Unstable blow-up and trees of bubbles for energy-critical wave equations

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Overview of the talk

1. Energy-critical wave equations
2. Soliton resolution
3. Examples of solutions with two bubbles
4. Formal computation
5. Elements of the proof
6. Construction of unstable type II blow-up solutions
7. Bounds on the speed of type II blow-up
8. Non-existence of two-bubbles with opposite signs
Critical NLW

Focusing energy-critical power nonlinearity in dimension $1 + N$ (with $N \geq 3$):

\[
\begin{aligned}
\partial_{tt} u(t, x) &= \Delta_x u(t, x) + |u(t, x)|^{\frac{4}{N-2}} u(t, x), \\
(u(t_0, x), \partial_t u(t_0, x)) &= (u_0(x), \dot{u}_0(x)).
\end{aligned}
\]  \hspace{2cm} \text{(NLW)}

Notation: $u_0 := (u_0, \dot{u}_0)$, $\mathcal{E} := \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$.

There is a natural energy functional defined on $\mathcal{E}$:

\[
E(u_0) = \int \frac{1}{2} |\dot{u}_0|^2 + \frac{1}{2} |\nabla u_0|^2 - \frac{N-2}{2N} |u_0|^{\frac{2N}{N-2}} \, dx.
\]

We consider solutions with radial symmetry: $u(t, x) = u(t, |x|)$. This model shares some features with critical equivariant wave maps:

\[
\partial_t^2 u(t, r) = \partial_r^2 u(t, r) + \frac{1}{r} u(t, r) - \frac{k^2}{2r^2} \sin(2u(t, r)). \hspace{2cm} \text{(WM)}
\]
Comments

- (Local well-posedness in $\mathcal{E}$)
  - Ginibre, Soffer, Velo (1992)
  - Shatah, Struwe (1994)
  \[ \forall u_0 \in \mathcal{E}, \exists! u \in C((T_-, T_+); \mathcal{E}), \quad T_- < t_0 < T_+. \]

- The energy is conserved; the flow is reversible.

- (Scaling) Let $\lambda > 0$. For $v = (v, \dot{v}) \in \mathcal{E}$ we denote
  \[ v_\lambda(x) := \left( \lambda^{-\frac{N-2}{2}} v\left(\frac{x}{\lambda}\right), \lambda^{-\frac{N}{2}} \dot{v}\left(\frac{x}{\lambda}\right) \right). \]

We have $\|v_\lambda\|_{\mathcal{E}} = \|v\|_{\mathcal{E}}$ and $E(v_\lambda) = E(v)$. Moreover, if $u(t)$ is a solution of (NLW) on the time interval $[0, T_+]$, then $w(t) := u\left(\frac{t}{\lambda}\right)_{\lambda}$ is a solution on $[0, \lambda T_+]$. 
Ground states

- Explicit radially symmetric solution of \( \Delta W(x) + |W(x)|^{\frac{4}{N-2}} W(x) = 0 \):

\[
W(x) := \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-\frac{N-2}{2}}, \quad \mathbf{W} := (W, 0) \in \mathcal{E}.
\]

- All the radially symmetric stationary states are obtained by rescaling: \( S := \{\mathbf{W}_\lambda\} \).


- \( S \) is a non-compact subset of \( \mathcal{E} \).

- All the elements of \( S \) have the same energy.

- “Building blocks” of every solution bounded in \( \mathcal{E} \).
For $N = 3$ in the radial case the answer is “yes”:

**Theorem – Duyckaerts, Kenig, Merle (2012)**

Let $u(t) : [0, T_+) \to \mathcal{E}$ be a radial solution of (NLW) in dimension $N = 3$.

- **Type II blow-up:** If $T_+ < \infty$ and $\|u(t)\|_{\mathcal{E}}$ is bounded, then there exist $v_0 \in \mathcal{E}$, $\iota_j \in \{\pm 1\}$, $\lambda_j(t)$ with $\lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_n(t) \ll T_+ - t$ as $t \to T_+$ such that

$$\lim_{t \to T_+} \left\| u(t) - \left( v_0 + \sum_{j=1}^{n} \iota_j W \lambda_j(t) \right) \right\|_{\mathcal{E}} = 0.$$ 

