LOCAL DEFORMATION RINGS FOR 2-ADIC REPRESENTATIONS OF $G_{Q_l}$, $l \neq 2$.

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Let $l$ and $p$ be distinct primes. Let $L/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$, uniformiser $\varpi$ and residue field $k$. Let $F/\mathbb{Q}_l$ be a finite extension with absolute Galois group $G_F$, inertia group $I_F$, and wild inertia group $P_F$. Let $\tilde{P}_F$ be the kernel of the maximal pro-$l$ quotient of $I_F$. Let $q$ be the order of the residue field of $F$. We assume that $L$ contains all $(q^2 - 1)$th roots of unity. Choose a pro-generator $\sigma$ of $I_F/\tilde{P}_F$ and $\phi \in G_F/\tilde{P}_F$ lifting the arithmetic Frobenius element of $G_F/I_F$. Then we have the relation

$$\phi \sigma \phi^{-1} = \sigma^q.$$  

If $\rho : G_F \to GL_2(k)$ is a continuous homomorphism, let $R^\square_\varpi$ be the universal framed deformation ring for $\rho$ parametrising lifts with coefficients in $O$-algebras. By [Sho16a] Theorem 2.5, $R^\square_\varpi$ is a reduced, $O$-flat complete intersection ring of relative dimension 4 over $O$.

If $\tau : I_F \to GL_2(L)$ is a continuous semisimple representation that extends to $G_F$, let $R^\square_\varpi(\tau)$ be the maximal reduced, $p$-torsion free quotient of $R^\square_\varpi$ such that, for every $O$-algebra homomorphism $x : R^\square_\varpi \to \mathcal{L}$, the corresponding representation $\rho_x : G_F \to GL_2(L)$ satisfies $(\rho_x|_{I_F})^{ss} \cong \tau$.

The goal of this appendix is to prove:

**Theorem 0.1.** For any $\rho$ and $\tau$ as above, the ring $R^\square_\varpi(\tau)$ is either Cohen–Macaulay or zero.

If $p > 2$, then this is the content of section 5.5 of [Sho16b]. If $p = 2$ and $\rho|_{\tilde{P}_F}$ is non-scalar, then the proof of proposition 5.1 of [Sho16b] shows that $R^\square_\varpi$ is a completed tensor product of deformation rings of characters, all of whose irreducible components are formally smooth, and that $R^\square_\varpi(\tau)$ is an irreducible component of $R^\square_\varpi$; thus $R^\square_\varpi(\tau)$ is formally smooth in this case. From now on, then, we assume that $p = 2$ and that $\rho|_{\tilde{P}_F}$ is scalar; by twisting, we may and do assume that $\rho|_{\tilde{P}_F}$ is trivial. In this case, we may list the semisimple inertial types $\tau$ for which $R^\square_\varpi(\tau)$ may be non-zero. They are determined by the eigenvalues of $\tau(\sigma)$, which must be of 2-power order and either fixed or interchanged by raising to the power $q$. Writing $a = v_2(q - 1)$ and $b = v_2(q^2 - 1)$, if $R^\square_\varpi(\tau)$ is non-zero then either

- $\tau = \tau_\xi$ is the inertial type in which the eigenvalues of $\tau(\sigma)$ are both equal to an $2^a$th root of unity, $\xi$;
- $\tau = \tau_{\xi_1, \xi_2}$ is the inertial type in which the eigenvalues of $\tau(\sigma)$ are equal to distinct $2^a$th roots of unity $\xi_1$ and $\xi_2$;
- $\tau = \tau_\xi$ is the inertial type in which the eigenvalues of $\tau(\sigma)$ are equal to $\xi$ and $\xi^q$ for $\xi$ an $2^b$th root of unity with $\xi \neq \xi^q$ (equivalently, with $\xi$ not an $2^a$th root of unity).
We also give a version with fixed determinant:

**Corollary 0.2.** If \( \psi \) is any lift of \( \det \) to \( O^X \) such that \( \psi|_{I_F} = \det \tau \), let \( R^\Box_{\tau}(\tau) \) be the universal framed deformation ring with determinant \( \psi \) and type \( \tau \). Then \( R^\Box_{\tau}(\tau) \) is Cohen–Macaulay.

