classes

$$\mathbf{p}_{R,\vec{x},i} = \{p_{R,\vec{x},i,0}, p_{R,\vec{x},i,1}\},$$

$$\mathbf{n}_{R,\vec{x},i} = \{n_{R,\vec{x},i,0}, n_{R,\vec{x},i,1}\},$$

for each $R^n \in \sigma$, $\vec{x} \in A^n$, $i \in \omega$, and one-element equivalence classes

$$\mathbf{p}'_{R,\vec{x},i} = \{p'_{R,\vec{x},i,0}\},$$

for each $i \in \omega$, $R^n \in \sigma$, $\vec{x} \in A^n$, such that $\mathcal{A} \models R(\vec{x})$, and one-element equivalence classes

$$\mathbf{n}'_{R,\vec{x},i} = \{n'_{R,\vec{x},i,0}\},$$

for each $i \in \omega$, $R^n \in \sigma$, $\vec{x} \in A^n$, such that $\mathcal{A} \models \neg R(\vec{x})$ (the index i here duplicates these one- and two-element equivalence classes infinitely many times);

3. $\mathcal{A}^{-1} \models P_R(\vec{x}, y) \iff y \in \bigcup_{i\in\omega} \mathbf{p}_{R,\vec{x},i} \vee y \in \bigcup_{i\in\omega} \mathbf{p}'_{R,\vec{x},i}$ & $\mathcal{A} \models R(\vec{x})$;
4. $\mathcal{A}^{-1} \models N_R(\vec{x}, y) \iff y \in \bigcup_{i\in\omega} \mathbf{n}_{R,\vec{x},i} \vee y \in \bigcup_{i\in\omega} \mathbf{n}'_{R,\vec{x},i}$ & $\mathcal{A} \models \neg R(\vec{x})$.

By the definition we have

$$\mathcal{A} \models R(\vec{x}) \iff (\exists y)(\forall z)[P_R(\vec{x}, y) \ \&\ [E(y, z) \implies y = z]]$$

and

$$\mathcal{A} \models \neg R(\vec{x}) \iff (\exists y)(\forall z)[N_R(\vec{x}, y) \ \&\ [E(y, z) \implies y = z]],$$

so that $\mathcal{A} \leq_\Sigma (\mathcal{A}^{-1})'$. It remains to prove that $\mathcal{A}^{-1} \leq_\Sigma \mathcal{M}$ for each countable structure $\mathcal{M}$ such that $\mathcal{A} \leq_\Sigma \mathcal{M}'$.

Suppose $\mathcal{A} \leq_\Sigma \mathcal{M}'$. Then there is a structure $\mathcal{B}$ in the signature $\Sigma \cup \{\equiv\}$ Σ_2-definable in $\mathbb{HF}(\mathcal{M})$ such that $(\mathcal{B}/\equiv) \cong \mathcal{A}$. Since

$$\mathcal{B} \restriction \sigma \cong \widetilde{\mathcal{A}} \cong \mathcal{A}$$

the structure $\mathcal{A}$ itself is Σ_2-definable in $\mathbb{HF}(\mathcal{M})$, and without loss of generality we can assume that the universe of $\mathcal{A}$ is $A = \mathrm{HF}(|\mathcal{M}|)$. For each $R \in \sigma$ we fix Δ_0-formulae Φ_R and Ψ_R such that

$$\mathcal{A} \models R(\vec{x}) \iff \mathbb{HF}(\mathcal{M}) \models (\exists \vec{y})(\forall \vec{z})\Phi_R(\vec{x}, \vec{y}, \vec{z}),$$

$$\mathcal{A} \models \neg R(\vec{x}) \iff \mathbb{HF}(\mathcal{M}) \models (\exists \vec{y})(\forall \vec{z})\Psi_R(\vec{x},\vec{y},\vec{z}).$$

Define a Σ-definable interpretation $\mathcal{D}$ of $\mathcal{A}^{-1}$ as follows:

1. the universe $D = \{\langle 0, x\rangle : x \in A\} \cup$
   $\{\langle 1, j, i, S, R, \vec{x}, 0\rangle : j \in \{0,1\}, i \in \omega, S \in \{P,N\}, R \in \sigma, \vec{x} \in A^{i \mapsto i}\} \cup$
   $\{\langle 2, 0, i, S, R, \vec{x}, \vec{y}\rangle : i \in \omega, S \in \{P,N\}, R \in \sigma, \vec{x}, \vec{y} \in A^{i \mapsto i}\} \cup$
   $\{\langle 2, 1, i, P, R, \vec{x}, \vec{y}\rangle : i \in \omega, R \in \sigma, \vec{x}, \vec{y} \in A^{i \mapsto i} : (\exists \vec{z})\neg\Phi_R(\vec{x},\vec{y},\vec{z})\} \cup$
   $\{\langle 2, 1, i, N, R, \vec{x}, \vec{y}\rangle : i \in \omega, R \in \sigma, \vec{x}, \vec{y} \in A^{i \mapsto i} : (\exists \vec{z})\neg\Psi_R(\vec{x},\vec{y},\vec{z})\}$
   is Σ-definable in $\mathbb{HF}(\mathcal{M})$ (where $\vec{x}$ is identified with the set-theoretic tuple function $\langle \vec{x}\rangle$, the finite set $\{P,N\}$ is identified with the set $\{0,1\}$, each element $R \in \sigma$ is identified by an unique natural number);
2. the predicate $W(t) \iff$ "$t = \langle 0, c\rangle$ for some c" is Δ on $\mathbb{HF}(\mathcal{M})$;
3. the predicate $E(s,t) \iff$ "$s = \langle m, j, i, S, R, c, d\rangle$ and $t = \langle m, k, i, S, R, c, d\rangle$ for some $m \in \{1,2\}, i, S, R, c, d$" is Δ on $\mathbb{HF}(\mathcal{M})$;
4. the predicate $P_R(\vec{s}, t) \iff$ "$\vec{s} = \langle 0, \vec{x}\rangle$ and $t = \langle m, j, i, P, R, \vec{x}, \vec{y}\rangle$ for some $m \in \{1,2\}, j, i, \vec{x}, \vec{y}$" is Δ on $\mathbb{HF}(\mathcal{M})$ for each $R \in \sigma$;
5. the predicate $N_R(\vec{s}, t) \iff$ "$\vec{s} = \langle 0, \vec{x}\rangle$ and $t = \langle m, j, i, N, R, \vec{x}, \vec{y}\rangle$ for some $m \in \{1,2\}, j, i, \vec{x}, \vec{y}$" is Δ-definable in $\mathbb{HF}(\mathcal{M})$ for each $R \in \sigma$;

It is easy to check that the Σ-definable structure $\mathcal{D}$ is isomorphic to $\mathcal{A}^{-1}$. Indeed, let

$$\vec{s} = (s_1, s_2, \ldots, s_k) \in A^{i \mapsto i}$$

be a tuple such that $W(s_i)$ holds for every i, $1 \le i \le k$. We can consider the tuple

$$\vec{x} = (x_1, x_2, \ldots, x_k) \in A^{i \mapsto i},$$

such that $s_i = \langle 0, x_i\rangle$. Then we have infinitely many two-element E-classes

$$\widehat{\mathbf{p}}_{R,\vec{s},i} = \{\langle 1, 0, i, P, R, \vec{x}, 0\rangle, \langle 1, 1, i, P, R, \vec{x}, 0\rangle\},$$

$$\widehat{\mathbf{n}}_{R,\vec{s},i} = \{\langle 1, 0, i, N, R, \vec{x}, 0\rangle, \langle 1, 1, i, N, R, \vec{x}, 0\rangle\}$$

such that $\mathcal{D} \models P_R(\vec{s}, p)$ and $\mathcal{D} \models P_N(\vec{s}, n)$ for $p \in \widehat{\mathbf{p}}_{R,\vec{s},i}$ and $n \in \widehat{\mathbf{n}}_{R,\vec{s},i}$.

If $\mathcal{A} \models R(\vec{x})$ then for some tuple $\vec{y} \in A^{i \mapsto i}$ we have $\mathcal{A} \models (\forall \vec{z})\Phi_R(\vec{x}, \vec{y}, \vec{z})$, so that $\langle 2, 1, i, P, R, \vec{x}, \vec{y} \rangle \notin D$ and, therefore, the elements

$$\widehat{\mathbf{p}}'_{R,\vec{x},i,\vec{y}} = \{\langle 2, 0, i, P, R, \vec{x}, \vec{y} \rangle\}, i \in \omega,$$

represent infinitely many one-element E-classes such that $\mathcal{D} \models P_R(\vec{s}, p)$, $p \in \widehat{\mathbf{p}}'_{R,\vec{s},i,\vec{y}}$. On another hand, if $\mathcal{A} \models R(\vec{x})$ then for every $\vec{y} \in A^{i \mapsto i}$ we have $\mathcal{A} \models (\exists \vec{z})\neg\Psi_R(\vec{x}, \vec{y}, \vec{z})$, so that $\langle 2, 1, i, N, R, \vec{x}, \vec{y} \rangle \in D$ and, therefore, each element n with $\mathcal{D} \models N_R(\vec{s}, n)$ belongs to some of two-element E-classes