- **Global solution:** If $T_+ = +\infty$, then there exist a solution $v_L$ of the linear wave equation, $\iota_j \in \{\pm 1\}$, $\lambda_j(t)$ with $\lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_n(t) \ll t$ as $t \to +\infty$ such that

$$\lim_{t \to +\infty} \left\| u(t) - \left( v_L(t) + \sum_{j=1}^{n} \iota_j W \lambda_j(t) \right) \right\|_{\mathcal{E}} = 0.$$
Comments

- Such a decomposition *for a sequence of times* holds in the nonradial case for $N \in \{3, 4, 5\}$
  - Duyckaerts, Jia, Kenig, Merle (2016)
- Similar results for critical wave maps with values in $S^2$ in the equivariant case
  - Côte (2015)
- The “Soliton Resolution Conjecture” originates in the theory of integrable systems
  - Eckhaus, Schuur (1983)
- Classical problem in the theory of the heat flow from $S^2$ to $S^2$
  - Struwe; Qing; Topping
Some natural questions

- Study the dynamics of solutions which remain close to \( \{ W_\lambda \} \) (for example in the energy space: \( \| u(t) - W_\lambda(t) \|_{\mathcal{C}} \leq \eta \ll 1 \) for \( t_0 \leq t < T_+ \)).

  How does \( \lambda(t) \) behave as \( t \to T_+ \) in this case?

  - Rodnianski, Raphaël (2012)
  - Hillairet, Raphaël (2012)
  - Donninger, Krieger (2013)
  - Krieger, Schlag (2014)
  - Donninger, Huang, Krieger, Schlag (2014)
  - Ortoleva, Perelman (2013)
  - Perelman (2014)

- Do there exist solutions decomposing into more than one bubble?
Examples of solutions with two bubbles

Unknown for $N \in \{3, 4, 5\}$. They exist for $N = 6$.

Theorem – J. (2016)

Let $N = 6$. There exists a solution $u : (-\infty, T_0] \to \mathcal{E}$ of (NLW) such that

$$\lim_{t \to -\infty} \left\| u(t) - (\mathcal{W} + \mathcal{W} \frac{1}{\kappa} e^{-\kappa |t|}) \right\|_{\mathcal{E}} = 0,$$

with $\kappa := \sqrt{\frac{5}{4}}$.

We have a similar result for wave maps:

Theorem – J. (2016)

Let $k \geq 3$. There exists a solution $u : (-\infty, T_0] \to \mathcal{E}$ of (WM) such that

$$\lim_{t \to -\infty} \left\| u(t) - (-\mathcal{W} + \mathcal{W} \frac{k-2}{2\kappa} (\kappa |t|)^{\frac{k-2}{k}}) \right\|_{\mathcal{E}} = 0,$$

$\kappa$ constant $> 0$. 
A similar result holds for the critical radial Yang-Mills equation (exponential concentration rate); the same would be the case for (WM) with $k = 2$.

In any dimension $N > 6$ we can expect an analogous result, with concentration rate $|t|^{-\frac{4}{N-6}}$.

Strong interaction of bubbles: the second bubble could not concentrate without being “pushed” by the first one,

- Martel, Raphaël (2015)
- Gérard, Lenzmann, Pocovnicu, Raphaël (in progress)

The two bubbles must have the same sign in the case of (NLW), and opposite orientations in the case of wave maps.
Asymptotic expansion, 1

We search a solution of the form \( u(t) \simeq W + W\lambda(t) \).

\[
u(t) = W + U^{(0)}_{\lambda(t)} + b(t) \cdot U^{(1)}_{\lambda(t)} + b(t)^2 \cdot U^{(2)}_{\lambda(t)} + \ldots,
\]

with \( U^{(0)} = W \), \( b(t) > 0 \) and \( \lambda(t), b(t) \to 0 \) as \( t \to -\infty \).