**Proof.** By Theorem 0.1, \( R^\Box_{\tau}(\tau) \) is Cohen–Macaulay. If we impose a single equation additional \( \det \rho(\phi) = \psi(\phi) \), then the ring will still be Cohen–Macaulay provided that \( \det \rho(\phi) - \psi(\phi) \) is a non-zero divisor — in other words, that it doesn’t vanish on any irreducible components of \( \text{Spec} R^\Box_{\tau}(\tau) \). This is the case, since the action of \( G^\wedge_m \) on \( \text{Spec} R^\Box_{\tau}(\tau) \) given by making unramified twists preserves irreducible components but varies the determinant. \( \square \)

Let \( \mathcal{X} \) be the affine \( O \)-scheme whose \( R \) points, for an \( O \)-algebra \( R \), are pairs

\[
\{(\Sigma, \Phi) \in GL_2(R) \times GL_2(R) : \Phi \Sigma = \Sigma^0 \Phi\}.
\]

Then \( \mathcal{X} \) is a reduced, \( O \)-flat complete intersection of relative dimension 4 over \( \text{Spec} O \) by the proof of Theorem 2.5 of [Sho16a]. Let \( A \) be the coordinate ring of \( \mathcal{X} \). We write \( \Sigma = \left( \begin{array}{cc} 1 + A & B \\ C & 1 + D \end{array} \right) \) and \( \Phi = \left( \begin{array}{cc} P & Q \\ R & T - P \end{array} \right) \), so that \( A \) is a quotient of

\[
S = O[\{A, B, C, D, P, Q, R, T\}] / ((\det \Sigma)^{-1}, (\det \Phi)^{-1}).
\]

For any continuous \( \overline{\rho} : G_F \rightarrow GL_2(k) \), the pair of matrices \( \overline{\rho}(\sigma) \) and \( \overline{\rho}(\phi) \) give rise to a closed point of \( \mathcal{X} \), and so a maximal ideal \( \mathfrak{m} \) of \( A \). Then \( R^\Box_{\tau}(\tau) = A^\wedge_m \). If \( C \) is a conjugacy class in \( GL_2(\overline{k}) \), then there is a unique irreducible component of \( \mathcal{X} \) such that, for a dense set of geometric points of that component, the corresponding matrix \( \Sigma \) has conjugacy class \( C \). This provides a bijection between the irreducible components of \( \mathcal{X} \) and the conjugacy classes of \( GL_2(\overline{k}) \) that are preserved under the \( q \)-power map (by [Sho16a] Proposition 2.6). If \( \tau \) is one of the above inertial types then we write \( \mathcal{X}(\tau) \) for the union of those irreducible components corresponding to conjugacy classes with the same characteristic polynomial as \( \tau(\sigma) \), with the reduced subscheme structure, and \( A(\tau) \) for its coordinate ring. Note that, since \( \mathcal{X} \) is \( O \)-flat and \( \mathcal{X}(\tau) \) is an irreducible component of \( \mathcal{X} \), \( \mathcal{X}(\tau) \) is also \( O \)-flat, so that \( A(\tau) \) is \( \tau \)-torsion free.

**Lemma 0.3.** If \( \tau = \tau_1, \tau_2, \tau_3 \) or \( \tau_4 \), then \( A(\tau)^\wedge_m = R^\Box_{\tau}(\tau) \).

**Proof.** Since \( A \) is \( O \)-flat and \( A(\tau) \) is the quotient of \( A \) by an intersection of minimal prime ideals, it is also \( O \)-flat. Thus \( A(\tau)^\wedge_m \) is also \( O \)-flat, by flatness of localisation and completion. Since \( A(\tau) \) is of finite type over a DVR it is Nagata by [Sta17, Tag 0335]. Since \( A(\tau) \) is reduced, the completion \( A(\tau)^\wedge_m \) is also reduced by [Sta17, Tag 07NZ]. The composite map \( A \rightarrow R^\Box_{\tau} \rightarrow R^\Box_{\tau}(\tau) \) factors through a map \( A(\tau) \rightarrow R^\Box_{\tau}(\tau) \), since any function in \( A \) that vanishes on all \( \overline{L} \)-points of type \( \tau \) must vanish in \( R^\Box_{\tau}(\tau) \) by definition. Thus we get a surjection \( A(\tau)^\wedge_m = A(\tau) \otimes_A R^\Box_{\tau} \rightarrow R^\Box_{\tau}(\tau) \). However, since \( A(\tau)^\wedge_m \) is reduced and \( \tau \)-torsion free, and has the property that every \( \overline{L} \)-point gives a Galois representation of type \( \tau \), this map is an isomorphism by the definition of \( R^\Box_{\tau}(\tau) \). \( \square \)

