CUTS OF LINEAR ORDERS

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Abstract. We study the connection between the number of ascending and descending cuts of a linear order, its classical size, and its effective complexity (how much [how little] information can be encoded into it).

1. Introduction

A fundamental question in effective algebra is which mathematical structures have effective copies. A related important question is what information can be encoded in the isomorphism type of a fixed structure. The *degree spectrum* of a structure, the set of Turing degrees that code a copy of the structure, provides a measure of the information that can be encoded in the isomorphism type.

For some classes of algebraic structures, the degree spectrum behaves in unusual ways. For example, the Low_n Conjecture for Boolean algebras is a well-known conjecture in effective algebra that, informally speaking, states if the amount of information encoded in the isomorphism type of a Boolean algebra is small, then in fact it encodes no information. In this case, the notion of being low_n formalizes the idea of having little information content.

Definition 1.1. A set $X \subseteq \omega$ is low_n if $X^{(n)} \equiv_T \emptyset^{(n)}$, where $A^{(n)}$ is the nth Turing jump of the set A.

Conjecture 1.2 (The Low_n Conjecture [DJ94])**.** *Every low_n Boolean algebra has a computable copy.*

This conjecture has been solved affirmatively up to level four by Knight and Stob (see [KS00], and also [DJ94] and [Thu95] for level one and two), and recent work by Harris and Montalbán demonstrates that an important new obstacle appears at level five (see [HMa] and [HMb]).

It is a common theme in effective mathematics to attempt to understand the properties of the degree spectra of mathematical structures. The question above falls inside this theme, but another motivation for posing this question is that an affirmative answer would say that the information content of the isomorphism type of a Boolean algebra has to be encoded in a rather uncommon way. However, we will show in this paper that the class of Boolean algebras is not the only example of this unusual behavior (if indeed it is an example), as this phenomena already

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occurs in the class of linear orders with only finitely many ascending or finitely many descending cuts.

Definition 1.3. A *cut* of a linear order $\mathcal{L}$ is a partition $(\mathcal{I}, \mathcal{J})$ of $\mathcal{L}$ where $\mathcal{I}$ is an initial segment of $\mathcal{L}$ and $\mathcal{J}$ is an end segment of $\mathcal{L}$. As every initial segment $\mathcal{I}$ of $\mathcal{L}$ determines a unique cut $(\mathcal{I}, \mathcal{J})$ of $\mathcal{L}$, we often denote a cut $(\mathcal{I}, \mathcal{J})$ by $\mathcal{I}$.

A cut $(\mathcal{I}, \mathcal{J})$ is an *ascending cut* if $\mathcal{I}$ is nonempty and has no greatest element and is a *descending cut* if $\mathcal{J}$ is nonempty and has no least element.

As $(\mathcal{I}, \mathcal{J})$ is an ascending cut of $\mathcal{L}$ if and only if $(\mathcal{J}, \mathcal{I})$ is a descending cut of $\mathcal{L}^*$ (where $a <_{\mathcal{L}^*} b$ if and only if $b <_{\mathcal{L}} a$), it makes no difference if ascending cuts or descending cuts are studied. Consequently, we consider only descending cuts hereafter.

The number of descending cuts in a countable linear order $\mathcal{L}$ provides a classical measure of the "size" of $\mathcal{L}$. We recall that a linear order $\mathcal{L}$ is *scattered* if the order type of the rationals does not embed into it.

Theorem 1.4. *If a countable linear order $\mathcal{L}$ has no ascending and no descending cuts, then $\mathcal{L}$ is finite. If a countable linear order $\mathcal{L}$ has no descending cuts, then $\mathcal{L}$ is well-ordered.*

If a countable linear order $\mathcal{L}$ has only one descending cut, then $\mathcal{L}$ is of the form

$$\beta + \sum_{j \in \omega^*} \omega^{\gamma_j} + \alpha$$

for some ordinals β, α and γ_j (possibly zero) with $\gamma_{j+1} \geq \gamma_j$ for all $j \in \omega$.

