Construction of concentrating bubbles for the energy-critical wave equation

Jacek Jendrej
University of Chicago

University of North Carolina
February 13th 2017
Critical NLW

Focusing energy-critical power nonlinearity in dimension $1 + N$ (with $N \geq 3$):

\[
\begin{aligned}
\partial_t^2 u(t, x) &= \Delta_x u(t, x) + |u(t, x)|^{\frac{4}{N-2}} u(t, x), \\
(u(t_0, x), \partial_t u(t_0, x)) &= (u_0(x), \dot{u}_0(x)).
\end{aligned}
\]  

(NLW)

Notation: $\vec{u}_0 := (u_0, \dot{u}_0)$, $\mathcal{H} := \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$.

There is a natural energy functional defined on $\mathcal{H}$:

\[
E(\vec{u}_0) = \int \frac{1}{2} |\dot{u}_0|^2 + \frac{1}{2} |\nabla u_0|^2 - \frac{N - 2}{2N} |u_0|^{\frac{2N}{N-2}} \, dx.
\]

We consider solutions with radial symmetry: $\vec{u}(t, x) = \vec{u}(t, |x|)$.

This model shares some features with critical equivariant wave maps:

\[
\partial_t^2 u(t, r) = \partial_r^2 u(t, r) + \frac{1}{r} u(t, r) - \frac{k^2}{2r^2} \sin(2u(t, r)).
\]  

(WM)
Local well-posedness in $\mathcal{H}$ (conditional)
- Ginibre, Soffer, Velo (1992)
- Shatah, Struwe (1994)

\[ \forall \tilde{u}_0 \in \mathcal{H}, \exists! \tilde{u} \in C((T_-, T_+); \mathcal{H}), \quad T_- < t_0 < T_+. \]

The energy is conserved; the flow is reversible.

Let $\lambda > 0$. For $\tilde{v} = (v, \dot{v}) \in \mathcal{H}$ we denote

\[ \tilde{v}_\lambda(x) := \left( \lambda^{-\frac{N-2}{2}} v \left( \frac{x}{\lambda} \right), \lambda^{-\frac{N}{2}} \dot{v} \left( \frac{x}{\lambda} \right) \right). \]

We have $\| \tilde{v}_\lambda \|_{\mathcal{H}} = \| \tilde{v} \|_{\mathcal{H}}$ and $E(\tilde{v}_\lambda) = E(\tilde{v})$. Moreover, if $\tilde{u}(t)$ is a solution of (NLW) on the time interval $[0, T_+]$, then $\tilde{w}(t) := \tilde{u} \left( \frac{t}{\lambda} \right) \lambda$ is a solution on $[0, \lambda T_+]$. 
Explicit radially symmetric solution of $\Delta W(x) + |W(x)|^{\frac{4}{N-2}} W(x) = 0$:

$$W(x) := \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-\frac{N-2}{2}}, \quad \vec{W} := (W, 0) \in \mathcal{H}.$$ 

All radial stationary states are obtained by rescaling: $S := \{\vec{W}_\lambda\}$.


“Building blocks” of every solution bounded in $\mathcal{H}$?
For \( N = 3 \) in the radial case the answer is "yes":

**Theorem – Duyckaerts, Kenig, Merle (2012)**

Let \( \vec{u}(t) : [0, T_+) \to \mathcal{H} \) be a radial solution of (NLW) in dimension \( N = 3 \).

- **Type II blow-up:** If \( T_+ < \infty \) and \( \|\vec{u}(t)\|_{\mathcal{H}} \) is bounded, then there exist \( \vec{u}_0^* \in \mathcal{H} \), \( \iota_j \in \{\pm 1\} \), \( \lambda_j(t) \) with \( \lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_n(t) \ll T_+ - t \) as \( t \to T_+ \) such that

  \[
  \lim_{t \to T_+} \|\vec{u}(t) - \left( \vec{u}_0^* + \sum_{j=1}^n \iota_j \vec{W}_{\lambda_j(t)} \right)\|_{\mathcal{H}} = 0.
  \]

- **Global solution:** If \( T_+ = +\infty \), then there exist a solution \( \vec{u}_{L}^* \) of the linear wave equation, \( \iota_j \in \{\pm 1\} \), \( \lambda_j(t) \) with \( \lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_n(t) \ll t \) as \( t \to +\infty \) such that

  \[
  \lim_{t \to +\infty} \|\vec{u}(t) - \left( \vec{u}_L^*(t) + \sum_{j=1}^n \iota_j \vec{W}_{\lambda_j(t)} \right)\|_{\mathcal{H}} = 0.
  \]
Comments