Let \( \mathcal{F} = S \otimes_O k, \overline{\mathcal{A}} = A \otimes_O k, \) and \( \overline{\mathcal{X}} = \text{Spec} \overline{\mathcal{A}} \). Then the irreducible components of \( \overline{\mathcal{X}} \) are in bijection with the conjugacy classes of \( GL_2(\overline{k}) \) that are stable under the
Let $\Sigma$ be the irreducible component corresponding to the trivial conjugacy class — this is just the locus where $\Sigma = 1$ — and let $\mathcal{X}_N$ be that corresponding to the non-trivial unipotent conjugacy class (we give the irreducible components the reduced subscheme structure). Let $I_1$ and $I_N$ be the prime ideals of $S$ cutting out $\mathcal{X}_1$ and $\mathcal{X}_N$; these correspond to minimal primes of $\mathcal{A}$. If $\tau$ is one of the above inertial types, then we write $I(\tau)$ for the ideal of $S$ cutting out $\mathcal{A}(\tau) \otimes_{\mathcal{O}} k$.

**Lemma 0.4.** The ideals $I_1$ and $I_N$ have generators

$$I_1 = (A, B, C, D)$$

$$I_N = (A^2 + BC, CQ + BR, T, A + D).$$

**Proof.** The presentation for $I_1$ is obvious. For $I_N$, the condition that $\Sigma$ is unipotent gives $A + D \in I_N$ and $A^2 + BC \in I_N$. If $N = \Sigma - 1$, then the relation $\Phi \Sigma = \Sigma^q \Phi$ becomes $\Phi N = qN\Phi = N\Phi$ (since we are working mod 2), which implies that $CQ + BR = 0$. At any closed point of $\mathcal{X}_N$ where $N \neq 0$, the eigenvalues of $\Phi$ must be in the ratio $1 : q = 1 : 1$, and so $T = 0$. As such closed points are dense on $\mathcal{X}_N$, we see that $T \in I_N$. Therefore

$$(A^2 + BC, CQ + BR, T, A + D) \subset I_N.$$ 

The ideal $I = (A^2 + BC, CQ + BR, T, A + D)$ is prime of dimension 4; indeed, $S/I$ is isomorphic to a localisation of

$$k[A, B, C]/(A^2 + BC) [P, Q, R]/(CQ + BR)$$

which is easily seen to be a 4-dimensional domain. Thus $I \subset I_N$ are prime ideals of $S$ of the same dimension, and so must be equal. □

**Proposition 0.5.** Let $\tau = \tau_\xi$. Then $I(\tau) = I_N$.

**Proof.** Write $\eta = \xi + \xi^q - 2$. The condition that $\Sigma$ has characteristic polynomial $(X - \xi)(X - \xi^q)$ shows that, on $X(\tau)$, we have the equations

$$A + D = \eta$$

$$A(A - \eta) + BC = \eta.$$ 

Using the first of these, we replace $D$ by $\eta - A$ everywhere. Now, if $x$ is an $\mathcal{T}$-point of $X(\tau)$ corresponding to a pair of matrices $(\Sigma_x, \Phi_x)$, then $\Phi_x$ exchanges the $\xi$ and $\xi^q$ eigenspaces of $\Sigma_x$ and so must have trace zero. Therefore on $X(\tau)$ we have the equation

$$T = 0.$$ 

Lastly, by the Cayley–Hamilton theorem, and the fact that

$$X^q \equiv \xi + \xi^q - X \mod (X - \xi)(X - \xi^q),$$

we see that $\Sigma^q = \left(\begin{array}{cc} 1 + \eta - A & -B \\ -C & 1 + A \end{array}\right)$ on $X(\tau)$. Equating matrix entries in the relation $\Phi \Sigma = \Sigma^q \Phi$, and noting that $T = 0$, we obtain one new equation