If a countable linear order $\mathcal{L}$ has only countably many descending cuts, then $\mathcal{L}$ is scattered. If a countable linear order $\mathcal{L}$ has uncountably many descending cuts, then $\mathcal{L}$ is nonscattered.

We defer the proof of Theorem 1.4 to Lemma 2.3 and Proposition 2.4. The main result of this paper is the following.

Theorem 1.5. *If $\mathcal{L}$ is a linear order with only finitely many descending cuts, then the existence of a low$_n$ copy implies the existence of a computable copy.*

Of course, if $\mathcal{L}$ is a linear order with no descending cuts, then the existence of a hyperarithmetic copy implies the existence of a computably copy (see [Spe55]). On the other hand, for every Turing degree $\mathbf{a} > \mathbf{0}$, there is a linear order with uncountably many descending cuts that has an $\mathbf{a}$-computable presentation but no computable presentation (see [AK00, Theorem 9.15] for this result by Knight that culminates a sequence of results by others).

In addition to showing Theorem 1.4 (Section 2) and Theorem 1.5 (Section 3), we discuss the optimality of the latter (Section 4). Amongst this discussion, we show the following.

Theorem 1.6. *There is a linear order with exactly one descending cut having a presentation of intermediate degree but no computable presentation.*

We leave open whether Corollary 4.6 is optimal.

Question 1.7. Does every low$_2$ linear ordering with only countably many descending cuts (i.e., scattered) have a computable copy?

Alaev, Thurber, and Frolov have obtained results related to this question in [ATF09].

2. Classical Study of Descending Cuts

In this section, we analyze the connection between the order type of a linear order and the cardinality of its ascending and descending cuts. Linear orders of the following form will be ubiquitous.

Definition 2.1. If $\Gamma = \{\gamma_i\}_{i\in\omega}$ is any nondecreasing sequence of countable ordinals, denote by $\mathcal{L}_\Gamma$ the linear order

$$\mathcal{L}_\Gamma := \cdots + \omega^{\gamma_i} + \cdots + \omega^{\gamma_1} + \omega^{\gamma_0}$$

with the convention that $\omega^0 = 1$.

The importance of the $\mathcal{L}_\Gamma$ is that they can be used to characterize the linear orders with only finitely many descending cuts.

Lemma 2.2. *If $\mathcal{L}$ is a countable linear order with no least element and such that every end segment $\mathcal{J}$ is well-ordered, then there is a nondecreasing sequence of countable ordinals $\Gamma = \{\gamma_i\}_{i\in\omega}$ and a countable ordinal β such that $\mathcal{L} = \mathcal{L}_\Gamma + \beta$.*

Proof. Since $\mathcal{L}$ has no least element, we can decompose $\mathcal{L}$ into an infinite ω^* sum

$$\mathcal{L} = \sum_{i\in\omega^*} \mathcal{L}_i = ... + \mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0,$$

where each $\mathcal{L}_i$ is an ordinal and ω^* is the order of the negative integers. As $\mathcal{L}_i$ is an ordinal, it can be written as $\omega^{\beta_{i,k}} + \cdots + \omega^{\beta_{i,1}} + \omega^{\beta_{i,0}}$ with $\beta_{i,0} \leq \beta_{i,1} \leq \cdots \leq \beta_{i,k}$. Thus by splitting each $\mathcal{L}_i$ as necessary, we may assume each $\mathcal{L}_i$ is of the form ω^{β_i}. Let $\gamma = \limsup_{i\in\omega} \beta_i$. We consider separately when there are only finitely many k with $\beta_k = \gamma$ from when there are infinitely many k with $\beta_k = \gamma$.

First, suppose there are infinitely many k with $\beta_k = \gamma$. Let k_0 be such that $\beta_{k_0} = \gamma$ and $\forall k \geq k_0$ $(\beta_k \leq \gamma)$; let $\{k_0 < k_1 < k_2 < ...\}$ be the set of all k greater than or equal to k_0 such that $\beta_k = \gamma$. By properties of ordinal addition, we have that for each i, $\omega^{\beta_{k_{i+1}-1}} + \omega^{\beta_{k_{i+1}-2}} + \cdots + \omega^{\beta_{k_i}} = \omega^\gamma$. Therefore

$$\mathcal{L} = \cdots + \omega^\gamma + \omega^\gamma + \omega^\gamma + \left(\sum_{j=k_0-1}^{j=0} \omega^{\beta_j}\right),$$

as desired.