For \( v(x) : \mathbb{R}^6 \to \mathbb{R} \) denote:

\[
v_\lambda(x) := \frac{1}{\lambda^2} v \left( \frac{x}{\lambda} \right), \quad \Lambda v := -\frac{\partial}{\partial \lambda} v_\lambda |_{\lambda=1} = (2 + x \cdot \nabla)v, \quad \Lambda_0 v := (3 + x \cdot \nabla)v.
\]

Since \( u(t) \simeq W + W\lambda(t) \), we have

\[
\dot{u}(t) = \partial_t u(t) \simeq -\frac{\lambda'(t)}{\lambda(t)} (\Lambda W)_{\lambda(t)} \quad \Rightarrow \quad b(t) = \lambda'(t), \quad U^{(1)} = (0, -\Lambda W).
\]

Can we find \( U^{(2)} = (U^{(2)}, \dot{U}^{(2)}) \)?
Asymptotic expansion, 2

Neglecting irrelevant terms and replacing $\lambda'(t)$ by $b(t)$, we compute

$$
\partial_t^2 u(t) = -\frac{b'(t)}{\lambda(t)} (\Lambda W)_{\lambda(t)} + \frac{b(t)^2}{\lambda(t)^2} (\Lambda_0 \Lambda W)_{\lambda(t)} + \ldots.
$$

Denote $L = -\Delta - f'(W)$ the linearization of $-\Delta u - f(u)$ near $u = W$. A simple computation yields

$$
\Delta u(t) + f(u(t)) = -\frac{b(t)^2}{\lambda(t)^2} (LU^{(2)})_{\lambda(t)} + f'(W)_{\lambda(t)} + \ldots,
$$

We discover that

$$
LU^{(2)} = -\Lambda_0 \Lambda W + \frac{\lambda(t)}{b(t)^2} (b'(t) \cdot \Lambda W + \lambda(t) \cdot f'(W)).
$$
Fredholm condition

\[ LU^{(2)} = -\Lambda_0 \Lambda W + \frac{\lambda(t)}{b(t)^2} (b'(t) \cdot \Lambda W + \lambda(t) \cdot f'(W)) \]  

(1)

Due to scaling invariance, \( \Lambda W \in \ker L \).

\[ \int_{\mathbb{R}^6} \Lambda W \cdot (b'(t) \cdot \Lambda W + \lambda(t) \cdot f'(W)) \, dx = 0 \]

\[ \Leftrightarrow \quad b'(t) = \frac{5}{4} \lambda(t) = \kappa^2 \lambda(t). \]

If this condition is satisfied, we can solve (1) and find \( U^{(2)} \). We take \( U^{(2)} = (U^{(2)}, 0) \).

\[
\begin{cases}
\lambda'(t) = b(t) \\
 b'(t) = \kappa^2 \lambda(t)
\end{cases}
\quad \text{leads to} \quad \begin{cases}
\lambda_{\text{app}}(t) = \frac{1}{\kappa} e^{-\kappa |t|} \\
 b_{\text{app}}(t) = e^{-\kappa |t|}.
\end{cases}
\]
Main scheme

- **Approximate solution:**

  \[ \varphi(\mu, \lambda, b) := \mathcal{W}_\mu + U^{(0)} + b \cdot U^{(1)} + b^2 \cdot U^{(2)} \]

- **Key idea:** construct a sequence of solutions \( u_n(t) \) such that we have *uniform* bounds

  \[ \|u_n(t) - \varphi(1, \lambda_{\text{app}}(t), b_{\text{app}}(t))\| \leq C e^{-\frac{3}{2} \kappa |t|} \]
  
  for \( t \in [T_n, T_0] \) with \( T_n \to -\infty \).

- **To do this,** we consider the initial data

  \( u_n(T_n) = \varphi(1, \lambda_{\text{app}}(T_n), b_{\text{app}}(T_n)) \) (with a correction due to the linear unstable direction).

- **Pass to a weak limit.**

  - Merle (1990); Martel (2005)

Note that time reversibility of the flow is crucial.
Control of the error term by the energy method

- Decompose $u_n(t) = \varphi(\mu(t), \lambda(t), b(t)) + g(t)$, with $\mu(t), \lambda(t)$ defined by natural orthogonality conditions and $b(t) := b_{\text{app}}(T_n) + \int_{T_n}^{t} \kappa^2 \lambda(\tau) \, d\tau$.

- We construct a functional $H(t)$ such that $H(t) \gtrsim \|g(t)\|_E^2$ and

$$\|g(t)\|_E \leq C_0 e^{-\frac{3}{2} \kappa |t|} \text{ for } t \in [T_n, T] \quad \Rightarrow \quad H'(t) \leq c \cdot C_0^2 \cdot e^{-3\kappa |t|},$$

with $c$ arbitrarily small. $H(t)$ is the energy functional, corrected using a localized virial term.