$$(2A - \eta)P + BR + CQ = 0.$$ 

Thus, letting

$$J = (A + D, T, A(A - \eta) + BC - \eta, (2A - \eta)P + BR + CQ)$$

$q$-power map (again by [Sho16a] Proposition 2.6). Let $\mathcal{X}_1$ be the irreducible component corresponding to the trivial conjugacy class — this is just the locus where $\Sigma = 1$ — and let $\mathcal{X}_N$ be that corresponding to the non-trivial unipotent conjugacy class (we give the irreducible components the reduced subscheme structure). Let $I_1$ and $I_N$ be the prime ideals of $S$ cutting out $\mathcal{X}_1$ and $\mathcal{X}_N$; these correspond to minimal primes of $\mathcal{A}$. If $\tau$ is one of the above inertial types, then we write $I(\tau)$ for the ideal of $S$ cutting out $\mathcal{A}(\tau) \otimes_{\mathcal{O}} k$.

**Lemma 0.4.** The ideals $I_1$ and $I_N$ have generators

$$I_1 = (A, B, C, D)$$

$$I_N = (A^2 + BC, CQ + BR, T, A + D).$$

**Proof.** The presentation for $I_1$ is obvious. For $I_N$, the condition that $\Sigma$ is unipotent gives $A + D \in I_N$ and $A^2 + BC \in I_N$. If $N = \Sigma - 1$, then the relation $\Phi \Sigma = \Sigma^q \Phi$ becomes $\Phi N = qN\Phi = N\Phi$ (since we are working mod 2), which implies that $CQ + BR = 0$. At any closed point of $\mathcal{X}_N$ where $N \neq 0$, the eigenvalues of $\Phi$ must be in the ratio $1 : q = 1 : 1$, and so $T = 0$. As such closed points are dense on $\mathcal{X}_N$, we see that $T \in I_N$. Therefore

$$(A^2 + BC, CQ + BR, T, A + D) \subset I_N.$$ 

The ideal $I = (A^2 + BC, CQ + BR, T, A + D)$ is prime of dimension 4; indeed, $S/I$ is isomorphic to a localisation of

$$k[A, B, C]/(A^2 + BC) [P, Q, R]/(CQ + BR)$$

which is easily seen to be a 4-dimensional domain. Thus $I \subset I_N$ are prime ideals of $S$ of the same dimension, and so must be equal. □

**Proposition 0.5.** Let $\tau = \tau_\xi$. Then $I(\tau) = I_N$.

**Proof.** Write $\eta = \xi + \xi^q - 2$. The condition that $\Sigma$ has characteristic polynomial $(X - \xi)(X - \xi^q)$ shows that, on $X(\tau)$, we have the equations

$$A + D = \eta$$

$$A(A - \eta) + BC = \eta.$$ 

Using the first of these, we replace $D$ by $\eta - A$ everywhere. Now, if $x$ is an $\mathcal{T}$-point of $X(\tau)$ corresponding to a pair of matrices $(\Sigma_x, \Phi_x)$, then $\Phi_x$ exchanges the $\xi$ and $\xi^q$ eigenspaces of $\Sigma_x$ and so must have trace zero. Therefore on $X(\tau)$ we have the equation

$$T = 0.$$ 

Lastly, by the Cayley–Hamilton theorem, and the fact that

$$X^q \equiv \xi + \xi^q - X \mod (X - \xi)(X - \xi^q),$$

we see that $\Sigma^q = \left(\begin{array}{cc} 1 + \eta - A & -B \\ -C & 1 + A \end{array}\right)$ on $X(\tau)$. Equating matrix entries in the relation $\Phi \Sigma = \Sigma^q \Phi$, and noting that $T = 0$, we obtain one new equation

$$(2A - \eta)P + BR + CQ = 0.$$ 

Thus, letting

$$J = (A + D, T, A(A - \eta) + BC - \eta, (2A - \eta)P + BR + CQ)$$
we obtain a surjection $S/J \to \mathcal{A}(\tau)$, and therefore a surjection

$$\mathcal{S}/J \to \mathcal{A}(\tau).$$

As $\eta$ is divisible by $\varpi$, we see that $J + (\varpi) = I_N$, and so we have a surjection $\mathcal{S}/I_N \to \mathcal{A}(\tau)$. This must be an isomorphism since $\mathcal{S}/I_N$ is a 4-dimensional domain and $\mathcal{A}(\tau)$ is a non-zero 4-dimensional ring. Therefore $I_N = I(\tau)$ as required. \qed

For the remaining types the following lemmas will be useful. If $R$ is a noetherian ring, $p$ is a minimal prime of $R$, and $M$ is a finitely-generated $R$-module, let $e_R(M, p) = i_{R_p}(M_p)$ (this is a special case of the Hilbert–Samuel multiplicity).