Second, suppose there are only finitely many k with $\beta_k = \gamma$. Let k_0 be such that $\forall k \geq k_0$ $(\beta_k < \gamma)$. We define a sequence $k_0 < k_1 < \ldots$ as follows. Let k_i be the least $k > k_{i-1}$ such that $\beta_k \geq \beta_{k_{i-1}}$. Let $\gamma_i = \beta_{k_i}$. Note that $\{\gamma_i\}_{i\in\omega}$ is a nondecreasing sequence. Again, by properties of ordinal addition, we have that for each i, $\omega^{\beta_{k_{i+1}-1}} + \omega^{\beta_{k_{i+1}-2}} + \cdots + \omega^{\beta_{k_i}} = \omega^{\gamma_i}$. Therefore

$$\mathcal{L} = \cdots + \omega^{\gamma_2} + \omega^{\gamma_1} + \omega^{\gamma_0} + \left(\sum_{j=k_0-1}^{j=0} \omega^{\beta_j}\right),$$

as desired. □

Lemma 2.3. *If a countable linear order $\mathcal{L}$ has only finitely many descending cuts, then $\mathcal{L}$ is of the form*

$$\mathcal{L} = \alpha + \mathcal{L}_{\Gamma_1} + \alpha_1 + \mathcal{L}_{\Gamma_2} + \alpha_2 + \cdots + \mathcal{L}_{\Gamma_n} + \alpha_n$$

for some (possibly 0) ordinals α and α_j and linear orders $\mathcal{L}_{\Gamma_j}$ for $1 \leq j \leq n$.

Proof. Let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be the initial segments of $\mathcal{L}$ that define the finitely many descending cuts; for notational convenience, let $\mathcal{I}_0 = \emptyset$ and $\mathcal{I}_{n+1} = \mathcal{L}$. Then $\mathcal{L} = \mathcal{L}_0 + \cdots + \mathcal{L}_n$, with $\mathcal{L}_j = \mathcal{I}_{j+1} \backslash \mathcal{I}_j$.

As $\mathcal{L}_0 = \mathcal{I}_1 \backslash \mathcal{I}_0$ cannot contain a descending cut, it must be an ordinal which we denote by α. It thus suffices to argue that for each $j \geq 1$, the linear order $\mathcal{L}_j$ is of the form $\mathcal{L}_{\Gamma_j} + \alpha_j$ for some nondecreasing sequence of countable ordinals Γ_j and ordinal α_j. Each $\mathcal{L}_j$ for $j \geq 1$ has no minimal element, else $\mathcal{I}_j$ would not define a descending cut. On the other hand, any proper end segment of $\mathcal{L}_j$ for any $j \geq 1$ must be well-ordered as a consequence of the hypothesis that the $\mathcal{I}_j$ define all the descending cuts. It follows from Lemma 2.2 that each $\mathcal{L}_j$ for $j \geq 1$ is of the form $\mathcal{L}_{\Gamma_j} + \alpha_j$ for some nondecreasing sequence of countable ordinals Γ_j and ordinal α_j. □

Having characterized the countable linear orders with finitely many descending cuts, we turn to characterizing the countable linear orders with countably many descending cuts. We recall the *block* $\mathbf{c}(x)$ of a point x in a linear order $\mathcal{L}$ is the set of points $\{y \in L : \text{the interval between } x \text{ and } y \text{ is finite}\}$. We define points $x_1, x_2 \in L$ as being in the same $\mathbf{c}$-equivalence class (*condensation class*) if $\mathbf{c}(x_1) = \mathbf{c}(x_2)$, noting this is indeed an equivalence relation on L. In a similar manner, we recall the α^{th} *condensation block* $\mathbf{c}^{(\alpha)}(x)$ of a point x in a linear order $\mathcal{L}$ is defined by recursion by $\mathbf{c}^{(0)}(x) = \{x\}$,

