DECIDABILITY AND COMPUTABILITY OF CERTAIN TORSION-FREE ABELIAN GROUPS

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Abstract. We study completely decomposable torsion-free abelian groups of the form $\mathcal{G}_S := \oplus_{n \in S} \mathbb{Q}_{p_n}$ for sets $S \subseteq \omega$. We show that $\mathcal{G}_S$ has a decidable copy if and only if S is Σ^0_2 and has a computable copy if and only if S is Σ^0_3.

1. Introduction

There are many natural ways to code a set $S \subseteq \omega$ into an algebraic structure $\mathcal{M}_S$. The history of encoding sets (effectively and noneffectively) into algebraic structures is, of course, rather old. The investigation of classical encodings goes back at least as far as Van der Waerden, who considered effective procedures in field theory, but without the language of computability theory (see [20]). Van der Waerden's analysis was formalized by Frölich and Shepherdson (see [8]); Mal′cev (see [15]) and Metakides and Nerode (see [19]) further analyzed the effective coding of sets into fields. There are now a large number of investigations into computable structure theory which rely on various codings into algebraic and combinatorial structures, and for a general reference here we refer the reader to various articles in the Handbook of Computability Theory (see [10]) and Volume 2 of the Handbook of Recursive Mathematics (see [5]).

A hallmark of these investigations was the work of Feiner, who demonstrated that sets more complicated than the Halting Problem could be effectively coded into algebraic structures. For example, Feiner coded Σ^0_3 sets into linear orderings (see [6]) and certain Δ^0_ω-computable sets into Boolean algebras (see [7]).

It seems that each familiar class of algebraic structures allows some natural encoding. Here are some examples:

(1) undirected graphs (e.g., via the presence or absence of n cycles),
(2) linear orders (e.g., via the presence or absence of maximal discrete blocks of size n),
(3) Boolean algebras (e.g., via the presence or absence of n in the measure),
(4) abelian groups (e.g., via the presence or absence of of elements of order p_n),
(5) rings (e.g., via the presence or absence of a p_n^{th} root of unity), and
(6) fields (e.g., via the presence or absence of a p_n^{th} root of unity).

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A common feature of these encodings is that there are natural sentences φ_n for $n \in \omega$ with the property that $\mathcal{M}_S \models \varphi_n$ if and only if $n \in S$. In the examples above, except for the case of Boolean algebras, the sentences are finitary; for Boolean algebras, the sentences are computable infinitary. The complexity of the sentences φ_n yields an upper bound on how complex S can be if $\mathcal{M}_S$ is to be decidable (computable). For undirected graphs, $\mathcal{M}_S$ has a decidable copy if and only if S is decidable, and $\mathcal{M}_S$ has a computable copy if and only if S is computably enumerable. For linear orders, $\mathcal{M}_S$ has a decidable copy if and only if S is decidable, and $\mathcal{M}_S$ has a computable copy if and only if S is Σ^0_3. For rings and fields, $\mathcal{M}_S$ has a decidable copy if and only if S is decidable, and $\mathcal{M}_S$ has a computable copy if and only if S is computably enumerable.

The purpose of this paper is to study an encoding of sets $S \subseteq \omega$ into completely decomposable torsion-free abelian groups.

Definition 1.1. A torsion-free abelian group $\mathcal{A}$ is *completely decomposable* if there is a collection of groups $(\mathcal{A}_i)_{i \in I}$ with

$$\mathcal{A} \cong \bigoplus_{i \in I} \mathcal{A}_i$$

and $\mathcal{A}_i \leq (\mathbb{Q} : +)$ for each $i \in I$.

It is well-known that the class of torsion-free abelian groups is classically quite complicated. Indeed, it has no simple invariants as a consequence of work by Downey and Montalbán (see [3]) and Hjorth (see [11]). Thus, special classes of torsion-free abelian groups classes are central objects of study in the area. The classic example of this is the collection of rank one torsion-free abelian groups, equivalently the subgroups of the additive group of the rationals. This was first studied by Baer (see [2]) and is easy to understand in the classical and effective settings.

The class of completely decomposable groups was also introduced by Baer in 1937 (see [2]), and seems the next most tractable class to understand after the rank one groups. This class has been well-studied and possesses a number of nice algebraic properties (see, e.g., [9]). One such property (which we use without further mention) is that a completely decomposable torsion-free abelian group has a unique direct decomposition (up to permutations of the summands).