$$\widehat{\mathbf{n}}_{R,\vec{s},i} = \{\langle 1, 0, i, N, R, \vec{x}, 0 \rangle, \langle 1, 1, i, N, R, \vec{x}, 0 \rangle\}$$

or

$$\widehat{\mathbf{n}}''_{R,\vec{s},i,\vec{y}} = \{\langle 2, 0, i, N, R, \vec{x}, \vec{y} \rangle, \langle 2, 1, i, N, R, \vec{x}, \vec{y} \rangle\}.$$

Similarly, if $\mathcal{A} \models \neg R(\vec{x})$ then we have infinitely many one-element E-classes

$$\widehat{\mathbf{n}}'_{R,\vec{s},i,\vec{y}} = \{\langle 2, 0, i, N, R, \vec{x}, \vec{y} \rangle\}$$

with $\mathcal{D} \models N_R(\vec{s}, n)$, $n \in \widehat{\mathbf{n}}'_{R,\vec{s},i,\vec{y}}$ and only two-element E-classes

$$\widehat{\mathbf{p}}_{R,\vec{s},i} = \{\langle 1, 0, i, P, R, \vec{x}, 0 \rangle, \langle 1, 1, i, P, R, \vec{x}, 0 \rangle\},$$

$$\widehat{\mathbf{p}}''_{R,\vec{s},i,\vec{y}} = \{\langle 2, 0, i, P, R, \vec{x}, \vec{y} \rangle, \langle 2, 1, i, P, R, \vec{x}, \vec{y} \rangle\}.$$

whose elements p satisfy $\mathcal{D} \models P_R(\vec{s}, p)$. Therefore, $\mathcal{D} \cong \mathcal{A}^{-1}$, and hence $\mathcal{A}^{-1} \leq_\Sigma \mathcal{M}$. $\square$

Stukachev [7] proved that if $\emptyset'$ is Δ on $\mathbb{HF}(\mathcal{A})$ then $\mathcal{A} \equiv_\Sigma \mathcal{B}'$ for some structure $\mathcal{B}$. Therefore,

Corollary 2. *If $\emptyset'$ is Δ on $\mathbb{HF}(\mathcal{A})$ for a countable $\mathcal{A}$ then $\mathcal{A} \equiv_\Sigma (\mathcal{A}^{-1})'$.*

4. The Least Inversion for Infinitely Iterated Jumps

The next theorem generalizes Theorem 2 for every successive computable ordinal. Now this result can be considered as a refinement of the results from [9].

Theorem 3. *For every countable structure $\mathcal{A}$ and computable successive ordinal α there is a countable structure $\mathcal{A}^{-\alpha}$ such that*

1. $\mathcal{A} \leqslant_\Sigma (\mathcal{A}^{-\alpha})^{(\alpha)}$*;*
2. $\mathcal{A} \leqslant_\Sigma \mathcal{M}^{(\alpha)} \Rightarrow \mathcal{A}^{-\alpha} \leqslant_\Sigma \mathcal{M}$ *for every countable structure* $\mathcal{M}$*.*

Proof. For simplicity we will consider only the case $\alpha = \omega + 1$. For arbitrary successive α the proof is almost the same. The next lemma formally allows to approximate the formulas from the ω-jump.

Lemma 1. *Let $\mathcal{M}$ be an algebraic structure of signature σ and $\Phi(x_1, \ldots, x_n)$ be a Σ-formula of signature $\sigma \cup \{T_\omega, \in, U\}$. Then there is a uniformly computable sequence of formulas $\{\Phi_k(x_1, \ldots, x_n)\}_{k\in\omega}$ such that $\Phi_k \in \Sigma_k$ for every k and for all $a_1, \ldots, a_n \in HF(\mathcal{M})$, $m \in \omega$*

$$\mathcal{M}^{(\omega)} \models \Phi(a_1, \ldots, a_n) \iff \exists k\, [\mathbb{HF}(\mathcal{M}) \models \Phi_k(a_1, \ldots, a_n)].$$