$$\mathbf{c}^{(\alpha+1)}(x) := \{y \in L : \text{the } \mathcal{L}\text{-interval between } x \text{ and } y \text{ contains at most finitely many } \mathbf{c}^{(\alpha)} \text{ classes}\},$$

and $\mathbf{c}^{(\alpha)}(x) := \cup_{\beta<\alpha} \mathbf{c}^{(\beta)}(x)$ for limit α, with $x_1, x_2 \in L$ belonging in the same $\mathbf{c}^{(\alpha)}$-equivalence class (α^{th} *condensation class*) if $\mathbf{c}^{(\alpha)}(x_1) = \mathbf{c}^{(\alpha)}(x_2)$.

Viewing these as (linearly ordered) equivalence classes, we can define the α^{th} *condensation* of a linear order $\mathcal{L}$, denoted $\mathcal{L}^{(\alpha)}$, to be the linear order $\mathcal{L}^{(\alpha)} := \mathcal{L}/\mathbf{c}^{(\alpha)}$. The *condensation rank* of a linear order $\mathcal{L}$ is the least ordinal α such that $\mathcal{L}^{(\alpha)} = \mathcal{L}^{(\alpha+1)}$ (i.e., the least ordinal α such that $\mathbf{c}^{(\alpha)}(x) = \mathbf{c}^{(\alpha+1)}(x)$ for all x).

Proposition 2.4. *[Folklore (Exercise 5.33(1) of [Ros82]] A countable linear order $\mathcal{L}$ is scattered if and only if it has only countably many descending cuts.*

Proof. If $\mathcal{L}$ is scattered, the condensation rank $r(\mathcal{L})$ of $\mathcal{L}$ is a countable ordinal α. We show by induction on α that a linear order of countable condensation rank α has only countably many descending cuts. If $\alpha = 0$, then $\mathcal{L} = 1$ and there are no descending cuts. If $\alpha > 0$, then $\mathcal{L} = \sum_{z \in \mathbb{Z}} \mathcal{L}_z$ for some (possibly empty) linear orders $\mathcal{L}_z$ with $r(\mathcal{L}_z) < r(\mathcal{L})$. Any descending cut in $\mathcal{L}$ is either a descending cut of some $\mathcal{L}_z$ or of the form $\sum_{z<n} \mathcal{L}_z$ for some $n \in \mathbb{Z} \cup \{-\infty\}$. Both of these are countable (the former by the inductive hypothesis), so $\mathcal{L}$ has only countably many descending cuts.

If $\mathcal{L}$ is nonscattered, the set of points X in the perfect kernel has the order type of a dense linear order, possibly with endpoints. Associate to each point $x \in X$ (excepting the least and greatest if they exist) a rational number $q = q_x \in \mathbb{Q}$ in an order-preserving manner. For each rational number q, let I_q to be the convex subset of $\mathcal{L}$ that is collapsed to x. Then, for each irrational number $r \in \mathbb{R} \backslash \mathbb{Q}$, let $\mathcal{I}_r = \bigcup_{q<r} I_q$. Each of the $\mathcal{I}_r$ defines a descending cut and $\mathcal{I}_r \neq \mathcal{I}_{r'}$ if $r \neq r'$. It follows that $\mathcal{L}$ has continuum many descending cuts. □

3. Effective Study of Descending Cuts

Having established the connection between the number of descending cuts and the classical "complexity" of the linear order, we turn to establishing the connection between the number of descending cuts and the minimal amount of strong jump inversion ensured. In doing so, we will move from $\mathcal{L}_\Gamma$ to condensations of $\mathcal{L}_\Gamma$.

Definition 3.1. Let $\mathcal{L}$ be any linear order. Write $x \equiv_n y$ if the order type of the interval $[x, y]$ can be embedded into a proper interval of ζ^n, where ζ^n is the lexicographic ordering of $\mathbb{Z}^n$ and a proper interval is an convex subset with endpoints in the linear ordering.

For a linear order $\mathcal{L}$, let $\mathcal{L}/\equiv_n$ be the linear order obtained from $\mathcal{L}$ under the equivalence relation $\equiv_n$.