However, not as much is known about the effective properties of completely decomposable torsion-free abelian groups. Mal′cev initiated the study of torsion-free abelian groups (see [16]). Khisamiev and Krykpaeva introduced the class of effectively (strongly) decomposable torsion-free abelian groups (see [14]). A completely decomposable torsion-free abelian group $\mathcal{A} \cong \bigoplus_{i \in I} \mathcal{A}_i$ is *effectively (strongly) decomposable* if it has a computable (decidable) copy in which the predicates $P_i(x) \leftrightharpoons x \in \mathcal{A}_i$ are uniformly computable. Khisamiev and Krykpaeva then studied a particular encoding of sets into completely decomposable torsion-free abelian groups.

Definition 1.2. Let $(p_n)_{n \in \omega}$ be the sequence of prime numbers, in ascending order. For each prime p, denote by $\mathbb{Q}_p$ the subgroup of $(\mathbb{Q} : +)$ generated by the numbers $1/p^k$ for $k \in \omega$. If $S \subseteq \omega$ is nonempty, denote by $\mathcal{G}_S$ the group

$$\mathcal{G}_S := \bigoplus_{n \in S} \mathbb{Q}_{p_n}.$$

This group is termed *S-divisible*.

Khisamiev and Krykpaeva showed that $\mathcal{G}_S$ is effectively decomposable if and only if S is Σ_2^0 (see [14]); Khisamiev showed that $\mathcal{G}_S$ is strongly decomposable if and only if S is Σ_2^0 and not quasihyperhyperimmune (see [13]).

Khisamiev, in personal correspondence with the sixth author, asked for necessary and sufficient conditions for $\mathcal{G}_S$ to have a computable (decidable) copy. We answer this question, showing $\mathcal{G}_S$ has a decidable copy if and only if S is Σ_2^0 (Theorem 2.2) and showing $\mathcal{G}_S$ has a computable copy if and only if S is Σ_3^0 (Theorem 3.3). We extend these results slightly to answer the same questions for completely decomposable torsion-free abelian groups of the form $\bigoplus_{j\in\omega}\mathbb{Q}_{p_{a_j}}$, where the primes p_{a_j} need not be distinct (this was Khisamiev's precise question).

For background on effective algebra, we refer the reader to [1] and [4]; for background on effective algebra and completely decomposable torsion-free abelian groups, we refer the reader to [9], [17], and [18].

2. Characterizing the Decidable $\mathcal{G}_S$

It is easy to see that if $\mathcal{G}_S$ has a decidable copy, then S is Σ_2^0. The reason, of course, is that $n \in S$ if and only if

$$\mathcal{G}_S \models (\exists x)\bigwedge_k(\exists y)\left[x = p_n^k y\right],$$

and we can decide satisfaction of the formulas $(\exists y)\left[x = p_n^k y\right]$ in a decidable copy. We will show that if S is Σ_2^0, then $\mathcal{G}_S$ has a decidable copy. The lemma below gives a sufficient condition for a copy to be decidable.

Lemma 2.1. *If $\mathcal{G}_S$ is an S-divisible group, a computable copy $\mathcal{G}$ of $\mathcal{G}_S$ is decidable if it is computable after expanding by the relations $p|x$ (x is divisible by p).*

Proof. As a consequence of an elimination of quantifiers result by Szmielew for abelian groups, it suffices to demonstrate that $\mathrm{Th}(\mathcal{G}_S)$ is decidable and the computability of the relations $p|x$ implies the computability of the relations $n|x$.

We first show that $\mathrm{Th}(\mathcal{G}_S)$ is decidable. It is not difficult to see that the Szmielew invariants of the groups $\mathcal{G}_S$ are the same as those for a direct sum of $|S|$ copies of $\mathbb{Z}$. It follows that the two theories are the same. The latter theory is decidable, since $\mathrm{Th}(\mathbb{Z})$ is decidable (see, for example, Corollary 1.2 and Proposition 1.3 of [12]).