- Such a decomposition *for a sequence of times* holds in the nonradial case for $N \in \{3, 4, 5\}$
  - Duyckaerts, Jia, Kenig, Merle (2016)
- Similar results for critical wave maps with values in $S^2$ in the equivariant case
  - Côte (2015)
- The “Soliton Resolution Conjecture” originates in the theory of integrable systems
  - Eckhaus, Schuur (1983)
- Classical problem in the theory of the heat flow from $S^2$ to $S^2$
  - Struwe; Qing; Topping
Some natural questions

- Study the dynamics of solutions which remain close to \( \{ \vec{W}_\lambda \} \) (for example in the energy space: \( \| \vec{u}(t) - \vec{W}_{\lambda(t)} \|_H \leq \eta \ll 1 \) for \( t_0 \leq t < T_+ \)). How does \( \lambda(t) \) behave as \( t \to T_+ \) in this case?
  - Rodnianski, Raphaël (2012)
  - Hillairet, Raphaël (2012)
  - Donninger, Krieger (2013)
  - Krieger, Schlag (2014)
  - Donninger, Huang, Krieger, Schlag (2014)
  - Ortoleva, Perelman (2013)
  - Perelman (2014)
  - Collot (2014)

- Do there exist solutions decomposing into more than one bubble?
Results

I considered two special cases of the (continuous time) soliton resolution in various dimensions:

- One bubble in the finite-time blow-up case:
  \[
  \vec{u}(t) = \vec{u}^*(t) + \vec{W}_\lambda(t) + \vec{h}(t), \quad \lim_{t \to T_+} \lambda(t) = 0, \quad \vec{u}^*(T_+) = \vec{u}_0^*,
  \]

- Two bubbles without remainder:
  \[
  \vec{u}(t) = \vec{W}_\mu(t) \pm \vec{W}_\lambda(t) + \vec{h}(t), \quad \lambda(t) \ll \mu(t) \text{ as } t \to T_+,
  \]

where \(\vec{h}(t)\) is an error term which satisfies \(\lim_{t \to T_+} \|\vec{h}(t)\|_{H^\lambda} = 0\).

Main idea – “modulation theory”

Try to understand the dynamics by “forgetting” \(\vec{h}(t)\), hence reducing the equation to an ODE. Then, control \(\vec{h}(t)\) using (modified) energy functionals.
Examples of solutions with two bubbles

**Theorem – J. (2016)**

Let $N = 6$. There exists a solution $\vec{u} : (-\infty, T_0] \to \mathcal{H}$ of (NLW) such that

$$\lim_{t \to -\infty} \left\| \vec{u}(t) - (\vec{W} + \vec{W} \frac{1}{\kappa} e^{-\kappa |t|}) \right\|_{\mathcal{H}} = 0,$$

with $\kappa := \sqrt{\frac{5}{4}}$.

We have a similar result for wave maps:

**Theorem – J. (2016)**

Let $k \geq 3$. There exists a solution $\vec{u} : (-\infty, T_0] \to \mathcal{H}$ of (WM) such that

$$\lim_{t \to -\infty} \left\| \vec{u}(t) - (-\vec{W} + \vec{W} \frac{k-2}{2\kappa} (\kappa |t|)^{\frac{2}{k-2}}) \right\|_{\mathcal{H}} = 0,$$

$\kappa$ constant $> 0$. 

Jacek Jendrej
Energy-critical wave equations
02/13/2017 9 / 20
A similar result holds for the critical radial Yang-Mills equation (exponential concentration rate); the same would be the case for (WM) with $k = 2$.

In any dimension $N > 6$ we can expect an analogous result, with concentration rate $\lambda(t) \sim |t|^{-\frac{4}{N-6}}$.

Strong interaction of bubbles: the second bubble could not concentrate without being “pushed” by the first one,
- Martel, Raphaël (2015)
We search a solution of the form $\vec{u}(t) \simeq \vec{W} + \vec{W}_\lambda(t)$.

$$\vec{u}(t) = \vec{W} + \vec{U}^{(0)}_\lambda(t) + b(t) \cdot \vec{U}^{(1)}_\lambda(t) + b(t)^2 \cdot \vec{U}^{(2)}_\lambda(t) + \ldots,$$

with $\vec{U}^{(0)} = \vec{W}$, $b(t) > 0$ and $\lambda(t), b(t) \to 0$ as $t \to -\infty$.