Proof. It is enough to prove the lemma for the case when Φ is a Δ-formula with parameters from $HF(\mathcal{M})$. Note that the negation $\neg T_n(x, y)$ is equivalent to $T_{n+1}(z, \langle x, y\rangle)$ for some z. Hence, without loss of generality we can also assume that Φ is in the prenex normal form and each instance of T_ω is positive (in the disjunctive normal form of the prefix-free part). Let $\widetilde{T}_n(k, x, y)$ be the Σ_n-formula for "$k \leq n$ & $T_\omega(k, x, y)$", and let Φ_n be the result of replacement of each instance of T_ω by $\widetilde{T}_n$. By hereditarily finiteness of $\mathbb{HF}(\mathcal{M})$ each Φ_n is equivalent to a Σ_n-formula. An easy induction shows that

$$\mathcal{M}^{(\omega)} \models \Phi(a_1, \ldots, a_m) \iff \exists n\, [\mathbb{HF}(\mathcal{M}) \models \Phi_n(a_1, \ldots, a_m)]$$

for every $a_1, \ldots, a_m \in HF(\mathcal{M})$. □

Now we are ready to proceed the proof of the theorem. Since every structure is Σ-equivalent to a graph (see [6]) we can assume for a simplicity that signature σ of $\mathcal{A}$ contains only symmetric binary predicate symbols R. As in the previous proof we can also assume that σ contains a congruence symbol $\sim$ with infinitely many congruence classes such that each congruence class is infinite. Also it is convenient to assume that for any $R \in \sigma$ there is an $R' \in \sigma$ such that

$$\mathcal{A} \models R'(x, y) \iff \mathcal{A} \models \neg R(x, y).$$

The definition of the structure $\mathcal{A}^{-(\omega+1)}$ will use the iterations $\mathcal{X}^{-2} = (\mathcal{X}^{-1})^{-1}$, $\mathcal{X}^{-3} = ((\mathcal{X}^{-1})^{-1})^{-1}, \ldots$ for the operator $\mathcal{X} \mapsto \mathcal{X}^{-1}$ defined in the previous theorem. The structures $\mathcal{X}$ here will code positive and negative facts about the edge relation on $\mathcal{A}$. For this reason we fix one-element structures $\mathcal{B}_0$ and $\mathcal{B}_1$ with an atomic predicate V such that $\mathcal{B}_0 \models \neg V$ and $\mathcal{B}_1 \models V$. Then to build $\mathcal{A}^{-(\omega+1)}$ we should replace each $R \in \sigma$ by a binary relation I_R which connects each pair $x, y \in |\mathcal{A}|$ with infinitely many ω-chains with corresponding sequence of graphs $\mathcal{B}^{-i}_{h(i)}, i \in \omega$ as it is shown below:

$$\begin{tikzcd}
y & & & & & \\
0 \arrow[u, "I_R"] \arrow[r, "I_R"] \arrow[d, "I_R"'] & 1 \arrow[r, "I_R"] & 2 \arrow[r, "I_R"] & \cdots & m \arrow[r, "I_R"] & \cdots \\
x & \mathcal{B}^{-1}_{h(1)} \arrow[u, "I_R"] & \mathcal{B}^{-2}_{h(2)} \arrow[u, "I_R"] & \arrow[u, phantom] & \mathcal{B}^{-m}_{h(m)} \arrow[u, "I_R"] &
\end{tikzcd}$$

where the elements of $\mathcal{B}^{-m}_{h(m)}$ are connected to corresponding elements from the chain via I_R, and h is any $\{0, 1\}$-valued non-increasing function such that $\lim_s h(s) = 1 \implies \mathcal{A} \models R(x, y)$. That is, h is either a one-step function with the limit 0, or it is the constant 1. The latter is possible only if $R(x, y)$ holds.

Each such function h is repeated in the ω-chains infinitely often. Note that for the case $\alpha > \omega + 1$ we should replace the ω-chains by the $(\alpha - 1)$-chains with additional predicates for the ordering and for limit element markers (in other words we should present the notation of α).

By the uniformity of $\mathcal{B}_{h(i)} \leq_\Sigma (\mathcal{B}^{-i}_{h(i)})^{(i)}$ from Theorem 2 there is a comuptable sequence of formulas $\{\Phi_i\}_{i\in\mathbb{N}}$ such that Φ_i is a Σ_i-formula and

$$h(i)=1 \iff \mathbb{HF}(\mathcal{B}_{h(i)}^{-i}) \models \Phi_i.$$

Therefore, the condition "h is the constant 1" is Π on $(\mathcal{A}^{-(\omega+1)})^{(\omega)}$, for a fixed chain. Since σ contains all negations from σ we can define all atomic relations in $\mathcal{A}$ using Σ_2-formulas in $(\mathcal{A}^{-(\omega+1)})^{(\omega)}$, and so $\mathcal{A} \leqslant_\Sigma (\mathcal{A}^{-(\omega+1)})^{(\omega+1)}$.