**Lemma 0.6.** Let $f : R \to S$ be a surjection of equidimensional rings of the same dimension, and suppose that $R$ is $S1$ and Nagata. Let $p_1, \ldots, p_n$ be the minimal primes of $R$. Suppose that, for $i = 1, \ldots, n$, there is a maximal ideal $m_i$ of $S$ such that $p_i \subset m_i$ but $p_j \not\subset m_i$ for $i \neq j$. If, for each $i$, we have

$$e_R(R, p_i) \leq e_{S_{m_i}}^\wedge(S_{m_i}^\wedge, q_i)$$

for some minimal prime $q_i$ of $S_{m_i}^\wedge$, then $f$ is an isomorphism.

**Remark 0.7.** For those primes $p_i$ such that $e_R(R, p_i) = 1$ — which is all of them if $R$ is reduced — the required inequality is implied simply by the existence of the $m_i$.

**Proof.** Since $R$ is $S1$, every associated prime of $S$ is minimal and so, by [Sta17, Tag 0311], it is enough to show that $f$ induces an isomorphism $f_{m_i} : R_{m_i} \to S_{p_i}$, for each $i$. Since $f$ is surjective and $R_{m_i}$ is artinian, it is enough to show that $e_R(R, m_i) \leq e_R(S, p_i)$. Let $i \in \{1, \ldots, n\}$. Choose $m_i$ and $q_i$ as in the hypotheses of the lemma. It is enough to show that for each $i$,

$$e_R(S, q_i) = e_{S_{m_i}}^\wedge(S_{m_i}^\wedge, q_i).$$

Since $m_i$ contains a unique minimal prime of $R$, after localising at $m_i$ we may assume that $R \to S$ is a local map of local rings, and that $p_i$ is the unique minimal prime of $R$, and drop $i$ from the notation. The hypothesis that $R$ and $S$ are equidimensional of the same dimension implies that $pS$ is the unique minimal prime of $S$, which we also denote by $p$. We have $e_R(S, p) = e_S(S, p)$ since both are just the length of $S_p$. Since $S \to S^\wedge$ is flat and $S^\wedge/p = (S/p)^\wedge$ is reduced because $R$ (and hence $S$) is Nagata, [Sta17, Tag 02M1] implies that $e_s(S, p) = e_{S^\wedge}(S^\wedge, q)$. So

$$e_R(S, p) = e_S(S, p) = e_{S^\wedge}(S^\wedge, q) \geq e_R(R, p)$$

as required. \qed

The S1 condition holds, in particular, if $R$ is reduced or Cohen–Macaulay, while the Nagata condition holds if $R$ is of finite type over a field or DVR.

**Proposition 0.8.** Let $\tau = \tau_\cdot$. Then

$$I(\tau) = I_N \cap I_1$$

$$= (A + D, AT, BT, CT, A^2 + BC, BR + CQ).$$
Proof. For simplicity, we twist so that \( \zeta = 1 \). Write \( N = \Sigma - 1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \). On \( \mathcal{A}(\tau) \), \( \Sigma \) has characteristic polynomial \((X - 1)^2\), and so the equations

\[
A + D = 0 \\
A^2 + BC = 0
\]

hold on \( \mathcal{A}(\tau) \). Moreover, since \((\Sigma - 1)^2 = 0\) on \( \mathcal{A}(\tau) \), by the Cayley–Hamilton theorem we have that \( \Sigma^q = 1 + q(\Sigma - 1) = 1 + qN \) on \( \mathcal{A}(\tau) \). The equation \( \Phi \Sigma = \Sigma^q \Phi \) becomes \( \Phi N = qN \Phi \), and comparing matrix entries we get equations

\[
qBR - CQ + (q - 1)AP = 0 \\
(q + 1)QA + B(qT - (q + 1)P) = 0 \\
(q + 1)RA + C(T - (q + 1)P)) = 0 \\
qCQ - BR + (q - 1)A(P - T) = 0.
\]

Summing the first and fourth of these gives \((q - 1)(BR + CQ + A(2P - T)) = 0\); since \( \mathcal{A}(\tau) \) is \((q - 1)\)-torsion free, we deduce that

\[
BR + CQ + A(2P - T) = 0
\]

in \( \mathcal{A}(\tau) \) and can replace the fourth of the above equations by this.