We next note that the decidability of the relations $p|x$ implies the decidability of the relations $n|x$. The reason is that in a torsion-free abelian group, if $p|x$, then there is a unique element y with $x = py$. Thus, for example if $n = p_1p_2$, then $n|x$ if and only if $p_2|y$, where y satisfies $x = p_1y$. This can be ascertained by asking if $p_1|x$, and, if so, searching for the (unique) element y with $x = p_1y$. $\square$

Theorem 2.2. *The group $\mathcal{G}_S$ has a decidable copy if and only if S is Σ_2^0.*

Proof. Fix an infinite Σ_2^0 set $S \subseteq \omega$. We must show that $\mathcal{G}_S$ has a decidable copy. By Lemma 2.1, it is enough to construct copy that is computable with the added predicates $p|x$. As preparation, fix an infinite Δ_2^0 set $S_1 \subseteq S$ and let $S_2 = S - S_1$, noting that S_2 is Σ_2^0. We assume the 0^{th} existential witness for membership of any number n in S_2 fails to witness $n \in S_2$. This assures we process all of S_1.

We use a standard computable approximation for S_2 such that if $n \in S_2$, then for all sufficiently large s, the number n appears to be in S_2 at stage s; and if

$n \notin S_2$, then there are infinitely many s such that n appears not to be in S_2 at stage s. When we believe $n \in S_2$, we work towards building a copy of $\mathbb{Q}_{p_n}$, using an element to which we give the label r_n. If later we believe that $n \notin S_2$, we *trash* this work by incorporating it into the integer part of $\mathbb{Q}_{p_a} \oplus \mathbb{Q}_{p_b}$ for some $a, b \in S_1$, using elements to which we have assigned the labels r_a and r_b. Since S_1 need not be computable, the integers a and b also need to be approximated, so we may *trash* this work as well. We then need a further pair of elements, carrying labels $r_{a'}$ and $r_{b'}$, representing integers a' and b' thought to be in S_1. This in turn may *injure* lower priority work, but as the set S_1 is Δ^0_2, this injury will be finitary. The pairs do not proliferate. That is, r_n and the first pair r_a, r_b are all generated by the second pair $r_{a'}, r_{b'}$. If later n reappears in S_2, we repeat this process afresh, working with new elements throughout, but re-using the labels as appropriate.

The *priority* of an element labeled r_n is the stage at which it was created; the *priority* of an element labeled r_a or r_b is the priority of the element labeled r_n with which the pair is associated.

Construction: At stage 0, we start with the trivial group.

At stage $s+1$, we introduce a new nonzero element carrying the label r_s. For each $n \leq s$, we act on behalf of the element carrying the label r_n as follows:

(1A) If n appears to be in S_2 and has at the previous k many stages, we introduce a solution z to the equation $r_n = p_n^k z$.

(2A) If n appears not to be in S_2 but appeared to be in S_2 at the previous k many stages, we *trash* the element r_n. This is done by guessing the lexicographically least (distinct) pair $(a, b) \in S_1$ for which neither a nor b is currently assigned to a higher priority element, introducing a pair of new elements carrying the labels r_a and r_b, and declaring $r_n = p_n^k r_a + p_n^k r_b$.

If either r_a or r_b is assigned to a lower priority element, both are *trashed* as described in (2B). We then introduce a new element carrying the label r_n (this label is removed from the old r_n) to approximate whether n is in S_2 via the next existential witness.

We also act on behalf of all pairs of elements carrying the labels r_a and r_b (associated with each other) in existence.

(1B) If a and b appear to be in S_1 and have at the previous k many stages, we introduce a solution z_a to $r_a = p_a^k z_a$ and a solution z_b to $r_b = p_b^k z_b$.

(2B) If either a or b (or both) appears not to be in S_1 but appeared to be in S_1 at the previous k many stages, the elements with the labels r_a and r_b are *trashed*. This is done as follows. We guess the lexicographically least (distinct) pair $(a', b') \in S_1$ such that neither a' nor b' is currently associated with a higher priority element (as compared to the elements with the labels r_a and r_b). Let z_a and z_b be such that $r_a = p_a^{k-1} z_a$ and $r_b = p_b^{k-1} z_b$.