Let $\mathcal{A} \leqslant_\Sigma \mathcal{M}^{(\omega+1)}$ for some countable structure $\mathcal{M}$. Without loss of generality we can assume $|\mathcal{A}| = HF(|\mathcal{M}|)$. Let us prove that $\mathcal{A}^{-(\omega+1)}$ is Σ-definable in $\mathbb{HF}(\mathcal{M})$. Fix an arbitrary $R \in \sigma$. To show that the relation I_R of $\mathcal{A}^{-(\omega+1)}$ is Σ-definable in $\mathbb{HF}(\mathcal{M})$ it is sufficient to establish the existence of a family of structures $\mathcal{P}_{x,y}^{a,k}$, $a,x,y \in HF(|\mathcal{M}|)$, $k\in\omega$, such that

1. for all $a,x,y\in HF(|\mathcal{M}|)$ there is a $\{0,1\}$-valued non-increasing function h such that $\mathcal{P}_{x,y}^{a,k} = \mathcal{B}_{h(k)}$ for every k;
2. for all $x,y\in HF(|\mathcal{M}|)$, $\mathcal{A}\models R(x,y)$ iff there is an $a\in HF(\mathcal{M})$ such that $\mathcal{P}_{x,y}^{a,k} = \mathcal{B}_1$ for every k;
3. there is a Σ-formula Φ such that for all $a,x,y \in HF(|\mathcal{M}|)$, $k\in\omega$, $\Phi(a,x,y,k)$ defines $(\mathcal{P}_{x,y}^{a,k})^{-k}$ in $\mathbb{HF}(\mathcal{M})$.

Since $\mathcal{A} \leqslant_\Sigma \mathcal{M}^{(\omega+1)}$ we have $\mathcal{A} \leqslant_{\Sigma_2} \mathcal{M}^{(\omega)}$. Let Ψ be a Δ_0-formula in $\mathcal{M}^{(\omega)}$ such that

$$\mathcal{A}\models R(x,y) \iff M^{(\omega)} \models \exists a \forall b\, \Psi(a,b,x,y).$$

Using Lemma 1 we can fix a sequence of formulas $\{\Theta_k\}_{k\in\omega}$ such that Θ_k is Π_k in $\mathbb{HF}(\mathcal{M})$ for every k and

$$\mathcal{M}^{(\omega)} \models \forall b\, \Psi(a,b,x,y) \iff \forall k\, [\mathbb{HF}(\mathcal{M}) \models \Theta_k(a,x,y)],$$

$$\mathcal{M}^{(\omega)} \models \neg\forall b\, \Psi(a,b,x,y) \iff \exists k_0 \forall k \geqslant k_0\, [\mathbb{HF}(\mathcal{M}) \models \neg\Theta_k(a,x,y)],$$

for all $a\in HF(|\mathcal{M}|)$. Define

$$\mathcal{P}_{x,y}^{a,k} = \begin{cases} \mathcal{B}_1, & \text{if } \mathbb{HF}(\mathcal{M}) \models \Theta_k(a,x,y), \\ \mathcal{B}_0, & \text{if } \mathbb{HF}(\mathcal{M}) \models \neg\Theta_k(a,x,y). \end{cases}$$

Note that $\mathcal{P}^{a,k}_{x,y} \leqslant_\Sigma \mathcal{M}^{(k)}$ uniformly by all parameters. Therefore, using Theorem 2 we have also uniform definabilities $(\mathcal{P}^{a,k}_{x,y})^{-k} \leqslant_\Sigma \mathcal{M}$. So that the structures $\mathcal{P}^{a,k}_{x,y}$ satisfy the conditions 1-3. This ends the proof. $\square$

For an analogue of Corollary 2 we need at least one structure $\mathcal{B}$ with $\mathcal{B}^{(\alpha)} \equiv_\Sigma \mathcal{A}$. An analysis of Σ^c_α-relations in the proofs from [9] gives us such an $\mathcal{B}$.