The ideal cutting out \( \mathcal{A}(\tau) \) therefore contains the ideal

\[
J = (A + D, A^2 + BC, qBR - CQ + (q - 1)AP, (q + 1)QA + B(qT - (q + 1)P), (q + 1)RA + C(T - (q + 1)P), CQ + BR + A(2P - T))
\]

which is equal to \((A + D, A^2 + BC, BR + CQ) + I_1 \cap (T) = I_N \cap I_1 \). Therefore there is a surjection

\[
f : \overline{S}/(I_N \cap I_1) \twoheadrightarrow \overline{\mathcal{A}}(\tau).
\]

Write \( \tilde{R} = \overline{S}/(I_N \cap I_1) \). Then \( \tilde{R} \) is reduced with two minimal primes, which we also call \( I_N \) and \( I_1 \). Let \( \rho_1 : G_F \to GL_2(\mathcal{O}) \) be diagonal unramified with distinct eigenvalues of Frobenius, and let \( \rho_N : G_F \to GL_2(\mathcal{O}) \) send \( \sigma \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( \phi \mapsto \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \). Let \( \mathfrak{m}_1 \) and \( \mathfrak{m}_N \) be the corresponding maximal ideals of \( \overline{\mathcal{A}}(\tau) \). Then \( I_1 \subset \mathfrak{m}_1, I_N \not\subset \mathfrak{m}_1, I_1 \not\subset \mathfrak{m}_N \) and \( I_N \subset \mathfrak{m}_N \), so \( f \) is an isomorphism by the remark following lemma 0.6. \( \square \)

Proposition 0.9. Let \( \tau = \tau_{\zeta_1, \zeta_2} \). Then

\[
I(\tau) = (A + D, BT, CT, CQ + BR, A^2 + BC).
\]

Proof. Write \( \mu = \zeta_1 + \zeta_2 - 2 \). The condition that \( \Sigma \) has characteristic polynomial \((X - \zeta_1)(X - \zeta_2)\) is equivalent to the equations

\[
A + D = \mu \\
A(A - \mu) + BC = \mu.
\]
As \( X^q \equiv X \mod (X - \zeta_1)(X - \zeta_2) \), we have by the Cayley–Hamilton theorem that \( \Sigma q = \Sigma \) on \( A(\tau) \). The equation \( \Phi \Sigma = \Sigma q \Phi \) therefore becomes \( \Phi \Sigma = \Sigma \Phi \), and comparing matrix entries we get three equations (the fourth being redundant):

\[
\begin{align*}
BR - CQ &= 0 \\
Q(2A - \mu) &= B(2P - T) \\
R(2A - \mu) &= C(2P - T).
\end{align*}
\]

Let

\[
J = (A + D - \mu, A(A - \mu) + BC - \mu, BR - CQ, Q(2A - \mu) - B(2P - T), R(2A - \mu) - C(2P - T)).
\]

Let \( I \) be the image of \( J \) in \( \mathcal{S} \), so that

\[
I = (A + D, BT, CT, CQ + BR, A^2 + BC).
\]

We have shown that there is a surjection \( \mathcal{S}/J \to A(\tau) \), and therefore there is a surjection \( f : \mathcal{S}/I \to \mathcal{A}(\tau) \). We have to show that \( f \) is an isomorphism. Write \( \bar{R} = \mathcal{S}/I \).

Then (see the proof of corollary 0.10 below) \( \mathcal{S}/I \) is Cohen–Macaulay, with minimal primes \( I_1 \) and \( I_N \), and it is easy to see that \( e_{\bar{R}}(\bar{R}, I_N) = 1 \) while \( e_{\bar{R}}(\bar{R}, I_1) = 2 \).

