On two-bubble solutions for energy-critical dispersive equations

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Critical wave maps
Wave maps from $\mathbb{R}^{1+2}$ to a Riemannian manifold $\mathcal{N}$:

$$\Box \psi \perp T_\psi \mathcal{N}.$$ 

Special case $\mathcal{N} = S^2 \subset \mathbb{R}^3$, $k$-equivariant solutions ($k \in \mathbb{Z}$):

$$\psi(t, r \cos \theta, r \sin \theta) = (\sin(u(t, r)) \cos k\theta, \sin(u(t, r)) \sin k\theta, \cos(u(t, r))).$$

Equation is reduced to a semi-linear one:

$$\begin{cases}
\partial_t^2 u(t, r) = \partial_r^2 u(t, r) + \frac{1}{r} \partial_r u(t, r) + \frac{k^2}{2r^2} \sin(2u(t, r)), \\
(u(t_0, r), \partial_t u(t_0, r)) = (u_0(r), \dot{u}_0(r)).
\end{cases}$$

(WMAP)

Notation: $u_0 := (u_0, \dot{u}_0)$, $\|\dot{u}\|_{L^2}^2 := \int_0^{+\infty} (\dot{u}^2) r \, dr$, $\|u\|_{\mathcal{H}}^2 := \int_0^{+\infty} \left( (\partial_r u)^2 + \frac{1}{r^2} u^2 \right) r \, dr$, $\mathcal{E} := \mathcal{H} \times L^2$.

$$E(u) := \pi \int_0^{+\infty} \left( \dot{u}^2 + (\partial_r u)^2 + \frac{k^2}{r^2} (\sin(u))^2 \right) r \, dr.$$
Related models

Energy critical focusing wave in dimension $N + 1$ ($N \geq 3$):

$$\begin{aligned}
\begin{cases}
\partial_t^2 u(t, x) &= \Delta_x u(t, x) + |u(t, x)|^{\frac{4}{N-2}} u(t, x), \\
(u(t_0, x), \partial_t u(t_0, x)) &= (u_0(x), \dot{u}_0(x)).
\end{cases}
\end{aligned}
$$

Energy-critical focusing Schrödinger:

$$\begin{aligned}
\begin{cases}
\partial_t u(t, x) &= i\Delta_x u(t, x) + i|u(t, x)|^{\frac{4}{N-2}} u(t, x), \\
u(t_0, x) &= u_0(x).
\end{cases}
\end{aligned}
$$

In this talk I always assume that solutions are spherically symmetric, meaning $u(t, x) = u(t, |x|)$.

In the sequel, I mainly consider (WMAP).
Local well-posedness in $\mathcal{E}$ (conditional)
- Ginibre, Soffer, Velo (1992)
- Shatah, Struwe (1994)

$$\forall u_0 \in \mathcal{E}, \exists! u \in C((T_-, T_+); \mathcal{E}), \quad T_- < t_0 < T_+.$$ 

- The energy is conserved; the flow is reversible.
- Let $\lambda > 0$. For $v = (v, \dot{v}) \in \mathcal{E}$ we denote

$$v_\lambda(x) := \left( v\left( \frac{x}{\lambda} \right), \frac{1}{\lambda} \dot{v}\left( \frac{x}{\lambda} \right) \right).$$

We have $\|v_\lambda\|_{\mathcal{E}} = \|v\|_{\mathcal{E}}$ and $E(v_\lambda) = E(v)$. Moreover, if $u(t)$ is a solution of (WMAP) on the time interval $[0, T_+]$, then $w(t) := u\left( \frac{t}{\lambda} \right)_\lambda$ is a solution on $[0, \lambda T_+]$. 

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Stationary states – $k$-equivariant harmonic maps

- Explicit radially symmetric solutions of
  \[ \partial_r^2 u(r) + \frac{1}{r} \partial_r u(r) + \frac{k^2}{2r^2} \sin(2u(r)) = 0: \]

  \[ W_\lambda(r) := 2 \arctan\left( \frac{r^k}{\lambda^k} \right), \quad W_\lambda := (W_\lambda, 0) \in \mathcal{E}. \]

- $E(W_\lambda) = 4k\pi$.
- $W_\lambda$ are, up to sign and translation by $\pi$, all the equivariant stationary states.
- There are alternative proofs by Sterbenz and Tataru, and Lawrie and Oh.
Theorem – Côte, Kenig, Lawrie, Schlag (2015)

Let $u_0$ be such that $E(u_0) < 8k\pi$ and $\lim_{r \to 0} u_0(r) = \lim_{r \to \infty} u_0(r)$. Then the solution $u(t)$ of (WMAP) with initial data $u(0) = u_0$ exists globally and scatters in both time directions.