Corollary 3. *If α is a successive computable ordinal and $\emptyset^{(\alpha)}$ is Δ on $\mathbb{HF}(\mathcal{A})$ for a countable $\mathcal{A}$ then $\mathcal{A} \equiv_\Sigma (\mathcal{A}^{-\alpha})^{(\alpha)}$.*

Proof. Let $\mathcal{A}^*$ be the structure constructed by Goncharov, Harrizanov, Knight, McCoy, Miller, Solomon [9]. It is not hard to see that within α jumps one can recover $\mathcal{A}$ from $\mathcal{A}^*$. For the other direction, it is clear that $\mathcal{A}^*$ has a Σ-definable copy $\mathcal{M}$ in $\mathbb{HF}(\mathcal{A})$, but we need to show that the Σ_α-diagram of $\mathcal{M}$ is also Σ-definable in $\mathbb{HF}(\mathcal{A})$. For this, we show that the Σ_α-diagram of $\mathcal{M}$ within $\mathcal{A}$ is r.i.c.e. (relatively intrinsically c.e.) in $\mathbb{HF}(\mathcal{A})$. Consider any presentation $\mathcal{A}_1$ of $\mathcal{A}$. It codes $\emptyset^{(\alpha)}$, so there is a real Y such that $Y^{(\alpha)}$ is Turing equivalent to the degree of the presentation. Using [9, Lemma 5.5], we get that since $\mathcal{A}_1$ is $Y^{(\alpha)}$-computable, the corresponding copy $\mathcal{M}_1$ of $\mathcal{A}_1^*$ is Y-computable, and the isomorphism between $\mathcal{M}$ and $\mathcal{M}_1$ is $Y^{(\alpha)}$-computable. Now, $Y^{(\alpha)}$ can compute the Σ_α-diagram of $\mathcal{M}_1$, and mapping it through the isomorphism, it can also compute the Σ_α-diagram of $\mathcal{M}_1$ as wanted. Thus, $(\mathcal{A}^*)^{(\alpha)} \equiv_\Sigma \mathcal{A}$, and also $\mathcal{A}^{-\alpha} \leqslant_\Sigma \mathcal{A}^*$ by Theorem 3. Therefore, $\mathcal{A} \leqslant_\Sigma (\mathcal{A}^{-\alpha})^{(\alpha)} \leqslant_\Sigma (\mathcal{A}^*)^{(\alpha)} \leqslant_\Sigma \mathcal{A}$. $\square$

5. Inversions of the ω-jump

Soskov [15] proved that for limit ordinal $\alpha = \omega$ some structures $\mathcal{A}$ can computably interpret $\emptyset^{(\omega)}$ having no ω-jump inversions $\mathcal{B}$, $\mathcal{B}^{(\omega)} \equiv_\Sigma \mathcal{A}$. In this section we prove that there are structures $\mathcal{A}$ which have such inversions but have no least one in the sense of Theorem 3. Namely, we can consider the word structures $\langle \omega, s(x) = x+1, A(x) \rangle$, where $A >_T \emptyset^{(\omega)}$. The ω-jump inversions of such structures exist since $B^{(\omega)} \equiv_T A$ for some set B.

To show that there is no Σ-least ω-jump inversions we need the following technical definition and lemma.

Definition 4 ([16]). *Fix a set Y. A sequence of sets $\{A_i\}_{i\in\mathbb{N}}$ is $\Sigma^0_{i\mapsto i}(Y)$ if there is an index e such that $A_i = W_e^{Y^{(i)}}$ for all i. In other words, the sequence of sets $\{A_i\}_{i\in\mathbb{N}}$ is $\Sigma^0_{i\mapsto i}(Y)$ if A_i is $\Sigma^0_{i+1}(Y)$ for all i, uniformly in i. If $Y=\emptyset$ we simply write $\Sigma^0_{i\mapsto i}$.*

Lemma 2. *Fix a set X. If a sequence of sets $\{A_i\}_{i\in\mathbb{N}}$ is $\Sigma^0_{i\mapsto i}(Y)$ for all Y with $X \leqslant_T Y^{(\omega)}$, then the sequence $\{A_i\}_{i\in\mathbb{N}}$ is $\Sigma^0_{i\mapsto i}$.*

Proof. Fix a set X and a sequence of sets $\{A_i\}_{i\in\mathbb{N}}$. We suppose the sequence $\{A_i\}_{i\in\mathbb{N}}$ is not $\Sigma^0_{i\mapsto i}$ and show there is a Y with $X \leqslant_T Y^{(\omega)}$ for which the sequence $\{A_i\}_{i\in\mathbb{N}}$ is not $\Sigma^0_{i\mapsto i}(Y)$. The construction of the set Y is done by finite extension: Depending on the parity (modulo three) of the stage, we work towards coding X into $Y^{(\omega)}$, towards making Y arithmetically generic, and towards making the sequence $\{A_i\}_{i\in\mathbb{N}}$ is not $\Sigma^0_{i\mapsto i}(Y)$.