At present, we have not said that z_a and z_b are divisible by any prime. We have said that r_a and r_b are *not* divisible by certain primes, so z_a and z_b must not be divisible by these primes. We will get rid of the labels r_a and r_b. We introduce a pair of new elements carrying the labels $r_{a'}$ and $r_{b'}$, with the intention of making these elements infinitely divisible by $p_{a'}$, $p_{b'}$, respectively, and not divisible by any other prime. We let $z_a = r_{a'} + q r_{b'}$ and $z_b = q r_{a'} + r_{b'}$, choosing q so that for $\alpha, \beta \in \mathbb{Z}$, $\alpha z_a + \beta z_b$ will be divisible by an arbitrary prime p only if p divides both α and β.

If either $r_{a'}$ or $r_{b'}$ is associated with a lower priority element, both are *trashed* as just described. This may, of course, recurse.

We also declare all *small* finite sums of elements with a label r_n, r_a, or r_b, not divisible by any prime p_i with $i \leq s$ if it is not already divisible by p_i. Here, a coefficient $r \in \mathbb{Q}$ is *small* if the Gödel code $||r||$ for r satisfies $||r|| \leq s$.

Finally, we introduce the sum of every two elements already in the group (if the sum does not already exist) and the inverse of every element already in the group (if the inverse does not already exist).

This completes the action at stage $s+1$.

Verification: It is clear that the group $\mathcal{G}$ constructed is computable.

It therefore suffices to demonstrate the relations $p|x$ are uniformly computable and $\mathcal{G} \cong \mathcal{G}_S$. The relation $p|x$ is clearly Σ^0_1, so it suffices to show it is Π^0_1. However, this is a consequence of the action at the end of every stage s. Of course, we never violate these declarations as divisors are only introduced in Step 1A, Step 2A, and Step 2B.

The group we are building, $\mathcal{G}$, is isomorphic to $\mathcal{G}_S$. We establish this via a sequence of claims. Before doing so, we make the (trivial) observation that every labeled element either carries its label for cofinitely many stages or is trashed.

Claim 2.2.1. For every $n \in S_2$, there is a unique element carrying the label r_n for cofinitely many stages. Moreover, this element is infinitely divisible by p_n, and it is not divisible by any other prime.

Proof. If $n \in S_2$, an existential witness will demonstrate this in a Π^0_1 fashion. The element created on behalf of the first such witness will carry the label r_n for cofinitely many stages. Moreover, this element is infinitely divisible by p_n, and it is not divisible by any other prime by the action at Step 1A. The uniqueness of this element is assured by the removal of the label r_n in Step 1B when the label is assigned to another element. □

Claim 2.2.2. For every $a \in S_1$, there is a unique element carrying the label r_a for cofinitely many stages. Moreover, this element is infinitely divisible by p_a, and it is not divisible by any other prime.

Proof. We show that there is a (unique) element carrying the label r_a for cofinitely many stages by induction. We consider a stage s_0 such that:

- for each $a' < a$ with $a' \in S_1$, an element carrying the label a' cofinitely has already been created,
- for some $b > a$ with $b \in S_1$, the approximation of all $b' \leq b$ in S_1 has converged.

At this stage, if an element already carrying the label r_a never gets trashed, then this element suffices. Otherwise, consider the currently existing highest priority element carrying a label that will eventually be trashed (the element carrying the label r_s ensures such an element exists, by our assumption on the zeroth existential witness). When this element is trashed, elements carrying the labels r_a and $r_{b'}$ for some $b' \leq b$ will be created, and these elements will never be trashed. By Step 2A, this element will be infinitely divisible by p_a. As no other divisors are introduced, this element is not divisible by any other prime. □

Claim 2.2.3. Every element in $\mathcal{G}$ is a linear combination of elements carrying a label for cofinitely many stages.

Proof. As every nonzero element in the group $\mathcal{G}$ is a linear combination of elements that carry a label at some stage, it suffices to consider elements that carry a label at some stage. Of course, we may further restrict our attention to those elements x which are later trashed.

If x was trashed by Step 2B, then x is a linear combination of elements carrying the labels $r_{a'}$ and $r_{b'}$. If these labels persist cofinitely, then this is the desired linear combination. Otherwise, the elements carrying the labels $r_{a'}$ and $r_{b'}$ are themselves trashed. However, this process is iterated at most finitely many times since S_1 is Δ^0_2 and a given element is only injured by higher priority elements.