We remark that *condensations* (also known as *splittings*) have been studied in more general contexts (see [Ros82] for a survey). In our context, the passage from $\mathcal{L}$ to $\mathcal{L}/\equiv_n$ is effective in the following sense.

Proposition 3.2. *Uniformly in an index for a computable linear order $\mathcal{L}_\Gamma$, there is an index for a Δ^0_{2n+1} presentation of the linear order $\mathcal{L}_\Gamma/\equiv_n$.*

Proof. The oracle $\emptyset^{(2n)}$ suffices to determine whether the order type of an interval $[x, y]$ is an ordinal less than ω^n (see [AK00, Proposition 7.2]). Noting that any proper interval $[x, y]$ of $\mathcal{L}_\Gamma$ is well ordered, it can be embedded into a proper interval of ζ^n if and only if it is an ordinal less than ω^n. □

Moving the opposite direction is also effective.

Proposition 3.3 (Watnick [Wat84])**.** *Uniformly in a Δ^0_{2n+1} index for the atomic diagram of a linear order $\mathcal{L}$ with distinguished least element, there is a Δ^0_1 index for the atomic diagram of the linear order $\omega^n \cdot \mathcal{L}$.*

Lemma 3.4. *If β is an ordinal and $\Gamma = \{\gamma_i\}_{i\in\omega}$ is a nondecreasing sequence of ordinals so that $\omega^\beta + \mathcal{L}_\Gamma$ is low$_n$, then $\omega^\beta + \mathcal{L}_\Gamma$ is computable.*

Proof. Fix a low$_n$ set A such that $\omega^\beta + \mathcal{L}_\Gamma$ is A-computable. From Proposition 3.2 relativized, the linear order $(\omega^\beta + \mathcal{L}_\Gamma)/\equiv_n$ is $A^{(2n)}$-computable. Note that we may assume $\sup\{\gamma_i : i \in \omega\}$ is infinite. For if it is finite, then Γ contains only finitely much information (namely, how many of the γ_i equal k for each $k \leq \sup\{\gamma_i : i \in \omega\}$), and thus $\omega^\beta + \mathcal{L}_\Gamma$ is computable.

Denote by $\hat{\mathcal{L}}$ the linear order $(\omega^\beta + \mathcal{L}_\Gamma)/\equiv_n$ with least point removed if $\beta < n$ and greatest point removed if $\gamma_0 < n$. Then $\hat{\mathcal{L}}$ is also $A^{(2n)}$-computable, and thus $\emptyset^{(2n)}$-computable as $A^{(2n)} \equiv_T \emptyset^{(2n)}$ since A was low$_n$. So by Proposition 3.3, the linear order $\omega^n \cdot \hat{\mathcal{L}}$ is computable.

Let N be maximal with respect to the property that $\gamma_i < n$ for all $i < N$. Then

$$\omega^\beta + \mathcal{L}_\Gamma = \omega^n \cdot \hat{\mathcal{L}} + \omega^{\gamma_{N-1}} + \cdots + \omega^{\gamma_0}.$$

if $\beta \geq n$ and

$$\omega^\beta + \mathcal{L}_\Gamma = \omega^\beta + \omega^n \cdot \hat{\mathcal{L}} + \omega^{\gamma_{N-1}} + \cdots + \omega^{\gamma_0}$$

if $\beta < n$. In either case, we see that $\omega^\beta + \mathcal{L}_\Gamma$ is computable as a consequence of $\omega^n \cdot \hat{\mathcal{L}}$ being computable. □

Theorem 3.5. *If $\mathcal{L}$ is a linear order having only finitely many descending cuts, then $\mathcal{L}$ admits strong nth jump inversion for every $n \in \omega$.*

Proof. Let $\mathcal{L}$ be a low$_n$ linear order with only finitely many descending cuts. Without loss of generality, we may assume that $\mathcal{L}$ has a least element. By Lemma 2.3, it is of the form $\alpha + \sum_{1 \le j \le n} (\mathcal{L}_{\Gamma_j} + \alpha_j)$ for some ordinals α and α_j and linear orders $\mathcal{L}_{\Gamma_j}$ for $1 \le j \le n$. We rewrite $\mathcal{L}$ as $\sum_{1 \le j \le n} (\alpha'_j + \omega^{\beta'_j} + \mathcal{L}_{\Gamma'_j}) + \alpha'_n$. Note that this is possible by considering an end segment of either $\mathcal{L}_{\Gamma_j}$ or α_j as $\omega^{\beta'_j}$.