- A continuity argument yields the required bounds on $\|g(t)\|_E$. This finishes the proof.
  - Raphaël, Szeftel (2011)
Take $u^*_0 \in \mathcal{E}$ and let $u^*(t)$ be the solution of (NLW) with $u^*(0) = u^*_0$. We wish to construct $u(t) \simeq u^*(t) + W\lambda(t)$ with $\lambda(t) \to 0$ as $t \to 0$. 

**Theorem – J. (2015)**

Let $N = 5$ and let $u^*_0 \in H^5 \times H^4$ with $u^*_0(0) > 0$. There exists a solution $u(t) : (0, T_0) \to \mathcal{E}$ of (NLW) such that

$$
\lim_{t \to 0^+} \left\| u(t) - (u^*_0 + W\lambda_{\text{app}}(t)) \right\|_{\mathcal{E}} = 0, \quad \lambda_{\text{app}}(t) := \left(\frac{32}{315\pi}\right)^2 (u^*_0(0))^2 t^4.
$$

**Theorem – J. (2015)**

Let $\nu > 8$. There exists a solution $u(t) : (0, T_0) \to \mathcal{E}$ of (NLW) such that

$$
\lim_{t \to 0^+} \left\| u(t) - (u^*_0 + W\lambda_{\text{app}}(t)) \right\|_{\mathcal{E}} = 0, \quad \lambda_{\text{app}}(t) := t^{\nu+1},
$$

where $u^*_0 = \left(\frac{315\nu(\nu+1)\pi}{32(\nu-1)(\nu+3)}|x|^\frac{\nu-3}{2}, 0\right)$ near $x = 0$. 

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Energy-critical wave equations
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Bounds on the speed of type II blow-up

It seems that there is a relationship between $u_0^*$ and the asymptotics of $\lambda(t)$. One can prove an upper bound:

**Theorem – J. (2015)**

Let $N \in \{3, 4, 5\}$ and $u_0^* \in H^3 \times H^2$ be a radial function. Suppose that $u(t)$ is a radial solution of (NLW) such that

$$\lim_{t \to T_+} \| u(t) - (u_0^* + W \lambda(t)) \|_E = 0, \quad \lim_{t \to T_+} \lambda(t) = 0, \quad T_+ < +\infty.$$ 

Then there exists a constant $C > 0$ depending on $u_0^*$ such that:

$$\lambda(t) \leq C(T_+ - t)^{\frac{4}{6-N}}.$$

- Main idea: $E(u^* + W \lambda + g) - E(u^*) - E(W) \geq -C \lambda \frac{N-2}{2} + c\|g\|_E^2.$
- As a by-product, if $u_0^*(0) < 0$, then such a solution cannot exist.
Non-existence of two-bubbles with opposite signs

**Theorem – J. (2015)**

Let $N \geq 3$. There exist no radial solutions $u : [t_0, T_+) \to \mathcal{E}$ of $(\text{NLW})$ such that

$$\lim_{t \to T_+} \| u(t) - (\mathcal{W}_\mu(t) - \mathcal{W}_\lambda(t)) \|_{\mathcal{E}} = 0$$

and

- in the case $T_+ < +\infty$, $\lambda(t) \ll \mu(t) \ll T_+ - t$ as $t \to T_+$,
- in the case $T_+ = +\infty$, $\lambda(t) \ll \mu(t) \ll t$ as $t \to +\infty$.

**Main idea:** $E(\mathcal{W}_\mu - \mathcal{W}_\lambda + \mathbf{g}) - 2E(\mathcal{W}) \geq c\lambda^{\frac{N-2}{2}} + c\|\mathbf{g}\|_{\mathcal{E}}^2$.

- Compare with an easy variational proof in the case of (WM).
- Negative eigenvalues related to (un)stable directions of the wave flow.
- We can recover coercivity by modulating around the stable variety.
Some open problems

- Construct two-bubble solutions for energy-critical NLS.
- Prove that $u^*$ determines the asymptotics of $\lambda(t)$.
- Classify the solutions at the energy level $2E(W)$ and topological degree 0 for $(WM)$.
  - Côte, Kenig, Lawrie, Schlag (2015) for $E < 2E(W)$,
  - Expect a unique non-dispersive solution for $k \geq 2$, dispersion for $k = 1$. 
Thank you!