Let \( \rho_1 : G_F \to GL_2(O) \) be diagonal such that the eigenvalues of \( \rho_1(\sigma) \) are \( \zeta_1 \) and \( \zeta_2 \), and the eigenvalues of \( \rho_1(\phi) \) are distinct modulo \( \varpi \). Let \( \rho_N : G_F \to GL_2(O) \) send \( \sigma \mapsto \begin{pmatrix} \zeta_1 & 1 \\ 0 & \zeta_2 \end{pmatrix} \) and \( \phi \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Let \( m_1 \) and \( m_N \) be the corresponding maximal ideals of \( \mathcal{A}(\tau) \). Then \( I_1 \cap m_1, I_N \not\subset m_1, I_1 \not\subset m_N \) and \( I_N \subset m_N \).

By [Sho16b] Proposition 5.3, which remains valid when \( p = 2 \), \( R^{\mathbb{Q}_2}_{\tau_1}(\tau) \) is formally smooth over

\[
\mathcal{O}[[X - 1]]/(X - \zeta_1)(X - \zeta_2).
\]

Therefore \( R^{\mathbb{Q}_2}_{\tau_1}(\tau) \otimes k \) has a unique minimal prime \( q \) and its multiplicity is 2. By lemmas 0.3 and 0.6, \( f \) is an isomorphism. □

**Corollary 0.10.** (of propositions 0.5, 0.8 and 0.9) For \( \tau = \tau_\xi, \tau_\zeta, \) or \( \tau_{\zeta_1, \zeta_2} \), \( A(\tau) \) is Cohen–Macaulay.

**Proof.** Since \( \varpi \) is a regular element of \( A(\tau) \), it suffices to prove that \( \mathcal{A}(\tau) \) is Cohen–Macaulay. This can easily be checked in magma; we sketch an alternative proof by hand. If \( \tau = \tau_\xi \), then by proposition 0.5, \( I(\tau) = I_N \). But \( \mathcal{S}/I_N \) is a complete intersection ring of dimension 4, and therefore is Cohen–Macaulay. If \( \tau = \tau_\zeta \), then by proposition 0.8, \( I(\tau_\zeta) = I_1 \cap I_N \). Now, \( \mathcal{S}/I_1 \) and \( \mathcal{S}/I_N \) are Cohen–Macaulay of dimension 4 (the latter by the previous case), while \( \mathcal{S}/(I_1 + I_N) \) is regular, and so Cohen–Macaulay, of dimension 3. By exercise 18.13 of [Eis95], \( \mathcal{S}/(I_1 \cap I_N) \) is also Cohen–Macaulay. Finally, if \( \tau = \tau_{\zeta_1, \zeta_2} \) then by proposition 0.9, \( I(\tau) = (A + D, A^2 + BC, BR + CQ, BT, CT) \). Let \( I = I(\tau) \). Since \( I + (AT) = I_1 \cap I_N \) and \( AT \cdot I_1 = 0 \), there is an exact sequence of \( \mathcal{S}/I \)-modules

\[
\mathcal{S}/I_1 \xrightarrow{AT} \mathcal{S}/I \to \mathcal{S}/(I_1 \cap I_N) \to 0.
\]

The first map must be injective, since \( I_1 \) is prime and \( e_{\mathcal{S}/I}(\mathcal{S}/I, I_1) = 2 > 1 = e_{\mathcal{S}/I}(\mathcal{S}/(I_1 \cap I_N), I_1) \). Since we have shown that \( \mathcal{S}/I_1 \) and \( \mathcal{S}/(I_1 \cap I_N) \) are maximal Cohen–Macaulay modules over \( \mathcal{S}/I \), so is \( \mathcal{S}/I \) (by [Yos90] Proposition 1.3). □
Since $\mathcal{R}_\mathfrak{p}^\mathcal{O}(\tau)$ is a completion of $\mathcal{A}(\tau)$ by lemma 0.3, and a completion of a Cohen–Macaulay ring is Cohen–Macaulay (by [Sta17, Tag 07NX]), we obtain Theorem 0.1.

References

[Sho16b] Jack Shotton, Local deformation rings for $GL_2$ and a Breuil-Mézard conjecture when $\ell \neq p$, Algebra Number Theory 10 (2016), no. 7, 1437–1475. MR 3554238