- The assumption $\lim_{r \to 0} u_0(r) = \lim_{r \to \infty} u_0(r)$ is equivalent to the topological degree of $u_0$ being equal to 0.
- For any $\eta > 0$ there exists $u_0$ such that $E(u_0) < 8k\pi + \eta$ and the solution with initial data $u(0) = u_0$ blows up in finite time.
- The possible behavior at the threshold $E(u) = 8k\pi = 2E(W)$ is currently not understood.
- The *threshold theorem* is a weakened version of what would be a *soliton resolution theorem*. 
Theorem – Duyckaerts, Kenig and Merle (2014)

Let \( u(t) : [0, T_+) \rightarrow \mathcal{E} \) be a radial solution of (NLW) in dimension \( N = 3 \).

- **Type II blow-up:** If \( T_+ < \infty \) and \( \|u(t)\|_{\mathcal{E}} \) is bounded, then there exist \( u_0^* \in \mathcal{E}, \nu_j \in \{\pm 1\}, \lambda_j(t) \) with
  \( \lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_n(t) \ll T_+ - t \) as \( t \to T_+ \) such that
  \[
  \lim_{t \to T_+} \left\| u(t) - \left( u_0^* + \sum_{j=1}^{n} \nu_j W_{\lambda_j(t)} \right) \right\|_{\mathcal{E}} = 0.
  \]

- **Global solution:** If \( T_+ = +\infty \), then there exist a solution \( u_L^* \) of the linear wave equation, \( \nu_j \in \{\pm 1\}, \lambda_j(t) \) with
  \( \lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_n(t) \ll t \) as \( t \to +\infty \) such that
  \[
  \lim_{t \to +\infty} \left\| u(t) - \left( u_L^*(t) + \sum_{j=1}^{n} \nu_j W_{\lambda_j(t)} \right) \right\|_{\mathcal{E}} = 0.
  \]
Such a decomposition *for a sequence of times* holds in the nonradial case for $N \in \{3, 4, 5, 6\}$

- Duyckaerts, Jia, Kenig and Merle (2016)

Similar results for critical wave maps with values in $S^2$ in the equivariant case

- Côte (2015)

Do there exist solutions decomposing into more than one bubble?
**Theorem – J. (2016)**

Let \( k \geq 3 \). There exists a solution \( u : (-\infty, T_0] \to \mathcal{E} \) of (WMAP),

\[
\lim_{t \to -\infty} \| u(t) - (-\mathcal{W} + \mathcal{W} \frac{k-2}{2\kappa} (\kappa|t|) - \frac{2}{k-2}) \|_{\mathcal{E}} = 0, \quad \kappa \text{ constant } > 0.
\]

**Theorem – J. (2016)**

Let \( N = 6 \). There exists a solution \( u : (-\infty, T_0] \to \mathcal{E} \) of (NLW),

\[
\lim_{t \to -\infty} \| u(t) - (\mathcal{W} + \mathcal{W} \frac{1}{\kappa} e^{-\kappa|t|}) \|_{\mathcal{E}} = 0, \quad \text{with } \kappa \text{ constant } > 0.
\]

**Theorem – J. (2016)**

Let \( N \geq 7 \). There exists a solution \( u : (-\infty, T_0] \to \mathcal{E} \) of (NLS),

\[
\lim_{t \to -\infty} \| u(t) - (-i\mathcal{W} + \mathcal{W} \frac{1}{\kappa} (\kappa|t|)^{-\frac{N}{6}}) \|_{\mathcal{E}} = 0, \quad \text{with } \kappa \text{ constant } > 0.
\]
An analogous result holds for the critical radial Yang-Mills equation (exponential concentration rate); the same would be the case for (WMAP) with $k = 2$.

For (NLW), in any dimension $N > 6$ we can expect an analogous result, with concentration rate $\lambda(t) \sim |t|^{-\frac{4}{N-6}}$. Similarly for (NLS) in dimension $N = 6$.

Strong interaction of bubbles: the second bubble could not concentrate without being “pushed” by the first one,
- Martel and Raphaël (2015)

Other related works – concentration of one bubble:
Main scheme

- Key idea: construct a sequence of solutions $u_n(t)$ such that we have uniform bounds

$$\|u_n(t) - (-W + W\lambda_{\text{app}}(t))\|_\infty \leq e(t) \quad \text{for } t \in [T_n, T_0],$$

where $T_n \to -\infty$, $\lambda_{\text{app}}(t)$ and $e(t)$ do not depend on $n$, $\lim_{t \to -\infty} \lambda_{\text{app}}(t) = \lim_{t \to -\infty} e(t) = 0$.

- Impose the initial condition at time $T_n$ and control the evolution until a fixed time $T_0$.

- Pass to a weak limit.
  - Merle (1990); Martel (2005).

Note that time reversibility of the flow is crucial.
Controlling $u_n$

Here $n$ is fixed and we write $u = u_n$.

- We decompose $u = -W + W_\lambda + g$, using well chosen orthogonality conditions. We obtain the following differential equation for the modulation parameter $\lambda(t)$:

$$\lambda'' = C_0 \lambda^{k-1} + \ldots \lambda, \lambda', g$$

- By the conservation of energy, $\|g\|_{\mathcal{E}}^2 \lesssim \lambda^k$.
- It turns out that this bound allows us to treat $g$ as a small perturbation in the modulation equations.
- The proof is reduced to a stability analysis of the modulation equations, which is standard.
Thank you!