If x was trashed by Step 1B, then x is a linear combination of elements carrying the labels r_a and r_b. If these exist cofinitely, then this is the desired linear combination; otherwise, the argument above assures the existence of such a linear combination. □

Claim 2.2.4. If $n \notin S$, then no element is infinitely divisible by p_n. Also, no element of the group is infinitely divisible by two distinct primes, and no element is divisible by infinitely many distinct primes.

Proof. This is an immediate consequence of the construction and the previous claim. □

It follows from these claims that $\mathcal{G}$ is a decidable copy of $\mathcal{G}_S$. □

Remark 2.3. Let $\mathcal{G}$ be a direct sum of groups of the form $\mathbb{Q}_{p_n}$. The *character* of $\mathcal{G}$ is the set χ consisting of the pairs (n,k) such that $\mathcal{G}$ has at least k direct summands of the form $\mathbb{Q}_{p_n}$. We write $\mathcal{G}_\chi$ for the group with character χ.

It is not difficult to see that the construction can be easily modified to show that $\mathcal{G}_\chi$ has a decidable copy if and only if the character χ is Σ^0_2.

3. Characterizing the Computable $\mathcal{G}_S$

It is easy to see that if $\mathcal{G}_S$ has a computable copy $\mathcal{G}$, then S is Σ^0_3. The reason is that $n \in S$ if and only if

$$\mathcal{G} \models (\exists x) \bigwedge_k (\exists y) \left[x = p_n^k y \right].$$

We show that if S is Σ^0_3, then $\mathcal{G}_S$ has a computable copy. This strengthens the result of Melnikov (Theorem 3 from [18]) showing that the group $\mathcal{G}_S \oplus \left(\bigoplus_{i\in\omega} \mathbb{Z} \right)$ has a computable copy if and only if S is Σ^0_3.

We shall use the following lemma on Π^0_2 approximations by pairs of Π^0_2 sets.

Definition 3.1. Let $[\omega]^2$ be the set of two-element subsets of ω, viewed as a set of pairs (a,b) with $a<b$. If $X \subseteq [\omega]^2$, let $\min X$ denote the reverse lexicographically least pair (a,b) in X.

Lemma 3.2. *For every infinite Π^0_2 set $T \subseteq \omega$, there is a uniformly computable sequence of Π^0_2 sets $(X_i)_{i\in\omega}$ such that the sets $\min X_i$, for $i \in \omega$, form a partition of T.*

Proof. Enumerate the elements of T in increasing order as $a_0 < b_0 < a_1 < b_1 < \dots$. Let $X_i = \{a_j : j \geq i\} \cup \{b_j : j \geq i\}$. It is not difficult to see that the sets X_i are Π^0_2 uniformly in i. Moreover, $\min X_i = (a_i, b_i)$. □

Theorem 3.3. *The group $\mathcal{G}_S$ has a computable copy if and only if S is Σ^0_3.*

Proof. Fix an infinite Σ^0_3 set $S \subseteq \omega$. We construct a computable copy $\mathcal{G}$ of $\mathcal{G}_S$. As preparation, we fix a Π^0_2 set $T \subset \omega$ such that $s \in S$ if and only if $\langle t, s\rangle \in T$ for some t. Further, if $s \in S$, we assume the witnessing t is unique. Let $(X_i)_{i\in\omega}$ be as in Lemma 3.2.

The idea for the construction is to add an element x to $\mathcal{G}$ and express x as a linear combination of elements u_0 and v_0 such that u_0 and v_0 are infinitely divisible by primes p_{a_0} and p_{b_0}, respectively, where $a_0, b_0 \in S$. Of course, we will make mistakes in approximating a_0 and b_0.

It may therefore become necessary to *recycle* the elements u_0 and v_0 when it appears $a_0 \notin S$ or $b_0 \notin S$. This will involve writing u_0 and v_0 as an internally consistent linear combination of x and another element w. We then continue to work for x using new (lower priority) elements u_1 and v_1 and primes p_{a_1} and p_{b_1} for which a_1 and b_1 appear in S. Similarly, we work for w using new elements u'_0 and v'_0. This process will, of course, possibly repeat itself in a recursive fashion.

As S is Σ^0_3, it will become necessary to return to a pair of elements u_i and v_i working on behalf of some element z with numbers a_i and b_i. When this happens, all work on behalf of z with elements u_j and v_j for $j > i$ is *trashed.* This includes not only the elements u_j and v_j, but also any elements created to recycle it (and so on). In addition, elements created to recycle u_i and v_i are also trashed.

