Generalized Witt Schemes

A New Perspective On Old Splittings

Justin Noel

Department of Mathematics
University of Chicago

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  - Generalized Husemoller-Witt splittings
  - Quillen's idempotent operation splitting $MU_{(p)}$.
- $\mathrm{Spf}(E^0(J)) \cong \widehat{\mathbb{W}}^{\mathbb{Z}_p^\times}$, for p odd and E Landweber exact or Morava K-theory.

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- Scheme theoretic perspective of formal groups and the Witt Vectors.
- Extending this correspondence to include $\widehat{\mathbb{W}}_{E_0} \cong \mathrm{Spf}(E^0(BU))$.
- Applications of this correspondence.

Generalized Chern Classes

- Assume E is an even-periodic cohomology theory, i.e. $\pi_* E$ is non-canonically isomorphic to $E_0[\lambda, \lambda^{-1}]$ with $|\lambda| = 2$.

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- This implies that $E^0(BU) \cong E_0[[c_1, c_2, \ldots]]$ is a coring object (without unit) in the category of E_0-algebras.
- Up to completion, $E^0(BU)$ is isomorphic to its dual $E_0(BU)$ as Hopf algebras.

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- Quillen's Theorem: The formal group associated to

$$E = MP \simeq \bigvee_{n \in \mathbb{Z}} \Sigma^{2n} MU$$

is universal:

$$\mathrm{Ring}(MP_0, R) \cong \text{Set of formal group laws defined over R.}$$

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- This example illustrates a correspondence between the algebra of formal groups and even-periodic ring spectra.
- This correspondence has been extended in many ways (the chromatic filtration, elliptic cohomology, Strickland's equivalence for Landweber exact formal groups, etc.)

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- The category of affine schemes is the subcategory of representable functors in $\mathrm{Set}^{\mathrm{Rings}}$.
- The Yoneda Lemma gives an equivalence between Affine schemes and $\mathrm{Ring}^{\mathrm{op}}$ and between formal schemes and pro–Ring$^{\mathrm{op}}$.

Some Affine Schemes

Example (Affine Line)

$$\mathbb{A}^1 = \mathrm{Spec}(\mathbb{Z}[x]) = \mathrm{Rings}(\mathbb{Z}[x], -).$$

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Example (The General Linear Group Scheme)

$$\mathbb{GL}_n = \mathrm{Spec}(\mathbb{Z}[a_{1,1}, \ldots, a_{n,n}][det(a_{i,j})^{-1}])$$

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Example (Formal Affine Line)

$$\begin{aligned} \hat{\mathbb{A}}^1 &= \operatorname{Spf}(\mathbb{Z}[[x]]) \\ &= \operatorname{colim} \operatorname{Spec}(\mathbb{Z}[x]/x^n) \\ &\cong \operatorname{colim} \operatorname{Nil}_n = \operatorname{Nil}. \end{aligned}$$

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Example (The Multiplicative Formal Group)

$$\begin{aligned}\hat{\mathbb{G}}_m &\cong \hat{\mathbb{A}}^1 \\ \Delta(x) &= x \otimes 1 + 1 \otimes x + x \otimes x \\ \epsilon(x) &= 0.\end{aligned}$$

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  - $\mathrm{Spf}(E^0(BU))$ is a formal ring scheme (without unit). If we forget the multiplication we have an infinite-dimensional formal group.
  - $\mathrm{Spec}(MP^0(S^0))$ is the functor that takes a ring to the set of formal group laws defined over that ring.
- Many natural objects in the theory of formal groups have analogues in topology.

The Witt Scheme

- The (big) Witt Scheme $\mathbb{W}$ is an affine ring scheme whose underlying scheme is $\mathbb{A}^{\infty}$.

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- The ring structure is not the componentwise structure that you have on $\mathbb{A}^\infty$.
- But the ring structure is determined by the ring structure on $\mathbb{A}^\infty$:

$$\begin{array}{rcl} \mathbb{W} & \hookrightarrow & \mathbb{A}^\infty \\ (\theta_1, \theta_2, \dots)_{\mathbb{W}} & \mapsto & (w_1, w_2, \dots)_{\mathbb{A}^\infty} \\ w_n & = & \sum_{d|n} d\theta_{n/d}^d \end{array}$$

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The Dual Witt Scheme

- The Cartier dual of the Witt Scheme $\mathbb{W}$ is an infinite-dimensional formal group $\widehat{\mathbb{W}}$.
- $\mathbb{W}$ and $\widehat{\mathbb{W}}$ are nearly the same (the representing Hopf-algebras are isomorphic after completion).
- This formal group represents the *Curves* functor:

$$\begin{aligned} \mathcal{C}(\widehat{\mathbb{G}}) &= \mathrm{FSch}_*(\hat{\mathbb{A}}^1, \widehat{\mathbb{G}}) \\ &\cong \mathrm{FGp}(\widehat{\mathbb{W}}, \widehat{\mathbb{G}}). \end{aligned}$$

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  - For all $n \in \mathbb{N}$,

$$F_n : t \mapsto \sum_{\xi^n = 1}^{\widehat{\mathbb{G}}} \xi t^{1/n}.$$

Structure of $\mathbb{W}_p$ and $\widehat{\mathbb{W}}_p$

- Over a p-local ring we can use these operations to construct the splitting:

$$\widehat{\mathbb{W}} \cong \prod_{\gcd(n,p)=1} \widehat{\mathbb{W}}_p$$

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- The splitting comes from the idempotent operation:

$$\epsilon(x) = \sum_{\gcd(n,p)=1}^{\widehat{\mathbb{G}}} \left[\frac{\mu(n)}{n}\right] V_n F_n(x)$$

$$\sum_{d|n} \mu(d) = \delta_{1,n}$$

Realizing $\widehat{\mathbb{W}}_{E_0}$ as $\mathrm{Spf}(E^0(BU))$

- $\mathbb{W}_R$ is represented by $R[\theta'_1, \theta'_2, \ldots]$ with

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- This makes its (topological) dual $R[[\theta_1, \theta_2, \ldots]]$ with analogous coproduct.