Construction: We define $Y[0]=\lambda$, the empty string.
At stage $s=3t$, we code another bit of X into Y, defining

$$Y[s+1] = Y[s]^\frown X(s).$$

At stage $s=3t+1$, we work towards arithmetic genericity, defining

$$Y[s+1] = \begin{cases} Y[s]^\frown\rho_s, & \text{if } \rho_s \text{ is the length-lexicographically least string } \rho \\ & \quad\text{with } m \in W_j^{Y[s]^\frown\rho_s\oplus\emptyset^{(k)}}, \\ Y[s], & \text{otherwise, i.e., if no such } \rho \text{ exists,} \end{cases}$$

where j, k, and m are such that $t=\langle j,k,m\rangle$.
At stage $s=3t+2$, we define an auxiliary sequence $\{B_i\}_{i\in\mathbb{N}}$ by

$$B_i = \{z : \exists\gamma\,[z \in W_t^{Y[s]^\frown\gamma\oplus\emptyset^{(i)}}]\}.$$

As $\{B_i\}_{i\in\mathbb{N}}$ is $\Sigma^0_{i\mapsto i}$ and $\{A_i\}_{i\in\mathbb{N}}$ is not, we let l be the least index i with $A_i \neq B_i$. We work towards making $\{A_i\}_{i\in\mathbb{N}}$ not $\Sigma^0_{i\mapsto i}(Y)$, defining

$$Y[s+1] = Y[s]^\frown\gamma_s,$$

where γ_s is an effective prefix-free description of the length-lexicographically least string γ witnessing $z_s \in B_l \setminus A_l$, where z_s is minimal witnessing $B_l \not\subseteq A_l$, if $B_l \not\subseteq A_l$; and where γ_s is the effective prefix-free description of the empty string otherwise.

We define $Y = \bigcup_s Y[s]$.

Verification: Together, the sets Y and $\emptyset^{(\omega)}$ can recover the action of the construction, stage by stage: Given $Y[s]$ and Y, if $s = 3t$, we have $X(s) = Y[|Y[s]|]$. Given $Y[s]$, if $s = 3t+1$, the oracle $Y[s] \oplus \emptyset^{(k+1)}$ can uniformly identify whether a string ρ exists and, if one does, the lexicographically-least string ρ_s. Given $Y[s]$ and Y, if $s = 3t+2$, by checking $Y[|Y[s]|]$ we can identify whether $A_l \not\subseteq B_l$ and, if not, identify the string γ_s (since it was encoded in an effective prefix-free manner).

Thus, together, the sets Y and $\emptyset^{(\omega)}$ can recover X. Consequently, we have $Y^{(\omega)} \geqslant_T Y \oplus \emptyset^{(\omega)} \geqslant_T X$ as needed. Indeed, more is true, namely that $Y \oplus \emptyset^{(n)} \geqslant_T Y^{(n)}$ for all n. For $n = 1$, this is a consequence of forcing the jump of Y at stages s of the form $s = 3\langle j, 0, m\rangle + 1$. For larger n, this is a consequence of the inductive hypothesis (the statement is true for $n-1$) and forcing the jump of Y at stages s of the form $s = 3\langle j, n, m\rangle + 1$.

It remains to show the $\{A_i\}_{i\in\mathbb{N}}$ are not $\Sigma^0_{i\mapsto i}(Y)$. Fixing an index j, we show there is an index k such that $A_k \neq W_j^{Y^{(k)}}$. Since $Y \oplus \emptyset^{(k)} \geqslant_T Y^{(k)}$, there is an index j' such that $W_j^{Y^{(k)}} = W_{j'}^{Y\oplus\emptyset^{(k)}}$. We consider the action of the construction at stage $s = 3j' + 2$. If $A_l \not\subseteq B_l$, then there is an integer $z \in A_l$ such that $z \notin W_{j'}^{Y[s]^\frown\gamma\oplus\emptyset^{(l)}}$ for any γ. Thus $z \notin W_{j'}^{Y\oplus\emptyset^{(l)}} = W_j^{Y^{(l)}}$, so the index l suffices. Otherwise, let z_s be the least integer z with $z \notin B_l \setminus A_l$. Let γ_s be the length-lexicographically least string γ with $z_s \in W_{j'}^{Y[s]^\frown\gamma\oplus\emptyset^{(l)}}$. Thus $z_s \notin W_{j'}^{Y\oplus\emptyset^{(l)}} = W_j^{Y^{(l)}}$, so the index l suffices.