Throughout the construction, certain elements will be termed *T-elements.* These will be the elements x and w discussed above. Finite sums of these elements are not so distinguished. At every stage, every T-element z will be associated with one of the sets X_i. The set X_i will control the primes p_a and p_b such that we are attempting to make the element z a sum of elements of $\mathbb{Q}_{p_a}$ and $\mathbb{Q}_{p_b}$. Though the index i may change finitely often for a T-element z, it will always reach a limit (provided z remains a T-element).

The *priority* of a T-element is the point in the construction at which it was introduced, with higher priority elements created earlier in the construction. In order of priority, T-elements will constantly seek to swap their X_i for an X_j with $j < i$.

Construction: At stage 0, we start with the zero group.

At stage $s + 1$, we introduce a new T-element. We also act on behalf of all existing T-elements.

We act on behalf of a T-element x by searching for the reverse lexicographically least pair $(\langle t_a, a\rangle, \langle t_b, b\rangle)$ that appears in the set X_i associated to x. If no u and v associated with this pair and x exists, we introduce new elements u and v to $\mathcal{G}$ with $u + v = x$, and associate them with this pair $(\langle t_a, a\rangle, \langle t_b, b\rangle)$ and x. Otherwise, we add to $\mathcal{G}$ a solution z_u to the equation $u = p_a^r z_u$ and a solution z_v to the equation $v = p_b^r z_v$, where u and v are the elements associated with the pair $(\langle t_a, a\rangle, \langle t_b, b\rangle)$ and the element x, and where r is the number of times we have already worked on behalf of these elements. We also *recycle* (as described below) any higher priority elements u' and v' that were introduced on behalf of x which are not already being recycled, *trash* (as described below) any lower priority elements u' and v' introduced on behalf of x, and, if u and v were being recycled at the previous stage, *trash* (as described below) the element w introduced on behalf of u and v.

For a pair (u', v') of elements associated with the pair $(\langle t_{a'}, a'\rangle, \langle t_{b'}, b'\rangle)$ and the element x, we *recycle* the work for (u', v') by:

(1A) finding integers α and β satisfying
$$\alpha p_{a'}^r + \beta p_{b'}^r = 1$$
where r is the number of times we have already worked on behalf of (u', v'),

(1B) introducing a new T-element w' satisfying
$$u' = p_{a'}^r(\alpha x + p_{b'}^r w') \qquad \text{and} \qquad v' = p_{b'}^r(\beta x - p_{a'}^r w')$$
into the group, and

(1C) associating the set $X_{i'}$ to w', where i' is minimal so that $X_{i'}$ is not associated to any other element.

Any T-element w previously introduced on behalf of u and v is *trashed* as follows. For each pair (u'', v'') created on behalf of w, we work by:

(2A) trashing any T-element w'' introduced on behalf of u'' and v'' (this may further recurse),

(2B) finding integers α and β satisfying
$$\alpha + \beta = 1 \qquad \text{and} \qquad p_{a''}^r \mid \alpha \qquad \text{and} \qquad p_{b''}^r \mid \beta$$
where r is the number of times we have already worked on behalf of (u'', v''),

(2C) declaring
$$u'' = \alpha w \qquad \text{and} \qquad v'' = \beta w.$$

We then remove the association of X_j with w, and no longer consider w to be a T-element.

For each pair (u', v') of T-elements associated the pair $(\langle t_{a'}, a'\rangle, \langle t_{b'}, b'\rangle)$ and the element x, we *trash* the work for (u', v') by:

(3A) trashing any T-element w' introduced on behalf of u' and v' (this may further recurse),

(3B) finding integers α and β satisfying
$$\alpha + \beta = 1 \qquad \text{and} \qquad p_{a'}^r \mid \alpha \qquad \text{and} \qquad p_{b'}^r \mid \beta$$
where r is the number of times we have already worked on behalf of (u', v'), and

(3C) declaring
$$u' = \alpha x \qquad \text{and} \qquad v' = \beta x.$$

If there is ever a T-element z associated with a set X_i and a set X_j for $j < i$ is unassociated (such a situation is possible whenever a T-element is trashed), the highest priority such T-element removes its association with X_i and associates itself with X_j.