Each of the $\omega^{\beta'_j} + \mathcal{L}_{\Gamma'_j}$ is also low$_n$ because each is an interval of $\mathcal{L}$ with endpoints in $\mathcal{L} \cup \{-\infty, +\infty\}$. Thus by Lemma 3.4, each of the $\omega^{\beta'_j} + \mathcal{L}_{\Gamma'_j}$ is computable. It follows that $\mathcal{L}$ is computable, being a finite sum of computable linear orders. □

4. Optimality of Strong Jump Inversion

Having demonstrated the strong jump inversion results, it is natural to ask whether these linear orders admit strong jump inversion at an even higher level. For linear orders with finitely many descending cuts (indeed, even a single descending cut), the question is answered by the following results of Hirschfeldt, Kach, and Montalbán and Kach and J. Miller.

Definition 4.1. Fix a set $X \subseteq \omega$. Define $\Sigma^0_{(2n+3)}(X)$ to be the class of all sets $S \subseteq \omega$ such that there is a c.e. operator W_e such that, for every $n \in \omega$,

$$n \in S \iff n \in W_e^{X^{(2n+2)}}.$$

A set X is *low for Σ-Feiner* if $\Sigma^0_{(2n+3)}(X) \subseteq \Sigma^0_{(2n+3)}(\emptyset)$.

We recall that a set X has *intermediate degree* if, for every $n \in \omega$, it satisfies $0^{(n)} <_T X^{(n)} <_T 0^{(n+1)}$.

Theorem 4.2 (Hirschfeldt, Kach, and Montalbán [HKM])**.** *There is a set X of intermediate Turing degree that is not low for Σ-Feiner.*

Theorem 4.3 (Kach and Miller [KM])**.** *If $A \subseteq \omega$ is any set, then $A \in \Sigma^0_{(2n+3)}(X)$ if and only if the linear order*

$$\mathcal{L}_{\omega,A} := \omega^\omega + (\cdots + \omega^n \cdot A(n) + \cdots + \omega \cdot A(1) + 1 \cdot A(0))$$

is X-computable.

Proof of Theorem 1.6. By Theorem 4.2, fix a set X of intermediate degree that is not low for Σ-Feiner. Let A witness this, i.e., let A belong to $\Sigma^0_{(2n+3)}(X)$ but not $\Sigma^0_{(2n+3)}(\emptyset)$. Then $\mathcal{L}_{\omega,A}$ is X-computable but not computable by Theorem 4.3. □

For linear orders with countably many descending cuts, the question is answered by the following result of [Ler81].

Definition 4.4. If $A = \{a_0 < a_1 < a_2 < \dots\} \subseteq \omega$ is any set, the linear order

$$\zeta + a_0 + \zeta + a_1 + \zeta + a_2 + \dots$$

is a *strong ζ-representation of A.*

Theorem 4.5 (Lerman [Ler81])**.** *A set A has a computable strong ζ-representation if and only if A is Σ^0_3.*

Corollary 4.6. *There is a* low$_3$ *linear order with descending cuts in order type* ω *having no computable presentation.*

Proof. Let $X \subseteq \omega$ be any low$_3$ set that is not low$_2$. Since $\Sigma^0_3 \subsetneq \Sigma^X_3$, by Theorem 4.5 (relativized) there is a set A having an X-computable strong ζ-representation with no computable copy. Any strong ζ-representation has descending cuts in order type ω. □

Finally, for linear orders with uncountably many cuts, the question is answered by the following result (see also, e.g., [Mil01]).

Theorem 4.7 (Knight [AK00])**.** *There is a linear order with a low presentation but no computable presentation.*

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