For $v(x) : \mathbb{R}^6 \to \mathbb{R}$ denote:

$$v_\lambda(x) := \frac{1}{\lambda^2} v \left( \frac{x}{\lambda} \right), \quad \Lambda v := -\frac{\partial}{\partial \lambda} v_\lambda|_{\lambda=1} = (2+x \cdot \nabla)v, \quad \Lambda_0 v := (3+x \cdot \nabla)v.$$

Since $u(t) \simeq W + W_\lambda(t)$, we have

$$\dot{u}(t) = \partial_t u(t) \simeq -\frac{\lambda'(t)}{\lambda(t)} (\Lambda W)_\lambda(t) \quad \Rightarrow \quad b(t) = \lambda'(t), \quad \vec{U}^{(1)} = (0, -\Lambda W).$$

Can we find $\vec{U}^{(2)} = (U^{(2)}, \dot{U}^{(2)})$?
Asymptotic expansion, 2

Neglecting irrelevant terms and replacing $\lambda'(t)$ by $b(t)$, we compute

$$\partial_t^2 u(t) = -\frac{b'(t)}{\lambda(t)}(\Lambda W)_{\lambda(t)} + \frac{b(t)^2}{\lambda(t)^2}(\Lambda_0 \Lambda W)_{\lambda(t)} + \ldots.$$  

Denote $f(u) := |u|u$ and $L = -\Delta - f'(W)$ the linearization of $-\Delta u - f(u)$ near $u = W$. A simple computation yields

$$\Delta u(t) + f(u(t)) = -\frac{b(t)^2}{\lambda(t)^2}(LU^{(2)})_{\lambda(t)} + f'(W)_{\lambda(t)} + \ldots,$$

We discover that

$$LU^{(2)} = -\Lambda_0 \Lambda W + \frac{\lambda(t)}{b(t)^2}(b'(t) \cdot \Lambda W + \lambda(t) \cdot f'(W)).$$
Fredholm condition

\[
L U^{(2)} = -\Lambda_0 \Lambda W + \frac{\lambda(t)}{b(t)^2} (b'(t) \cdot \Lambda W + \lambda(t) \cdot f'(W)).
\]  

(1)

Due to scaling invariance, \( \Lambda W \in \text{ker } L \).

\[
\int_{\mathbb{R}^6} \Lambda W \cdot (b'(t) \cdot \Lambda W + \lambda(t) \cdot f'(W)) \, dx = 0
\]

\[\iff \quad b'(t) = \frac{5}{4} \lambda(t) = \kappa^2 \lambda(t).\]

If this condition is satisfied, we can solve (1) and find \( U^{(2)} \). We take \( \vec{U}^{(2)} = (U^{(2)}, 0) \).

\[
\begin{align*}
\lambda'(t) &= b(t) \\
\kappa^2 \lambda(t) &= b'(t)
\end{align*}
\]

leads to

\[
\begin{align*}
\lambda_{\text{app}}(t) &= \frac{1}{\kappa} e^{-\kappa |t|} \\
b_{\text{app}}(t) &= e^{-\kappa |t|}.
\end{align*}
\]
Main scheme

- **Approximate solution:**

\[
\tilde{\varphi}(\mu, \lambda, b) := \tilde{W}_\mu + \tilde{U}_\lambda^{(0)} + b \cdot \tilde{U}_\lambda^{(1)} + b^2 \cdot \tilde{U}_\lambda^{(2)}
\]

- **Key idea:** construct a sequence of solutions \(\tilde{u}_n(t)\) such that we have *uniform* bounds

\[
\|\tilde{u}_n(t) - \tilde{\varphi}(1, \lambda_{\text{app}}(t), b_{\text{app}}(t))\|_{\mathcal{H}} \leq C e^{-\frac{1}{2} \kappa |t|}
\]

for \(t \in [T_n, T_0]\) with \(T_n \to -\infty\).

- To do this, we consider the initial data \(\tilde{u}_n(T_n) = \tilde{\varphi}(1, \lambda_{\text{app}}(T_n), b_{\text{app}}(T_n))\) (with a correction due to the linear unstable direction).

- Pass to a weak limit.
  - Merle (1990); Martel (2005)

Note that time reversibility of the flow is crucial.
Control of the error term by the energy method

- Decompose $\vec{u}_n(t) = \vec{\varphi}(\mu(t), \lambda(t), b(t)) + \vec{g}(t)$, with $\mu(t)$, $\lambda(t)$ defined by natural orthogonality conditions and $b(t) := b_{\text{app}}(T_n) + \int_{T_n}^{t} \kappa^2 \lambda(\tau) \, d\tau$.

- If we assume that $\|\vec{g}(t)\|_{\mathcal{H}} \leq C e^{-\frac{3}{2} \kappa |t|}$, we can solve differential inequalities and find $\lambda(t) \approx \lambda_{\text{app}}(t)$, $\mu(t) \approx 1$ and $b(t) \approx b_{\text{app}}(t)$.

- We construct a functional $H(t)$ such that $H(t) \gtrsim \|\vec{g}(t)\|_{\mathcal{H}}^2$ and

$$\|\vec{g}(t)\|_{\mathcal{H}} \leq C_0 e^{-\frac{3}{2} \kappa |t|} \text{ for } t \in [T_n, T] \Rightarrow H'(t) \leq c \cdot C_0^2 \cdot e^{-3\kappa |t|},$$

with $c$ arbitrarily small. $H(t)$ is the energy functional, corrected using a localized virial term.