We conclude that the $\{A_i\}_{i\in\mathbb{N}}$ are not $\Sigma^0_{i\mapsto i}(Y)$. $\square$

Now we are ready to prove the main result of the section, which gives us not only non-existence of the Σ-least inversions of the ω-jump, but also the non-existence of the structures with the degree spectrum $\{\mathbf{d} : \mathbf{d}^{(\omega)} \geqslant \mathbf{a}\}$.

Theorem 4. *Fix a degree* $\mathbf{a}$ *with* $\mathbf{a} > \mathbf{0}^{(\omega)}$*. There is no structure* $\mathcal{S}$ *such that for every Turing degree* $\mathbf{d}$

$$\mathcal{S} \textit{ has a } \mathbf{d}\textit{-computable copy} \iff \mathbf{d}^{(\omega)} \geqslant \mathbf{a}.$$

Proof. Assume that there is a structure $\mathcal{S}$ such that for every Turing degree $\mathbf{d}$

$$\mathcal{S} \text{ has a } \mathbf{d}\text{-computable copy} \iff \mathbf{d}^{(\omega)} \geqslant \mathbf{a}.$$

Let $\mathbf{a} = deg_T(A)$. Then by Theorem 5 from [17] the sets A and $\overline{A}$ are Σ-definable in $S^{(\omega)}$. Fix a Σ-formula Φ and a tuple $\vec{a} \in HF(\mathcal{S})$ such that for every integer n

$$n \in A \iff \mathcal{S}^{(\omega)} \models \Phi(\vec{a}, n).$$

By Lemma 1, there is a uniformly computable sequence of formulas $\{\Phi_k\}_{k\in\omega}$ such that $\Phi_k \in \Sigma_k$ for every k and

$$n \in A \iff \exists k\, [\mathbb{HF}(\mathcal{S}) \models \Phi_k(\vec{a}, n)]$$

for all $n \in \omega$. For every $k \in \omega$ define A_k by

$$A_k = \{n : \mathbb{HF}(\mathcal{S}) \models \Phi_k(\vec{a}, n)\}.$$

It's easy to see that $A = \bigcup_k A_k$, and for every Y which computes a copy of $\mathcal{S}$ we have $A_k \in \Sigma_k(Y)$ uniformly by k. Therefore, the sequence $\{A_k\}_{k\in\omega}$ is $\Sigma^0_{i\mapsto i}(Y)$ for all Y with $A \leqslant_T Y^{(\omega)}$. By Lemma 2, $\{A_k\}_{k\in\omega}$ is $\Sigma^0_{i\mapsto i}$. Hence, $A \in \Sigma^0_1(\emptyset^{(\omega)})$. Similarly, we obtain $\overline{A} \in \Sigma^0_1(\emptyset^{(\omega)})$. This contradicts the fact that $\mathbf{a} > \mathbf{0}^{(\omega)}$. □

It follows immediately from the last theorem, that for $\mathbf{a} > \mathbf{0}^{(\omega)}$ the structures with the degree spectra $\{\mathbf{d} : \mathbf{d} \geqslant \mathbf{a}\}$ (i.e, the $\mathbf{a}$-computable structures in which an of element of $\mathbf{a}$ is Δ-definable) have no Σ-least ω-jump inversion.

Corollary 4. *Let* $\mathbf{a} > \mathbf{0}^{(\omega)}$ *and let* $\mathcal{A}$ *be any countable structure which has a* $\mathbf{d}$*-computable copy iff* $\mathbf{d} \geqslant \mathbf{a}$ *for every Turing degree* $\mathbf{d}$ *(for example,* $\mathcal{A} = \langle \omega, s(x) = x+1, A(x)\rangle$*, where* $A \in \mathbf{a}$*). Then there is no structure* $\mathcal{S}$ *such that*

1. $\mathcal{A} \leqslant_\Sigma \mathcal{S}^{(\omega)}$*;*

2. $\mathcal{A} \leqslant_\Sigma \mathcal{M}^{(\omega)} \Rightarrow \mathcal{S} \leqslant_\Sigma \mathcal{M}$ *for every countable structure* $\mathcal{M}$.

Indeed, to deduce the corollary from Theorem 4 it is enough to consider every degree $\mathbf{d}$ as a $\mathbf{d}$-computable structure $\mathcal{M}_{\mathbf{d}}$ in which an element of $\mathbf{d}$ is Δ-definable. Also it easy to check that all our arguments for ω can be adapted to any computable limit ordinal.

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