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- The isomorphism betweeen this Hopf algebra and $E^0(BU)$ is given by the following generating function:

$$\prod_{i\geq 1}(1-\theta_i t^i)^{-1} = 1 + \sum_{i\geq 1} c_i t^i$$

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- For example:

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- This is an isomorphism mod decomposables, so $\mathrm{Spec}(E_0(BU)) \cong \mathbb{W}_{E_0}$.

$Spf(E^0(BU))$ Represents Curves

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- Recall $\operatorname{colim} \mathbb{C}P^n \cong \mathbb{C}P^\infty$ and $\operatorname{colim} \Omega SU(n+1) \simeq BU$.
- Then the topological analogue is

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- Moreover, if $\widehat{\mathbb{G}} \cong \operatorname{Spf}(E^0(BU(1)))$ then we can trace through a series of adjunctions (after dualizing) to see that a curve (which induces an isomorphism $\hat{\mathbb{A}}^1 \cong \widehat{\mathbb{G}}$) defines a map of *ring spectra* $MU \to E$.

Making the Identification

Proof.

$$\mathrm{fSch}_*(\hat{\mathbb{A}}^1, \hat{\mathbb{G}}) \cong aug\text{–}Alg(E^0(\mathbb{C}P^\infty), E^0(\mathbb{C}P^\infty))$$

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&\cong E_0\text{–}HAlg(E_0(BU), E_0(\mathbb{C}P^\infty)) \\
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&\cong \mathrm{FGpSch}(\mathrm{Spf}(E^0(BU)), \mathrm{Spf}(E^0(\mathbb{C}P^\infty))).
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□

Quillen's Splitting

- This splitting of $\widehat{\mathbb{W}}_{\mathbb{Z}_{(p)}}$ is constructed in the *same* way as the splitting

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- Partial result: (Basterra-Mandell) BP is E_4.

Condition for Quillen's splitting to be H_∞.

Theorem (Johnson-Noel)

The projection $r : MU_{(p)} \to BP$ is H_∞ iff the map $r \circ P_p : MU^*_{(p)}(\mathbb{C}P^\infty) \to BP^{2p*}_{C_p}(\mathbb{C}P^\infty)$ determines a p-typical formal group law.

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The projection $r : MU_{(p)} \to BP$ is H_∞ iff the map $r \circ P_p : MU^*_{(p)}(\mathbb{C}P^\infty) \to BP^{2p*}_{C_p}(\mathbb{C}P^\infty)$ determines a p-typical formal group law.

Corollary

The projection map is H_∞ iff $F_q(r) \in \mathcal{C}(\widehat{\mathbb{G}})$ vanishes for all primes $q \neq p$. Where $\widehat{\mathbb{G}}$ is the formal group $\mathrm{Spf}(BP^{2p*}_{C_p}(\mathbb{C}P^\infty))$.

Condition for Quillen's splitting to be H_∞.

Theorem (Johnson-Noel)

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Computer calculations support the hypothesis that BP is H_∞.

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- We can see that this splitting induces the maps that give the splitting of $MU_{(p)}$.
- The first splitting is *purely algebraic* while the second is topological, but they both arise from the same construction.

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- Each formal group law F, associated to E (given by an isomorphism with the formal affine line) determines a multiplicative (without unit) structure on the $\widehat{\mathbb{W}}_{E_0}$.
- This structure is determined by the formula for the total E-theory Chern class for a tensor product of two stable bundles.

$$c((\eta_1 - [n]) \otimes (\eta_2 - [m])) = \frac{c(\eta_1 \otimes \eta_2)}{c(\eta_1)^m c(\eta_2)^n}$$

- If $\eta_1 = \sum_{j=1}^{n} \eta_{1,j}$ and $\eta_2 = \sum_{k=1}^{m} \eta_{2,k}$ then

$$c(\eta_1 \otimes \eta_2) = \prod_{i,j} (1 + (c_1(\eta_{1,i}) +_F c_1(\eta_{2,j})))$$

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- Take $k \in \mathbb{Z}_p^* \cong \mathbb{Z}/(p-1) \times \mathbb{Z}_p$ such that $\langle \bar{k} \rangle = \mathbb{Z}_p^*$.
- Then the space $J^{\oplus}$ is defined by the fiber sequence:

$$J^{\oplus} \to BU \xrightarrow{\psi^k - 1} BU$$

- The Adams operation ψ^k acts on $E^0(BU)$ by

$$\prod_{i\geq 1}(1+c_1(\xi_i)t) = 1+\sum_{i\geq 1} c_i t^i \mapsto \prod_{i\geq 1}(1+[k]c_1(\xi_i)t) = 1+\sum_{i\geq 1} P_k t^i$$

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- The fixed points of this group action on $\widehat{\mathbb{W}}_{E_0}$ give the formal group scheme $\operatorname{Spf} E^0(J)$.
- This follows from the work of RWY that give us the short exact sequence of (Hopf) algebras:

$$E^0(J) \leftarrow E^0(BU) \xleftarrow{\psi^k - 1} E^0(BU)$$

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- We've used the algebraic geometry to *simultaneously* construct the algebraic splitting of $E^0(BU)$ and the topological splitting of $MU_{(p)}$.
- The topology constructs a product structure on $\widehat{\mathbb{W}}_{E_0}$ that is not part of the theory of Witt schemes.