Finally, we introduce the sum of every two elements already in the group (if the sum does not already exist) and the inverse of every element already in the group (if the inverse does not already exist).

This completes the action at stage $s + 1$.

Verification: It is clear the group $\mathcal{G}$ is computable, provided that integers α and β can always be found. We demonstrate this and that $\mathcal{G} \cong \mathcal{G}_S$ via a sequence of claims. We let U be the set of elements u and v that are never trashed and are not recycled for cofinitely many stages.

Claim 3.3.1. Integers α and β always exist (and thus are found) satisfying the desired constraints.

Proof. Elementary number theory assures the existence of integers α and β since powers of distinct primes are relatively prime. □

Claim 3.3.2. For each integer i, the set X_i will be associated with a fixed T-element for cofinitely many stages.

Proof. Fix the i^{th} highest priority T-element that is never trashed, the existence of which is ensured by the new T-element introduced at every stage. Once all higher priority T-elements that will ever be trashed are, the set X_i will be associated with this element from now on. □

Claim 3.3.3. The integers in S are in $1-1$ correspondence with the elements of U.

Proof. Fixing an integer $n \in S$, let i be such that $\langle t_n, n\rangle \in \min X_i$ for some t_n. By Claim 3.3.2, the set X_i will be associated with a fixed T-element x for cofinitely many stages. Consider a stage after which elements less than $\min X_i$ never appear in X_i. Then the elements u and v created on behalf of $\min X_i$ and x will never be trashed, and they will not be recycled for cofinitely many stages. One of these elements will be working on behalf of n.

This correspondence is $1-1$ because there exist a unique t_n such that $\langle t_n, n\rangle \in T$, a unique X_i such that $\langle t_n, n\rangle \in X_i$, and a unique T-element x cofinitely associated with X_i.

If $n \notin S$, then $\langle t_n, n\rangle \notin \min X_i$ for any i and t_n. Thus, any u or v associated with n will be trashed when its associated T-element is trashed, trashed when a smaller pair appears in X_i, or recycled for cofinitely many stages when its pair never again appears in X_i. Therefore, this correspondence is surjective. □

Claim 3.3.4. If an element $u \in U$ is in correspondence with n, then u is infinitely divisible by p_n, and it is not divisible by any other primes.

Proof. Solutions to $u = p_n^{r+1} z_u$ (or $v = p_n^{r+1} z_v$ as the case may be) will be introduced for arbitrarily large r. No other prime will divide u, by the choice of α and β. □

Claim 3.3.5. If $n \notin S$, then no nonzero element is infinitely divisible by p_n. Also, no element of the group is either infinitely divisible by distinct primes or divisible by infinitely many primes.

Proof. As a consequence of the construction, no element is infinitely divisible by p_n unless that element is a rational multiple of the element of U that corresponds to p_n. By construction, no nonzero element is a rational multiple of two distinct elements of U. □

Claim 3.3.6. Every element of $\mathcal{G}$ is a linear combination of elements in U.

Proof. We argue by induction, treating several cases separately. It suffices to treat those elements z that are explicitly added to the group (i.e., not implicitly added to the group as a sum of existing elements).

If z is a T-element that is never trashed, fix the set X_i cofinitely associated with it. Let u and v be the elements associated with $\min X_i$ and z. Then $z = u + v$ and $u, v \in V$.

If z is an element u/p_n^r created for a T-element x, and z is trashed, then $z = \alpha x$ for some integer α.

If z is an element u/p_n^r created for a T-element x, and z is cofinitely recycled, then $z = \alpha x + \beta w'$ for appropriate integers α and β, and x and w' are T-elements which are never trashed.

If z is a T-element introduced because of the recycling of a pair (u, v) associated with a T-element x and a pair $(\langle t_a, a\rangle, \langle t_b, b\rangle)$, and z is trashed, then $z = \beta\frac{u}{p_a^r} - \alpha\frac{v}{p_b^r}$, for appropriate integers r, α, and β. □

From the claims, we conclude that $\mathcal{G} \cong \mathcal{G}_S$. □

Remark 3.4. Again, it is not difficult to see that the construction can be easily modified to show that $\mathcal{G}_\chi$ (see Remark 2.3) has a computable copy if and only if the character χ is Σ^0_3.

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