- A continuity argument yields the required bounds on $\|\vec{g}(t)\|_{\mathcal{H}}$. This finishes the proof.

  ▶ Raphaël, Szeftel (2011)
Take $\vec{u}_0^* \in \mathcal{H}$ and let $\vec{u}^*(t)$ be the solution of (NLW) with $\vec{u}^*(0) = \vec{u}_0^*$. We wish to construct $\vec{u}(t) \simeq \vec{u}^*(t) + \vec{W}_\lambda(t)$ with $\lambda(t) \to 0$ as $t \to 0$. 

**Theorem – J. (2015)**

Let $N = 5$ and let $\vec{u}_0^* \in H^5 \times H^4$ with $u_0^*(0) > 0$. There exists a solution $\vec{u}(t) : (0, T_0) \to \mathcal{H}$ of (NLW) such that

$$\lim_{t \to 0^+} \| \vec{u}(t) - (\vec{u}_0^* + \vec{W}_{\lambda_{\text{app}}}(t)) \|_{\mathcal{H}} = 0, \quad \lambda_{\text{app}}(t) := \left( \frac{32}{315\pi} \right)^2 (u_0^*(0))^2 t^4.$$ 

**Theorem – J. (2015)**

Let $\nu > 8$. There exists a solution $\vec{u}(t) : (0, T_0) \to \mathcal{H}$ of (NLW) such that

$$\lim_{t \to 0^+} \| \vec{u}(t) - (\vec{u}_0^* + \vec{W}_{\lambda_{\text{app}}}(t)) \|_{\mathcal{H}} = 0, \quad \lambda_{\text{app}}(t) := t^{\nu+1},$$ 

where $\vec{u}_0^* = \left( \frac{315\nu(\nu+1)}{32(\nu-1)(\nu+3)} |x|^{\nu-3} \frac{\nu-3}{2}, 0 \right)$ near $x = 0$. 
Bounds on the speed of type II blow-up

It seems that there is a relationship between $\vec{u}_0^*$ and the asymptotics of $\lambda(t)$. One can prove an upper bound:

**Theorem – J. (2015)**

Let $N \in \{3, 4, 5\}$ and $\vec{u}_0^* \in H^3 \times H^2$ be a radial function. Suppose that $\vec{u}(t)$ is a radial solution of (NLW) such that

$$\lim_{t \to T_+} \|\vec{u}(t) - (\vec{u}_0^* + \vec{W}_{\lambda(t)})\|_H = 0, \quad \lim_{t \to T_+} \lambda(t) = 0, \quad T_+ < +\infty.$$

Then there exists a constant $C > 0$ depending on $\vec{u}_0^*$ such that:

$$\lambda(t) \leq C(T_+ - t)^{\frac{4}{6-N}}.$$

- **Main idea:** $E(\vec{u}^* + \vec{W}_{\lambda} + \vec{h}) - E(\vec{u}^*) - E(\vec{W}) \geq -C\lambda^{\frac{N-2}{2}} + c\|\vec{h}\|^2_H$.
- As a by-product, if $u_0^*(0) < 0$, then such a solution cannot exist.
Non-existence of two-bubbles with opposite signs

**Theorem – J. (2015)**

Let $N \geq 3$. There exist no radial solutions $\vec{u} : [t_0, T_+) \to H$ of (NLW) such that

$$\lim_{t \to T_+} \| \vec{u}(t) - (\vec{W}_\mu(t) - \vec{W}_\lambda(t)) \|_H = 0$$

and

- in the case $T_+ < +\infty$, $\lambda(t) \ll \mu(t) \ll T_+ - t$ as $t \to T_+$,
- in the case $T_+ = +\infty$, $\lambda(t) \ll \mu(t) \ll t$ as $t \to +\infty$.

**Main idea:** $E(\vec{W}_\mu - \vec{W}_\lambda + \vec{h}) - 2E(\vec{W}) \geq c\lambda \frac{N-2}{2} + c\|\vec{h}\|_H^2$. Compare with an easy variational proof in the case of (WM).

- Negative eigenvalues related to (un)stable directions of the wave flow.
- We can recover coercivity by modulating around the stable variety.
Some open problems

- Prove that $\vec{u}_0^*$ determines the asymptotics of $\lambda(t)$.
- Classify the solutions at the energy level $2E(\vec{W})$ and topological degree 0 for (WM).
  - Côte, Kenig, Lawrie, Schlag (2015) for $E(\vec{u}) < 2E(\vec{W})$,
  - Expect a unique non-dispersive solution.
Thank you!